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Equilibrium Valuation of Currency Options in a Small Open Economy

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Abstract

The log-normal Garman and Kohlhagen (1983) currency option model usually creates pricing biases when matched with the market prices. The observed price bias pattern is generally consistent with the mixed jump-diffusion distribution for exchange rates. Various studies have provided evidence of jump risks in exchange rate movements. This paper argues that the jump risk in the exchange rates may be correlated with the market. Thus an equilibrium framework is needed to price the systematic jump components in currency option prices. I propose an equilibrium model to investigate the dynamics of the exchange rate in a small open monetary economy with stochastic jump-diffusion processes for both the money supply and aggregate dividend. It is shown that the exchange rate is affected by government monetary policies, aggregate dividends and the level of investment in foreign assets. As a result, the exchange rate exhibits more discontinuities than stock prices as empirically documented. Since the jump in the exchange rate is correlated with aggregate consumption, the jump risk in the exchange rate derived from aggregate dividend must be priced for currency options. I further derive the foreign agents' risk-neutral valuation of the European currency option and provide restrictions to ensure the parity conditions in currency option market. In general, these restrictions depend on the agent's risk preference.

1. Introduction

Currency options are typically priced using the model of Garman and Kohlhagen (1983) (GK hereafter), which extends Black's (1976) commodity option model to currency options. A deficiency of the GK model is that it underprices out-of-the-money currency calls as compared with market prices. In particular, for out-of-the-money calls with short maturity, the average price bias is about 29% (Bodurtha and Courtadon 1987). In an attempt to find models that correct this deficiency, various empirical studies on exchange rates suggest that there may be jump risks in exchange rate movements (See Akgiray and Booth (1988), Jorion (1988), Tucker (1991), and Ball and Roma (1993)). A number of scholars, such as Bodurtha and Courtadion (1987), Jorion (1988) and Dumas, Jennergren and Näsland (1995), suggest that replacing the Brownian motion in the GK model by Merton's (1976) mixed jump-diffusion process would improve the performance of the model.

However, there are three reasons why directly applying Merton's jump-diffusion stock option model to currency options is unsatisfactory. First, Merton's formula is derived under the assumption that the Brownian motion risk is arbitraged away, while the jump risk is uncorrelated with the market. The assumption of uncorrelated jump risks may be reasonable if the concern were firm specific stocks, but is problematic for currency markets. Since the exchange rate reflects one nation's purchasing power relative to another nation, the exchange rate is inherently correlated with aggregate fundamental forces that affect the market. For example, movements in aggregate dividends must simultaneously affect aggregate consumption and the exchange rate.

Second, the information arrival process in the foreign exchange market differs from that in the stock market, since exchange rates are directly influenced by monetary policies that do not have apparent counterparts in the stock market. Such an inappropriateness of applying stock option models to currency options has been recognized by researchers such as Jorion (1988, pp427-428):

> Many financial models rely heavily on the assumption of a particular stochastic process, while relatively little attention is paid to the empirical fit of the postulated distribution. As a result, models like option pricing models are applied indiscriminately to various markets such as the stock market and the foreign exchange market when the underlying processes may be fundamentally different.

In fact, the price bias pattern of the GK model for currency options is opposite to the bias pattern of the Black-Scholes model (1973) (BS hereafter) for stock options. The BS model generally overprices out-of-the-money calls and underprices in-the-money calls (MacBeth and Merville 1979), but the GK model usually underprices out-of-the-money currency calls (Bodurtha and Courtadon 1987). Thus, if Merton's jump-diffusion stock option model with non-systematic jumps can eliminate the price distortion by the BS model for stock options, adding a similar non-systematic jump process into the GK model may not be sufficient to correct the price bias for currency options. Jorion (1988) empirically compares the short-maturity out-of-the-money call prices given by Merton's (1976) formula with these given by the GK model. Merton's formula generates 17% of the pricing correction over the GK model. Still there are 12% pricing bias left unexplained. Such gap may be explained by systematic jump risks. Recent study by Cao (1997) show that it is necessary to incorporate both systematic and non-systematic jump risks into the currency option pricing.

A third argument against directly applying Merton's stock option model to currency options is that such an application generates seemingly paradoxical results such as the analog of the so-called Siegel's paradox in currency options. In particular, if both domestic and foreign investors maintain Merton's risk-neutral formulation of the exchange rate process, then the jump-diffusion model delivers option values that are different for the two investors. This shows that the jump risk cannot be unpriced for both investors.[1]

To eliminate these unsatisfactory consequences of applying Merton's model requires an equilibrium model of currency options, as suggested by Dumas and Näslund (1995). This paper employs a continuous-time extension of the Lucas (1978) asset pricing model to a small open monetary economy, where money has a non-trivial role in the agents' utility function. Based on utility maximization, the equilibrium analysis solves the first problem above by endogenizing the relationship between the exchange rate and the fundamental forces that underlie the market. The equilibrium exchange rate, expressed as the relative price of foreign currency in terms of home currency, is a function of the domestic money supply, aggregate dividend and the level of investments in foreign assets. This indicates that the dynamics of exchange rates and aggregate dividend are strongly correlated and such a correlation arises endogenously.

The explicit modelling of the relationship between the exchange rate and monetary policies also helps to uncover the distinct nature of the exchange rate process that differs from the stock price process. In contrast to the exchange rate, the real price of the domestic stock is only affected by aggregate dividend under logarithmic utility. Equilibrium formulation also enables me to price options on the exchange rate and stock accordingly. As a result, currency option prices are affected by the dynamics of the money supply, aggregate dividend and investments in foreign asset, but only the parameters underlying aggregate dividend affect stock option values. Moreover, the parameters underlying aggregate dividend affect currency options and stock options differently.

When an investor's intertemporal decision is explicitly formulated, the specification of the utility function abandons the assumption of unpriced jump risks and hence solves the third problem with

[1]The analog of Siegel's paradox for currency options is illustrated by Dumas and Näslund (1995). The paradox (Siegel 1972) originally illustrates the discrepancy between the forward exchange rate and the expected future exchange rate. That is, when the exchange rate is expressed as the price of the domestic currency in terms of the foreign currency, the forward exchange rate is always less than the expected future rate.

application of Merton's model. Equilibrium analysis provides consistent restrictions to eliminate the analog of Siegel's paradox in currency options. Besides the directional adjustments suggested by Bardhan (1995), the current analysis suggests that an additional adjustment be made on the jump size to reflect the fact that the jump component in the exchange rate is related to aggregate dividend. In general, all these adjustments depend on the domestic agent's risk preference.

There is a voluminous literature on currency option valuations, which exogenously specifies the spot exchange rate.[2] As pointed out by Bailey and Stulz (1989), the arbitrary choice of the exogenous process for any security price in the partial equilibrium models is unlikely to be consistent with the equilibrium condition or to provide important insights into how derivative prices may respond to changes in any fundamental economic variables. The main improvement of the current paper over these models is that assets prices and the exchange rate here are endogenously derived from agent's maximization behavior and market clearing conditions.

On the other hand, most international equilibrium models of exchange rates do not examine currency options.[3] An exception is Bakshi and Chen (1996), who extend Lucas's (1982) two-country endowment economy from a discrete-time environment to a continuous-time environment and study the exchange rate derivative valuations.[4] The current paper and the Bakshi and Chen paper both share the equilibrium approach but differ in many important aspects. These important differences indicate that these two studies are complementary to each other. First, the focus of this paper is how systematic jump risks in the exchange rate inherited from aggregate dividend affect currency option prices. As discussed earlier, the presence of jump risks in exchange rates is apparent and has important implications on currency options. On the other hand, Bakshi and Chen examine

[2]Notable examples are Amin and Jarrow (1991), Biger and Hull (1983), Chesney and Scott (1989), Dumas, Jennergren and Näslund (1995), GK (1983), Grabbe (1983), Heston (1993), and Melino and Turnbull (1990, 1991).

[3]Examples include Bekaert (1994), Stulz (1987), Svensson (1985), and Grinols and Turnovsky (1994), who focus on issues such as equilibrium interest rate, exchange rate premia and forward exchange biases.

[4]I became aware of this independent work by Bakshi and Chen (1996) after completing the first draft of this paper.

the effects of endogenized stochastic volatility on currency option prices in an equilibrium context. Second, this paper examines currency options in a small open economy, while Bakshi and Chen study a two-country economy. The distinction between a small open economy model and a two-country model has important implications on the equilibrium exchange rate. For example, in Lucas (1982) and Bakshi and Chen (1996), the equilibrium portfolio of each country is identical to its initial endowment and so the net trading in assets between the two countries is zero in equilibrium. In contrast, the net trading in foreign bonds here must be non-zero in equilibrium in order to clear the goods market. Specifically, the domestic agents in the small open economy can finance their consumption through both domestic aggregate dividend and the return to holding foreign bonds. Since the exchange rate clears the goods market, the net trading volume affects the exchange rate. Third, the role of money is introduced here through agents' utility function rather than through a cash-in-advance constraint as in Bakshi and Chen (1996) since these two approaches of modelling money are shown to be equivalent in a discrete-time setting by Feenstra (1986).[5]

The remainder of this paper is organized as follows. Section 2 describes the economy and presents general equilibrium results. Section 3 examines the endogenized exchange rate and derives equilibrium pricing formulas for European currency options from the view of the domestic risk-averse agent. Section 4 identifies the adjustments on the risk-neutral process of the exchange rate that help to solve the analog of Siegel's paradox in currency options. Section 5 extends the model to allow for a correlation between the money supply and aggregate dividends. Section 6 concludes the paper and the appendices provide necessary proofs.

[5]The money-in-the-utility approach is also technically convenient in a continuous-time setting. On the other hand, the cash-in-advance approach depends crucially on the timing of events and hence on the discrete-time structure, as stated in Sargent (1987, pp 157). For the cash-in-advance constraint to bind, all financial markets must be temporarily shut down when consumption goods are purchased with money. In a continuous-time setting where agents can instantaneously sell goods and assets for money, the cash-in-advance constraint is technically difficult to implement.

2. A Small Open Monetary Economy

Consider a small open economy with perfect capital mobility between itself (termed the domestic country) and the rest of the world (termed the foreign country). This economy consists of a single risk-averse representative agent whose lifetime horizon is infinite. I adopt the standard formulation of a small open economy, which employs the following characteristics.[6] First, the agent in the small economy has perfect access to the international goods and assets markets. Since the small economy has little influence on the foreign country, it takes the price of any foreign asset as given. Second, the domestic currency and domestic assets held by the foreign country are assumed to be negligible, implying that the supplies of these assets or currency are cleared by domestic demands. Third, domestic aggregate consumption is financed through both domestic aggregate output (dividend) and the return to holding foreign assets (which is paid in consumption goods). When the sum of aggregate dividend and the return to foreign assets exceeds aggregate consumption, the goods market is cleared by an increased holding of foreign assets (i.e., a current account surplus); when the sum of aggregate dividend and the return to foreign assets falls short of aggregate consumption, the residual is financed by a reduction in the holding of foreign assets (i.e., a current account deficit). This feature distinguishes a small open economy from a closed economy.

I will first describe the primitives of the economy and then solve the agent's maximization problem. Equilibrium asset prices, including the domestic nominal interest rate and the exchange rate, are determined by requiring goods, money and financial markets to clear, as in Lucas (1982).

[6]For a reference to a deterministic model of a small open economy, see Obstfeld (1982). An example in the stochastic environment is Grinols and Turnovsky (1994).

2.1. Structure of the Economy

There is a single good traded worldwide with no barriers, which can be used for consumption and investment. The nominal price of the good at home at time t is denoted p_t. Let P^* be the foreign price level measured in the foreign currency. According to the law of one price in the good market, p equals the spot exchange rate times P^*. Since the home country is small, it takes P^* as given and so we can simplify the discussions by normalizing $P^* = 1$.[7] Then, p_t equals the spot exchange rate expressed as the relative price of the foreign currency in terms of the home currency.

The home government controls the domestic money supply, which is taken as given by each domestic agent. The real money balance held by the domestic agent at time t is defined as $m_t = M_t/p_t$, where M_t is the domestic money demanded by home agents. To assign a non-trivial role to money, I follow Sidrauski (1967) to assume that real money balances yield utility to agents in addition to their purchasing power. In particular, the agent's period utility function, $U(c_t, m_t, t)$, depends positively on the agent's real money balance, m_t, as well as consumption, c_t. The rationale is that a larger real money balance reduces the transaction time in the goods market and hence allows the agent to enjoy more leisure. As long as leisure yields positive marginal utility to the agents, real money balances yield utility.

The government's purchase of goods and services is assumed to be constant and so the change in the money supply is injected into the economy as lump-sum monetary transfers. As in Lucas (1982), I assume that the agent is endowed with one unit of a claim on these monetary transfers. Denote the real price of this equity claim at time t as L_t. The money transfer measured in real terms, l, can be understood as the "dividend" for this claim. Therefore, L is the present value of future real monetary transfers. Note that monetary transfers are lump-sum and hence are taken

[7] Allowing P^* to follow a stochstic process complicates the analysis without changing the qualitative results, provided that the process for P^* is independent of the processes for domestic dividends and domestic money supply.

as given by individual agents. The dynamics of the domestic money supply are described in the following assumption.

Assumption 1. *The domestic money supply, M^s, is assumed to evolve according to the following mixed diffusion-jump process:*

$$\frac{dM^s}{M^s} = (\mu_m - \lambda_m k_m)dt + \sigma_m dz_1 + (Y_m - 1)dQ_m, \quad \forall \quad t \in (0, \infty). \tag{2.1}$$

Here, μ_m is the instantaneous expected growth rate of the money supply; σ_m^2 is the instantaneous variance of the growth rate, conditional on no arrivals of new important shock and dz_1 is a one-dimensional Gauss-Wiener process. The element dQ_m is a jump process with a jump intensity parameter λ_m and $Y_m - 1$ is the random variable percentage change in the money supply if the Poisson event occurs. The logarithm of Y_m is normally distributed with mean θ_m and variance ϕ_m^2. The expected jump amplitude, $k_m = E(Y_m - 1)$, is equal to $\exp(\theta_m + \phi_m^2/2) - 1$. Also, $\overline{k}_m = E(\frac{1}{Y_m} - 1)$ is equal to $\exp(-\theta_m + \phi_m^2/2) - 1$. The random variables $\{z_{1t}, t \geq 0\}$, $\{Q_{mt}, t \geq 0\}$ and $\{Y_{mj}, j \geq 1\}$ are assumed to be mutually independent. Also, Y_{mj} is independent of $Y_{mj'}$ for $j \neq j'$. The parameters $(\mu_m, \sigma_m, \lambda_m, \theta_m, \phi_m)$ are constant.

The above money supply process incorporates both frequent fluctuations in the money supply, which correspond to the diffusion part dz_1, and infrequent large shocks to the money supply, which correspond to the jump part dQ_m. Both capture changes in government monetary policies.

There is only one domestic risky stock, which represents the ownership of the home productive technology for the single good. The total supply of this risky stock is normalized to one. Denote its real price at time t as S_t and the dividend as δ_t. The dividend stream $\{\delta_t\}$ can be understood as aggregate dividends in this small economy, which are exogenously given as:[8]

$$\frac{d\delta}{\delta} = \mu(\delta)dt + \sigma(\delta)dz_2 + (Y_\delta - 1)dQ_\delta, \tag{2.2}$$

where dz_2 is a one-dimensional Gauss-Wiener process and dQ_δ is an independent jump process, described more precisely later.

The specification of aggregate dividend process corresponds to an economy which is infrequently subject to real shocks of unpredictable magnitude. The shocks on dividends could result from output shocks or shocks due to technological innovations. The mixed jump-diffusion formulation of the aggregate dividend process is also adopted by Naik and Lee (1990) and Ma (1994). For most of the discussion, the dividend process and the money supply process are assumed to be independent, measured with respect to a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Section 5 will extend the discussion to allow for a correlation between the two processes.

There are foreign pure discount bonds available for trading to the home agent at any time. A foreign pure discount bond pays 1 unit of consumption goods at maturity and 0 at all other time. The agent can internationally diversify his portfolio by holding the foreign bonds and the domestic financial assets. That is, the net trading in assets between this small economy and the foreign country is positive and time-varying. Since the country is small, the real price of the foreign bond at time t, F_t, is taken as exogenous by the home agent. The dynamics of F_t are assumed below.

Assumption 2. *F_t evolves as $dF = rFdt$, where r is a positive constant.*

The processes for the money supply, the foreign bond price and the aggregate dividend are the primitives of the economy. Together with the specification of the utility function described below, they induce equilibrium prices for other assets. Among these other assets, there are domestic

[8] Although dividends are not continuously distributed in reality, one may be able to find reasonable proxies for aggregate dividends used here. Aggregate output and dividends on stock indices are the examples.

nominal pure discount bonds in zero net supply, with nominal rate of return i. A domestic nominal discount bond pays 1 unit of domestic currency at maturity and 0 at all other time. Denote B_t as the nominal price of the discount bond at time t. Then, $\mathrm{d}B = iB\mathrm{d}t$, where i is endogenously determined in equilibrium. The real price of the domestic bond at time t, b_t, is given as $b_t = B_t/p_t$. In addition, there are many other contingent claims on the risky domestic stock and the spot exchange rate available for trading at any time in the economy. These contingent claims are all in zero net supply. Denote the real prices of the contingent claims at time t by a vector x_t and the corresponding vector of real dividends by δ_t^x.

2.2. The Agent's Optimization Problem

The representative agent's information structure is given by the filtration $\mathcal{F}_t \equiv \sigma(M_\tau^s, \delta_\tau; 0 \leq \tau \leq t)$. As described earlier, the period utility at time t is $U(c_t, m_t, t)$, where $U(\cdot, \cdot, t) : \mathcal{R}_+^2 \rightarrow \mathcal{R}$ is increasing and strictly concave and satisfies the following properties:

$$\lim_{x_j \rightarrow \infty} U_j(x_1, x_2) = 0 \quad \text{and} \quad \lim_{x_j \rightarrow 0} U_j(x_1, x_2) = \infty, \quad j = 1, 2.$$

The agent's intertemporal utility is described by

$$V(c, m) = E_0 \int_0^\infty U(c_t, m_t, t)\mathrm{d}t.$$

Initially, the agent is endowed with N_0^F units of the foreign bond, one share of the domestic risky stock, money holdings M_0 and one share of the equity claim for domestic monetary transfer. His consumption over time is financed by a continuous trading strategy $\{M_t, N_t, \forall\ t \geq 0\}$, where M_t is the money holding at time t and $N_t = (N_t^L,\ N_t^F,\ N_t^S,\ N_t^b,\ N_t^{x\prime})'$ is a vector which represents the portfolio holdings consisting of all the financial assets traded in financial markets at time t. For example, N_t^F is the quantity of foreign bonds held by the domestic agent at time t. Denote the real prices of all financial assets at time t by a vector $X_t = (L_t,\ F_t,\ S_t,\ b_t,\ x_t')'$ and the corresponding vector of real dividends by q_t. The cumulative dividends up to t are defined as $D_t = \int_0^t q_\tau d\tau$. At any point $\tau \geq 0$, the agent's wealth is $W_\tau = N_\tau \cdot X_\tau + M_\tau / p_\tau$ and the flow budget constraint is

$$c_\tau d\tau = M_\tau d(\frac{1}{p_\tau}) + N_\tau^X \cdot (dD_\tau + dX_\tau) - dW_\tau. \tag{2.3}$$

This constraint intuitively states that the sum of the wealth increase (dW_τ) and consumption flow ($c_\tau d\tau$) is bounded by the dividend and capital gain from the portfolio $\{M_\tau, N_\tau\}$.

With this flow budget constraint, one can use the technique of optimal control to derive the partial differential equations that are satisfied by the assets prices. In the presence of the jump components in the money supply process and the dividend process, these partial differential equations turn out to be very complicated. In contrast, the Euler equation approach appears much simpler and is adopted here.[9] To do so, transform the flow budget constraint into an integrated one (see Duffie (p. 110) for a similar formulation):

$$\int_0^t c_\tau d\tau = \frac{M_0}{p_0} + \int_0^t M_\tau d(\frac{1}{p_\tau}) - \frac{M_t}{p_t} + N_0^X \cdot X_0 + \int_0^t N_\tau^X \cdot (dD_\tau + dX_\tau) - N_t^X \cdot X_t. \tag{2.4}$$

The agent chooses an optimal portfolio trading strategy $\{M_t, N_t, \forall\, t \geq 0\}$ so as to maximize his expected lifetime utility. Precisely, he solves:[10]

$$\max_{\{c_t, M_t, N_t\}} E \int_0^\infty U(c_t, m_t, t) dt \qquad \text{s.t. (2.4) holds.}$$

The first order conditions (Euler equations) are:

$$\frac{1}{p_t} = \frac{1}{U_c(c_t, m_t, t)} E_t \left(\int_t^\infty U_m(c_\tau, m_\tau, \tau) \frac{1}{p_\tau} d\tau \right) \tag{2.5}$$

$$X_t = \frac{1}{U_c(c_t, m_t, t)} E_t \left(\int_t^\infty U_c(c_\tau, m_\tau, \tau) dD_\tau \right). \tag{2.6}$$

[9]The Euler equation approach has been used in Naik and Lee (1990) and the two approaches are equivalent in the sense that they lead to the same assets prices. I have also used the optimal control rule to derive the partial differential equations, which are omitted here but are available upon request.

[10]All expectations in the paper are taken with respect to the filtration specified earlier.

That is, the reciprocal of the exchange rate equals the expected discounted sum of future real wealth of one dollar, with the state price deflator being the marginal rate of substitution between consumption and the real money balance. The price of any other asset equals the expected discounted sum of dividends, with the stochastic state price deflator being the marginal rate of substitution between consumption at different dates.

As is typical for a small open economy, the exogenous foreign interest rate, the rate of time preference and the parameters describing consumption must satisfy certain restriction in order to ensure existence of an equilibrium. Such a restriction can be obtained by examining an agent's trade-off between consuming at time t and purchasing the foreign bond. The net utility gain from purchasing bond is $\frac{dF/F}{dt} + E_t(\frac{dU_c/U_c}{dt})$, where $\frac{dF/F}{dt} = r$ is the rate of return to holding the bond and $E_t(\frac{dU_c/U_c}{dt})$ is the utility loss due to the delay in consumption. Since optimality requires the net utility gain to be zero, the equilibrium restriction is $r = -E_t(\frac{dU_c/U_c}{dt})$.[11]

2.3. Equilibrium Exchange Rate and Asset Prices under Logarithmic Utility

Market clearing conditions are described as follows. The domestic currency held by the foreign country is negligible, so the money market is cleared by the domestic money demand. That is, $M^s = M$. Similarly, the demand for the risky stock equals the supply of shares, which is one share, and the demand for the claim on monetary transfers equals the supply, which is also one. Also, equilibrium prices are such that the representative agent holds neither the domestic nominal bonds nor any other contingent claims, because the net supply of each such asset is zero. Note that the supply of a domestic asset (or money) equals the domestic demand for the asset (or money). This equality holds here not because the economy is closed but rather because the economy is small relative to the outside world and so the foreign demand for its asset (or money) is negligible, as

[11]This restriction can be formally derived from the Euler equation (2.6). When there is no uncertainty, this restriction becomes the well-known equality between the real interest rate (r) and the rate of time preference (ρ).

discussed at the beginning of Section 2.

On the other hand, the goods market clearing condition is quite different here from that in a closed economy. Since the country can have current account surplus or deficit, as discussed in the introduction, aggregate consumption does not necessarily equal the aggregate dividend generated from the domestic stock. Since the country can export the goods to the foreign country to increase its holdings on foreign bonds, the total expenditure on goods is $cdt+\,\mathrm{d}f$, where $f_t = N_t^F F_t$ is the value of foreign bonds. The total supply of goods is the sum of domestic dividends, $\delta \mathrm{d}t$, and return to holding foreign bonds, $rf\mathrm{d}t$. Thus, the goods market clearing condition is

$$\mathrm{d}f = (\delta + rf - c)\mathrm{d}t.$$

This goods market clearing conditions also differs from that in Lucas's (1982) two-country assets pricing model and its application in currency options by Bakshi and Chen (1996). In these models, the equilibrium portfolio of each country is identical to its initial endowment and so the net trading in assets between the two countries is zero in equilibrium. In contrast, here the net trading in foreign bonds must be non-zero in equilibrium as δ and c vary over time. This difference not only makes it more challenging to solve for the equilibrium portfolio here but also leads to important differences in the behavior of the exchange rate: Since the exchange rate clears the goods market, the net trading volume affects the exchange rate.

For analytical tractability, I assume that preferences are given by:[12]

Assumption 3. *The risk-averse agent's period utility is described by*

$$U(c_t, m_t, t) = e^{-\rho t}[\alpha \ln c_t + (1-\alpha)\ln m_t], \qquad \alpha \in (0,1). \tag{2.7}$$

[12] Bakshi and Chen (1996) adopt the logarithmic utility function for a similar reasoning.

The goods market clearing condition implies that the real wealth, $f_t + S_t$, is equal to the expected present value of future consumption stream, c_t/ρ. This condition, together with (2.6), helps to determine the equilibrium price of the domestic risky stock, S_t, and the equilibrium quantity of the foreign bonds held by the domestic agent, f_t (See Appendix A for a proof).

Proposition 2.1. *Under Assumptions 1-3, the equilibrium real price of the domestic risky stock at time t, S_t, is $S_t = S(\delta_t) = \frac{\delta_t}{\rho}$, $\forall \quad t \in (0, \infty)$ and the equilibrium value of foreign bonds held by the domestic agent is $f_t = N_t^F F_t = f_0 e^{(r-\rho)t}$.*

Given the logarithmic utility function in Assumption 3, the real price of the risky stock is only affected by aggregate dividend. Precisely, the stock price equals the present value of future dividends discounted at the rate of time preference. The quantity of foreign bonds held by the domestic agent in equilibrium evolves at a constant rate of $r - \rho$. Equivalently, the level of investment in foreign bonds at time t in equilibrium is determined as $N_t^F = N_0^F e^{-\rho t}$. Therefore, the market portfolio in this small open economy is internationally diversified and consists of the domestic risky stock and f_t amount of foreign bonds.

Using (2.5), (2.6) and the money market clearing condition $M^s = M$, we can derive the equilibrium exchange rate, the nominal interest rate and the real price of the claim on monetary transfers, L_t (see Appendix B for proof).

Proposition 2.2. *Under Assumptions 1-3, the equilibrium exchange rate is $p_t = \frac{\alpha}{1-\alpha}\frac{iM_t}{\delta_t + \rho f_t} = \frac{\alpha}{1-\alpha}\frac{iM_t}{c_t}$. The nominal interest rate is $i = \rho + \beta_m$ where $\beta_m \equiv \mu_m - \sigma_m^2 - \lambda_m k_m - \lambda_m \bar{k}_m$ is defined through $E_t(\frac{M_t}{M_T}) = e^{-\beta_m(T-t)}$. The equilibrium real price at any time t of the claim for monetary transfers is $L_t = (i - \rho)m_t/\rho$.*

In contrast to the real price of the risky stock, the exchange rate is a function of money supply, aggregate dividend and the level of investment in the foreign bonds, as in a typical small open economy model. This is a consequence of the representative agent's optimal condition, $\frac{U_m}{U_c} = i$, which states intuitively that the marginal rate of substitution between the real money balance and consumption must equal the opportunity cost of holding money (the foregone nominal interest income). Under the logarithmic utility function form, this general relation implies that the flow of services derived from holding money is proportional to the level of consumption. That is, $i\frac{M_t}{p_t} = \frac{1-\alpha}{\alpha}c_t$, which leads to the expression for equilibrium exchange rate in Proposition 2.2.

The nominal interest rate is constant and equal to the sum of the rate of time preference and the expected growth rate of money supply after adjusting the uncertainties, β_m in Proposition 2.2. This relation arises from the agent's optimal trade-off between consuming today and purchasing a nominal bond today. Holding a nominal bond for an arbitrarily short period time and then spending the return on consumption goods has a net gain $i + E_t(\frac{\mathrm{d}p^{-1}/p^{-1}}{\mathrm{d}t}) + E_t(\frac{\mathrm{d}U_c/U_c}{\mathrm{d}t})$, where $E_t(\frac{\mathrm{d}p^{-1}/p^{-1}}{\mathrm{d}t})$ is the capital loss resulted from inflation and $E_t(\frac{\mathrm{d}U_c/U_c}{\mathrm{d}t})$ is the utility loss from the delay in consumption. Since optimality requires the agent to be indifferent between consuming now and holding a nominal bond at the margin, $i = -E_t(\frac{\mathrm{d}p^{-1}/p^{-1}}{\mathrm{d}t}) - E_t(\frac{\mathrm{d}U_c/U_c}{\mathrm{d}t})$. Under the logarithmic utility function and the exchange rate p in Proposition 2.2, this implies $i = \rho + \beta_m$.

Also, the real price of the claim on monetary transfers is proportional to the real money balance, i.e., the present value of future real monetary transfers is proportional to current real money balances in equilibrium.

Since $c_t = \delta_t + \rho f_t$ and since real prices of the stock and foreign bonds are independent of the money supply process, equilibrium consumption is independent of the money supply process. The domestic agent consumes the dividends generated from the domestic risky stock and the foreign bond. Since the foreign bond price evolves exogenously in equilibrium, equilibrium consumption is determined by the stock dividend process. Under the general process for dividends (2.2), consumption follows a complicated stochastic process. This makes it difficult to compare the results of the current model with those in previous models such as GK (1983) and Merton (1976), who assume that the exchange rate follows a diffusion or jump-diffusion process. To facilitate comparison, let us restrict the dividend process by the following assumption, which allows me to derive currency option pricing formulas that encompass GK (1983) and Merton (1976) as special cases.

Assumption 4. *The dividend process (2.2) evolves as:*

$$d\delta = (\mu_\delta - \lambda_\delta k_\delta)(\delta + \rho f)dt - \rho(r - \rho)f dt$$
$$+\sigma_\delta(\delta + \rho f)dz_2 + (Y_\delta - 1)(\delta + \rho f)dQ_\delta.$$

Assumption 4 implies the following mixed jump-diffusion process for consumption:

$$\frac{\mathrm{d}c}{c} = (\mu_\delta - \lambda_\delta k_\delta)\mathrm{d}t + \sigma_\delta \mathrm{d}z_2 + (Y_\delta - 1)\mathrm{d}Q_\delta. \tag{2.8}$$

Here, μ_δ is the instantaneous expected growth rate; σ_δ^2 is the instantaneous variance of the growth rate, conditional on no arrivals of new important shock. The element $\mathrm{d}Q_\delta$ is a jump process with a jump intensity parameter λ_δ and $Y_\delta - 1$ is the random variable percentage change in aggregate consumption if the Poisson event occurs. The logarithm of Y_δ is normally distributed with mean θ_δ and variance ϕ_δ^2. The expected jump amplitude, $k_\delta = E(Y_\delta - 1)$, is equal to $\exp(\theta_\delta + \phi_\delta^2/2) - 1$. Also $\overline{k}_\delta = E(\frac{1}{Y_\delta} - 1)$, is equal to $\exp(-\theta_\delta + \phi_\delta^2/2) - 1$. The random variables $\{z_{2t},\, t \geq 0\}$, $\{Q_{\delta t},\, t \geq 0\}$ and $\{Y_{\delta j},\, j \geq 1\}$ are assumed to be mutually independent. Also, $Y_{\delta j}$ is independent of $Y_{\delta j'}$ for $j \neq j'$. The parameters $(\mu_\delta,\, \sigma_\delta,\, \lambda_\delta,\, \theta_\delta,\, \phi_\delta)$ are constant.

Under the logarithmic utility function and the above assumption, the restriction on the foreign interest rate, discussed at the end of subsection 2.2, becomes $r = \rho + \beta_\delta$, where $\beta_\delta \equiv \mu_\delta - \sigma_\delta^2 - \lambda_\delta k_\delta - \lambda_\delta \overline{k}_\delta$ is defined through $E_t(\frac{c_t}{c_T}) = e^{-\beta_\delta(T-t)}$.

3. Pricing Currency Options

3.1. Dynamics of the Exchange Rate

Let us examine the dynamics followed by the exchange rate from the domestic agent's perspective. Since p_t is a function of M and c, applying Ito's Lemma yields

$$\begin{aligned}\frac{dp}{p} &= (\mu_p - \lambda_m k_m - \lambda_\delta \overline{k}_\delta)dt + \sigma_m dz_1 - \sigma_\delta dz_2 \\ &+ (Y_m - 1)dQ_m + (Y_\delta^{-1} - 1)dQ_\delta,\end{aligned} \tag{3.1}$$

where $\mu_p = \mu_m - \beta_\delta$. Under the equilibrium conditions for the nominal interest rate and the rate of time preference, the exchange rate dynamics can be rewritten as:

$$\frac{dp}{p} = (i - r + \sigma_m^2 + \lambda_m \overline{k}_m - \lambda_\delta \overline{k}_\delta)dt + \sigma_m dz_1 - \sigma_\delta dz_2 + (Y_m - 1)dQ_m + (\frac{1}{Y_\delta} - 1)dQ_\delta.$$

The key feature of the above exchange rate is that it is derived endogenously from the underlying processes for the money supply, aggregate dividend and the foreign bond price. This endogeneity is in stark contrast with the arbitrariness in the existing currency option models mentioned in the introduction. Clearly, the exchange rate is affected by the domestic government monetary policy, aggregate dividend and the level of foreign investment. In this sense, Merton's assumption that the jump risk is uncorrelated with aggregate consumption is inappropriate for the exchange rate.

The domestic government monetary policy and aggregate dividend affect the real price of the domestic risky stock and the exchange rate differently. The difference is crystal clear under the logarithmic utility. The real price of the domestic risky stock is solely determined by the aggregate dividend, where monetary policies play no role. The exchange rate incorporates jump components from both aggregate consumption and the money supply, while the stock price is only affected by the jump risk from the aggregate dividend. Thus, the current model is able to explain why discontinuities in exchange rate movements are more valent than in stock prices, a feature empirically documented by Jorion (1988). Examining the sample paths of exchange rates and a value-weighted

U. S. stock market index, Jorion finds that exchange rates display significant jump components, while discontinuities are harder to detect in the stock market.

Specifically, the expected growth rate of the exchange rate, μ_p, is associated with the drifts of the money supply and aggregate consumption. It is also affected by the instantaneous variance of the growth rate of consumption and the jump component in consumption. The exchange rate dynamics incorporate the two independent jump components from the money supply and aggregate dividend. Obviously, the jump in the exchange rate generated by aggregate dividends must be priced. The instantaneous variance of the growth rate of the exchange rate is the sum of the variances in the money supply and consumption, $\sigma_m^2 + \sigma_\delta^2$. On the other hand, the stock price is completely described by the parameters underlying aggregate consumption. These requirements suggest that cross-equation restrictions must be imposed on the coefficients when the processes for the exchange rate and the stock price are to be estimated.[13]

3.2. Domestic Risk-Averse Agent's Equilibrium Valuation of Currency Options

Now consider the valuation of European style currency options. According to the agent's maximization condition (2.6), for any contingent claim with maturity T and dividend q_T, its real price at time $t \leq T$, $x_t(T)$, is

$$x_t(T) = \frac{1}{U_c(c_t, m_t, t)} E_t \left(q_T U_c(c_T, m_T, T) \right).$$

For a European call written on the spot exchange rate with a striking price K that matures at time T, its nominal price at time $t \leq T$, $CC_t^D(p_t, T)$, is

$$CC_t^D(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left(\frac{1}{c_T} \frac{\max(p_T - K, 0)}{p_T} \right).$$

[13]In a sequel work studying the empirical implication of the systematic jump risks in currency option prices, I have used these cross-equation restrictions for estimating the parameters of the underlying processes for exchange rates and stock indices (see Cao 1997).

Similarly, for a European put written on the spot exchange rate with a striking price K that matures at time T, its nominal price at time $t \leq T$, $CP_t^D(p_t, T)$, is

$$CP_t^D(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left(\frac{1}{c_T} \frac{\max(K - p_T, 0)}{p_T} \right).$$

The joint density function for (c_T, M_T) conditional on (c_t, M_t), $f(c_T, M_T, T \mid c_t, M_t, t)$, is known. We can explicitly compute the prices of the European call and put, since the exchange rate is a function of c and M. To facilitate the presentation of equilibrium prices of call and put options, let C_{GK} and P_{GK} be, respectively, the currency call and put prices derived by GK (1983) with the following expressions

$$C_{GK}(p_t, \tau; K, r_F, r_D, \sigma_E) = p_t e^{-r_F \tau} N(d_1) - K e^{-r_D \tau} N(d_2),$$

$$P_{GK}(p_t, \tau; K, r_F, r_D, \sigma_E) = K e^{-r_D \tau} N(-d_2) - p_t e^{-r_F \tau} N(-d_1),$$

where

$$\tau = T - t, \quad d_1 = \frac{\ln p_t/K + (r_D - r_F + \frac{1}{2}\sigma_E^2)\tau}{\sigma_E \sqrt{\tau}}, \quad d_2 = d_1 - \sigma_E \sqrt{\tau}.$$

Then, the option prices in the current model are described as follows (See Appendix C for a proof):

Proposition 3.1. *Under Assumptions 1-4, equilibrium nominal prices of currency call and put are:*

$$CC_t^D(p_t, T) = \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) C_{GK}(p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m}) \tag{3.2}$$

and

$$CP_t^D(p_t, T) = \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) P_{GK}(p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m}) \tag{3.3}$$

where $P(\cdot, \cdot)$ is defined as

$$P(a, b) = \frac{e^{-a\tau}(a\tau)^{n_\delta}}{n_\delta!} \frac{e^{-b\tau}(b\tau)^{n_m}}{n_m!}$$

and

$$r_\delta = r + \lambda_\delta \bar{k}_\delta + \frac{n_\delta(\theta_\delta - \frac{1}{2}\phi_\delta^2)}{\tau} = \rho + \mu_\delta - \lambda_\delta k_\delta - \sigma_\delta^2 + \frac{n_\delta(\theta_\delta - \frac{1}{2}\phi_\delta^2)}{\tau},$$

$$i_m = i + \lambda_m \bar{k}_m + \frac{n_m(\theta_m - \frac{1}{2}\phi_m^2)}{\tau} = \rho + \mu_m - \lambda_m k_m - \sigma_m^2 + \frac{n_m(\theta_m - \frac{1}{2}\phi_m^2)}{\tau},$$

$$\sigma_{\delta,m} = \sqrt{\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau} + \sigma_m^2 + \frac{n_m \phi_m^2}{\tau}},$$

Consider the call price for example. C_{GK} is an increasing function of the conditional domestic interest rate, i_m, and the conditional exchange rate volatility, $\sigma_{\delta,m}$, but a decreasing function of the conditional foreign interest rate, r_δ. The currency option prices depend intuitively on the fundamental parameters. First, an increase in the conditional consumption volatility, σ_δ, or the volatility of jump size, ϕ_δ, induces a lower r_δ and a higher $\sigma_{\delta,m}$: the joint consequence is a higher currency call price. Second, a higher conditional volatility of money supply, σ_m, or higher volatility of the corresponding jump, ϕ_m, does not necessarily imply a higher call price. This is because an increase in σ_m or ϕ_m reduces i_m and increases $\sigma_{\delta,m}$ simultaneously, while the increase in $\sigma_{\delta,m}$ tends to increase the call price and the reduction in i_m tends to reduce the call price. Further, the call value is positively related to the instantaneous expected growth rate of the money supply, μ_m, and negatively related to the instantaneous expected growth rate of aggregate consumption, μ_δ. Call prices also depend ambiguously on $(\lambda_\delta, \lambda_m, \theta_\delta, \theta_m)$.

Note that if there were no jump component in aggregate dividend, the currency call and put prices in Proposition 3.1 would reduce to Merton's (1976) price equations. In this case, the only jump uncertainty underlying the exchange rate would be from the money supply and this jump uncertainty is not priced.

The Euler equation (2.6) can also be used to price European style options on the domestic risky stock. Denote the real price of a call (put) on the risky stock at time t with a striking price k and an expiration date T by $C_t(k, S_t, T)$ $(P_t(k, S_t, T))$. As shown in Appendix D, the stock option prices are completely described by the parameters underlying aggregate dividend. The explicit valuations are stated in the following proposition.

Proposition 3.2. *Under Assumptions 1-4, $C_t(k, S_t, T)$ and $P_t(k, S_t, T)$ are:*

$$C_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} C_{GK}(S_t + f_t, \tau; k + f_t e^{(r-\rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau})$$

and

$$P_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} P_{GK}(S_t + f_t, \tau; k + f_t e^{(r-\rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau})$$

where r_δ is defined in Proposition 3.1.

In contrast to currency options, real prices of stock options are independent of the uncertainty underlying the domestic money supply. Although aggregate consumption affects both the stock price and the exchange rate, the parameters describing the dynamics of consumption affect stock options and currency options differently. For example, the instantaneous expected growth rate of consumption, μ_δ, positively affects the price of a call on the stock but negatively affects the price of a call on the exchange rate. An increase in σ_δ or ϕ_δ increases the currency call prices as discussed earlier, but does not necessarily increase the stock call price. For the call price on the stock, increasing σ_δ or ϕ_δ implies a higher instantaneous stock volatility $\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}$, which in turn induces a higher call price. However, an increase in σ_δ or ϕ_δ also reduces r_δ at the same time. Since r_δ is positively related to the call price, the joint effect of a lower r_δ and a higher $\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}$ on the call price, is ambiguous. This further illustrates the difference between currency options and stock options.[14]

[14]The common belief is that an increase in stock volatility will be accomplished by an increase in call price according to the risk-neutral based Black-Scholes model (1973). Bailey and Stulz (1989) show that this common belief is not necessarily supported in an equilibrium context. Our result confirms the observation made by Bailey and Stulz (1989).

Note that the market portfolio in this small open economy consists of the domestic stock and the foreign bond. If the domestic agent did not hold foreign bond in equilibrium, this small open economy would be similar to a closed economy in which the market portfolio is the domestic stock. In this case, the stock option formulas in Proposition 3.2 would reduce to those on the market portfolio in Naik and Lee (1990) with jump risks and logarithmic utility.

4. Foreign Agent's Risk-Neutral Valuation

I now use the above framework to examine the analog of Siegel's paradox in currency option valuation. The purpose is to identify the necessary restrictions that must be imposed on the risk-neutral process of the exchange rate if foreign agents use the risk-neutral approach.

The analog of Siegel's paradox in currency option valuation refers to the violation of the parity conditions between domestic and foreign investors' valuations. A call option from the domestic agent's point view is a put option from the foreign investor's perspective. A call gives the domestic agent the right to buy the foreign currency from the foreign agent. On the other hand, a put option from the point of view of the foreign agent is an option to sell the domestic currency to obtain the foreign currency. In fact, the expression of "the call option value from the domestic agent's view" is the same as the expression of "the put option value from the foreign agent's view". The foreign agent's risk-neutral valuation of the put option is

$$CP_t^F(1/p_t, T) = e^{-r(T-t)} E_t^F\left(\max(1 - \frac{K}{p_T}, 0)\right),$$

where $E_t^F(\cdot)$ is the risk-neutral expectation operator conditional on the information at time t available to the foreign investor. According to the law of one price, $CP_t^F(1/p_t, T)$ converted into the domestic currency at the spot exchange rate should be the same as $CC_t^D(p_t, T)$. That is

$$p_t CP_t^F(1/p_t, T) = CC_t^D(p_t, T). \tag{4.1}$$

Similarly, the put value from the domestic agent's point should equal the call value from the foreign agent's point, once the price is converted into the domestic currency at the spot exchange rate. That is

$$p_t CC_t^F(1/p_t, T) = CP_t^D(p_t, T), \tag{4.2}$$

where $CC_t^F(1/p_t, T) = e^{-r(T-t)} E_t^F \left(\max(\frac{K}{p_T} - 1, 0)\right)$.

As pointed out by Dumas and Näslund (1995), if both the domestic and foreign investors assume their own risk neutral processes, even in the case where the jump component in the exchange rate is uncorrelated with the consumption, applying Merton's formula generates an analog to Siegel's paradox that either (4.1) or (4.2) is violated. The reason is that both investors use different probability measures for the exchange rate. To see this, let x be the risk-neutral exchange rate expressed as the relative price of the foreign currency in terms of the home currency. The risk-neutral process is usually assumed to be

$$\frac{\mathrm{d}x}{x} = (i - r - \lambda_x E(Y_x - 1))\mathrm{d}t + \sigma \mathrm{d}w_x + (Y_x - 1)\mathrm{d}Q_x,$$

where the difference between the domestic and the foreign interest rate, $i - r$, is the risk-neutral drift rate. The foreign agent observes the same exchange rate dynamics but instead expresses the spot rate as $y = 1/x$, the relative price of the home currency expressed in terms of the foreign currency. The risk-neutral process for y is usually assumed by the foreign investor to be

$$\frac{\mathrm{d}y}{y} = (r - i - \lambda_y E(Y_y - 1))\mathrm{d}t + \sigma \mathrm{d}w_y + (Y_y - 1)\mathrm{d}Q_y,$$

where the difference between the foreign and the domestic interest rate, $r - i$, is the risk-neutral drift rate. Obviously,

$$\frac{\mathrm{d}x^{-1}}{x^{-1}} = \left(r - i + \sigma_x^2 + \lambda_x E(\frac{(Y_x - 1)^2}{Y_x}) - \lambda_x E(\frac{1}{Y_x} - 1)\right) \mathrm{d}t - \sigma_x \mathrm{d}w_x + (\frac{1}{Y_x} - 1)\mathrm{d}Q_y \neq \frac{\mathrm{d}y}{y},$$

with $\sigma_x = \sigma_y$ and $Y_y = 1/Y_x$. The extra term, $\sigma_x^2 + \lambda_x E[(Y_x - 1)^2/Y_x]$, appears in the drift for dx^{-1}/x^{-1}. Bardhan (1995) calls this extra term the "directional adjustments" and suggests that the foreign investor use dx^{-1}/x^{-1} as his risk-neutral process for y, or vice verse.[15] Strictly speaking, dx^{-1}/x^{-1} is not the risk-neutral process for y since the drift for y is no longer the risk-neutral drift $r - i$. Instead, the drift is $r - i + \sigma_x^2 + \lambda_x E[(Y_x - 1)^2/Y_x]$. One may interpret dx^{-1}/x^{-1} as the domestic risk-neutral process for y. Bardhan's directional adjustments would eliminate the paradox if the jump risk in the exchange rate were uncorrelated with consumption. However, they are insufficient to eliminate the paradox when the exchange rate is correlated with consumption, as in our case.

To examine the necessary restrictions on the risk-neutral process of the exchange rate when the jump component in the exchange rate is correlated with aggregate consumption, denote $\omega_t = 1/p_t = \frac{1-\alpha}{\alpha i}\frac{c_t}{M_t}$. The actual process for ω viewed by both domestic and foreign investors is

$$\begin{aligned}\tfrac{d\omega}{\omega} = (r - i + \sigma_\delta^2 + \lambda_\delta \overline{k}_\delta - \lambda_m \overline{k}_m)dt + \sigma_\delta dz_2 - \sigma_m dz_1 \\ +(Y_\delta - 1)dQ_\delta + (\tfrac{1}{Y_m} - 1)dQ_m.\end{aligned} \tag{4.3}$$

If the foreign agent uses the risk-neutral valuation to price the currency options, we can identify the restrictions on the risk-neutral process for ω by comparing the risk-neutral valuation of the options with (3.2) and (3.3). Denote the risk-neutral process for ω as follows:

$$\begin{aligned}\tfrac{d\omega^*}{\omega^*} = (r - i - \lambda_\delta^* k_\delta^* - \lambda_m^* \overline{k}_m^*)dt + \sigma_\delta dz_2^* - \sigma_m dz_1^* \\ +(Y_\delta^* - 1)dQ_\delta^* + (\tfrac{1}{Y_m^*} - 1)dQ_m^*.\end{aligned} \tag{4.4}$$

The following proposition details the foreign agents' risk-neutral valuations of the corresponding currency options (See Appendix E for proof):

Proposition 4.1. *Under the risk-neutral process of the exchange rate (4.4), the foreign agents'*

[15] The "directional adjustments" are sometimes referred to as the quanto adjustments or the convexity effects.

valuations of $CP_t^F(1/p_t,T)$ and $CC_t^F(1/p_t,T)$ are:

$$CP_t^F(1/p_t,T)=\sum_{n_\delta=0}^{\infty}\sum_{n_m=0}^{\infty}P(\lambda_\delta^*(k_\delta^*+1),\lambda_m^*)C_{GK}(1,\tau;\frac{K}{p_t},r_\delta^*,i_m^*,\sigma_{\delta,m}^*) \tag{4.5}$$

and

$$CC_t^F(1/p_t,T)=\sum_{n_\delta=0}^{\infty}\sum_{n_m=0}^{\infty}(\lambda_\delta^*(k_\delta^*+1),\lambda_m^*)P_{GK}(1,\tau;\frac{K}{p_t},r_\delta^*,i_m^*,\sigma_{\delta,m}^*) \tag{4.6}$$

where $C_{GK}(\cdot)$, $P_{GK}(\cdot)$ and $P(\cdot,\cdot)$ are defined in previous section and

$$r_\delta^*=r+\lambda_\delta^*k_\delta^*+\frac{n_\delta(\theta_\delta^*+\frac{1}{2}\phi_\delta^{*2})}{\tau};$$

$$i_m^*=i+\lambda_m^*k_m^*+\frac{n_m(\theta_m^*-\frac{1}{2}\phi_m^{*2})}{\tau},$$

$$\sigma_{\delta,m}^*=\sqrt{\sigma_\delta^2+\sigma_m^2+\frac{n_\delta\phi_\delta^{*2}}{\tau}+\frac{n_m\phi_m^{*2}}{\tau}}.$$

In order to ensure the parity conditions (4.1) and (4.2), the following restrictions on the risk-neutral process (4.4) must be satisfied :

$$\begin{array}{ll} \lambda_m^*=\lambda_m, & \lambda_\delta^*=\lambda_\delta(1+\bar{k}_\delta), \\ \bar{k}_m^*=\bar{k}_m, & k_\delta^*=E(Y_\delta^*-1)=-\frac{\bar{k}_\delta}{\bar{k}_\delta+1}, \\ \theta_m^*=\theta_m, & \theta_\delta^*=\theta_\delta-\phi_\delta^2, \\ \phi_m^*=\phi_m, & \phi_\delta^*=\phi_\delta. \end{array} \tag{4.7}$$

Under these restrictions, the actual probability is transformed into the risk-neutral or the equivalent martingale measure. In this case, the risk-neutral process can be expressed as:

$$\frac{\mathrm{d}\omega^*}{\omega^*}=\frac{\mathrm{d}\omega}{\omega}-\sigma_\delta^2\mathrm{d}t-Y_\delta(1-e^{-\phi_\delta^2})\mathrm{d}Q_\delta.$$

In light of (4.3) and (4.4), this implies $\mathrm{d}z_2^*=\mathrm{d}z_2-\sigma_\delta dt$, $\mathrm{d}z_1^*=\mathrm{d}z_1$, $(Y_\delta^*-1)\mathrm{d}Q_\delta=(Y_\delta e^{-\phi_\delta^2}-1)\mathrm{d}Q_\delta$. In fact, no adjustment is needed for the money supply process since it is assumed to be independent of the consumption process. For the consumption process, one needs to adjust not only the risk from the diffusion ($\mathrm{d}z_2$) and jump intensity parameters (λ_δ, k_δ), but also from the jump size (θ_δ). The adjustments on ($\mathrm{d}z_2$, λ_δ, k_δ) are the directional adjustments suggested by Bardhan (1995) for the case where the jump in the exchange rate is not correlated with aggregate consumption. The additional adjustment on θ_δ reflects the fact that the jump risk in exchange rate is related to aggregate consumption. Note that in the special case where the jump size in consumption is certain, i.e., $\phi_\delta = 0$, no adjustment is needed for the jump size and so the jump component in consumption can be hedged away (see Bardhan 1995).

The above adjustments are specific to the utility function (2.7), but the general message of the exercise should be valid for a wider class of utility functions. That is, if the jump components in the exchange rate are related to those in consumption, at least one investor (either the domestic or the foreign investor) must use the utility-based equilibrium model to price the currency options. The appropriate risk-neutral or the equivalent martingale process for the exchange rate should be based on an equilibrium model in an international context in order to ensure the parity conditions (4.1) and (4.2). Adjustments for the risk-neutral process must be made on all uncertainties, including the Brownian motion, the jump intensity and the jump size. Making only the directional adjustments is not enough.

5. An Extension of the Model

The above discussions have employed the assumption that the government monetary policy is independent of aggregate dividend. In this section, I extend the framework in previous sections to incorporate a correlation between the money supply and aggregate dividend. This correlation arises when the government uses the monetary policy to react to shocks in aggregate output. I capture this possible active monetary policy by allowing for a correlation between the shock dz_1 in the money supply and the shock dz_2 in aggregate dividends to be correlated, with a correlation coefficient ρ_{12}.[16]

With this correlation structure, the jump component in the money supply is still independent of aggregate dividend. Because of the separability between consumption and real money balances in the utility function, the exchange rate, the nominal interest rate, the restriction on the rate of time preference, the risky stock price and the equilibrium quantity of foreign bonds held by the domestic agent are the same as in previous sections. More importantly, the stock option valuation in Proposition 3.2 is unchanged and so is still independent of the money supply. In contrast, the correlation between dz_1 and dz_2 affects currency option valuations from the domestic agent's view. To see this, one can verify that Proposition 3.1 still holds with the following modification:

$$\sigma_{\delta,m}{}^2 = \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau} - 2\rho_{12}\sqrt{(\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau})(\sigma_m^2 + \frac{n_m \phi_m^2}{\tau})} + \sigma_m^2 + \frac{n_m \phi_m^2}{\tau}.$$

Since the parameter ρ_{12} influences the currency option price only through $\sigma_{\delta,m}$, a call on the exchange rate with $\rho_{12} < 0$ will have a higher value than when $\rho_{12} = 0$, because the call price is an increasing function of $\sigma_{\delta,m}$.

One can also examine the analog of Siegel's paradox through the hypothetical exercise in Section 4. The risk neutral valuations in Proposition 4.1 are modified through the conditional instantaneous variance below:

$$\sigma_{\delta,m}^*{}^2 = \sigma_\delta^2 + \frac{n_\delta \phi_\delta^{*2}}{\tau} - 2\rho_{12}\sqrt{(\sigma_\delta^2 + \frac{n_\delta \phi_\delta^{*2}}{\tau})(\sigma_m^2 + \frac{n_m \phi_m^{*2}}{\tau})} + \sigma_m^2 + \frac{n_m \phi_m^{*2}}{\tau}.$$

The restrictions imposed on the risk-neutral process of the exchange rate are the same as in (4.7). The risk-neutral process now is expressed as:

$$\frac{d\omega^*}{\omega^*} = \frac{d\omega}{\omega} - (\sigma_\delta^2 - \rho_{12}\sigma_\delta\sigma_m)dt - Y_\delta(1 - e^{-\phi_\delta^2})dQ_\delta.$$

[16] **I thank John Hull for suggesting this extension. Although in principle one can also allow the money supply and aggregate dividends to be correlated through the jumps, analyzing this type of correlation is not tractable.**

In light of (4.3), this implies $dz_2^* = dz_2 - \sigma_\delta dt$, $dz_1^* = dz_1 - \rho_{12}\sigma_\delta dt$, $(Y_\delta^* - 1)dQ_\delta = (Y_\delta e^{-\phi_\delta^2} - 1)dQ_\delta$. Compared with the adjustments made for the risk-neutral process (4.4) where the correlation is zero, an additional adjustment on dz_1 in the magnitude of $-\rho_{12}\sigma_\delta dt$ is needed to reflect the fact that the money supply is correlated with aggregate consumption. In this case, the exchange rate is correlated with aggregate consumption, not only directly, but also indirectly through the correlation between the money supply and aggregate consumption. Both correlations must be priced in currency option valuations. This exercise reinforces the key message that currency options must be priced by means of utility maximization if the risks in exchange rates are correlated with aggregate consumption.

6. Conclusion

This paper uses an equilibrium model to investigate exchange rates and currency options in a small open monetary economy where the jump-diffusion money supply and the jump-diffusion aggregate dividend processes are the sources of uncertainties. It is known that the exchange rate is affected by government monetary policies, aggregate dividends and the level of foreign investments, while the real price of the domestic market portfolio is determined only by aggregate dividends. The model is thus able to capture the empirical feature that discontinuities in exchange rates are more manifest than in stock prices (see Jorion 1988). Since the jump in the exchange rate is correlated with aggregate consumption, ignoring the systematic jump risks in exchange rates would be inappropriate and so directly applying Merton's jump-diffusion stock option model to currency options would be deficient. The European currency option formulas derived in this paper incorporate both systematic and non-systematic jump risks.

An empirical investigation of the model's predictions is a natural step to take and has been completed in Cao (1997). The detailed empirical procedure is not presented here because of lack of space. Some of the findings can be summarized here. I first extend the equilibrium conditions imposed on the joint distribution of the exchange rate and the price of the domestic market portfolio to a general economy and empirically estimate the parameters underlying the joint distribution through the maximum likelihood method. Then I compare unrestricted case, where there are both systematic and non-systematic jump risks in the exchange rate, with the restricted case, where there is only non-systematic risk in the exchange rate. The likelihood ratio tests strongly reject the hypothesis that there is no systematic jumps in the exchange rate. Further, I compare the current model with Merton's formula and the GK model with the estimated parameter values. The current model can perform better than both the GK pure diffusion model and Merton's non-systematic jumps model. For example, for short-maturity call options written on the three exchange rates (C$/US$, US$/DM and C$/DM), the current model provides a 28 % upward correction on the price generated by the GK model, a magnitude close to eliminating the price bias (29%) suggested by evidence (see introduction). In contrast, Merton's formula provides only 16% upward price correction on the GK model. The 12% difference in price bias correction shows that systematic jump risks are important for currency option pricing and reinforces the central message of the current paper that it is important to incorporate systematic jumps as well as non-systematic jump risks in modelling of exchange rate options.

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Appendices

A. Proof of Proposition 2.1:

Proof. The risky stock price and the foreign bond price must satisfy the first order condition (2.6). Thus

$$S_t = \frac{1}{U_{c_t}} E_t\left(\int_t^\infty U_{c_T} \cdot \delta_T \mathrm{d}T\right) = c_t E_t\left(\int_t^\infty e^{-\rho(T-t)} \frac{\delta_T}{c_T} \mathrm{d}T\right),$$
$$F_t = e^{-r(T-t)} = \frac{1}{U_{c_t}} E_t(U_{C_T}). \tag{A.1}$$

Since the real wealth in equilibrium, $S + f$, equals to the expected present value of future consumption stream, c/ρ, thus $c = \rho S + \rho f$. From the flow budget constraint (2.3), we have

$$\mathrm{d}f = (\delta + rf - c)\mathrm{d}t = (\delta + rf - \rho S - \rho f)\mathrm{d}t. \tag{A.2}$$

The stock price S_t and the quantity of foreign bonds held by the domestic agent f are solved from the above equations (A.1) and (A.2). The solutions are $S = \delta/\rho$ and $f_t = f_0 e^{(r-\rho)t}$. ■

B. Proof of Proposition 2.2 :

Proof. Since the money supply process (2.1) is independent of the consumption process (2.8), the joint distribution of (M_T, c_T) conditional on (M_t, c_t) is:

$$f(M_T, c_T, T \mid M_t, c_t, t) = g(M_T, T \mid M_t, t) h(c_T, T \mid c_t, t),$$

where

$$g(M_T, T \mid M_t, t) = \sum_{n_m=0}^{\infty} \frac{e^{-\lambda_m \tau}(\lambda_m \tau)^{n_m}}{n_m!} \frac{1}{\sqrt{2\pi \Sigma_m}} e^{-\frac{(\ln M_T - \psi_m)^2}{2\Sigma_m}},$$

$$h(c_T, T \mid c_t, t) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} \frac{1}{\sqrt{2\pi \Sigma_\delta}} e^{-\frac{(\ln c_T - \psi_\delta)^2}{2\Sigma_\delta}},$$

with

$$\psi_m = \ln M_t + (\mu_m - \lambda_m k_m - \tfrac{1}{2}\sigma_m^2)(T-t) + n_m \theta_m, \qquad \Sigma_m = \sigma_m^2 (T-t) + n_m \phi_m^2,$$

$$\psi_\delta = \ln c_t + (\mu_\delta - \lambda_\delta k_\delta - \tfrac{1}{2}\sigma_\delta^2)(T-t) + n_\delta \theta_\delta, \qquad \Sigma_\delta = \sigma_\delta^2 (T-t) + n_\delta \phi_\delta^2.$$

According to the first order condition (2.5) and utility function (2.7),

$$\frac{1}{p_t} = \frac{1}{U_{c_t}} E_t \left(\int_t^{\infty} U_{m_T} \frac{1}{p_T} \mathrm{d}T \right) = \frac{1-\alpha}{\alpha} c_t e^{\rho t} \int_t^{\infty} e^{-\rho T} E_t(\frac{1}{M_T}) \mathrm{d}T.$$

Since $E_t(\frac{1}{M_T}) = \frac{1}{M_t} e^{-(\mu_m - \sigma_m^2 - \lambda_m k_m - \lambda_m \bar{k}_m)(T-t)}$, then we have

$$p_t = \frac{\alpha}{1-\alpha} \frac{M_t}{c_t} (\rho + \mu_m - \lambda_m k_m - \sigma_m^2 - \lambda_m \bar{k}_m), \quad \forall \quad t \in (0, \infty).$$

The first order conditions (2.5) and (2.6) imply $i = \frac{U_m}{U_c}$. Under the logarithmic utility function (2.7), $i = \frac{\alpha}{1-\alpha} \frac{m_t}{c_t}$. Therefore, $i = \rho + \mu_m - \lambda_m k_m - \sigma_m^2 - \lambda_m \bar{k}_m$.

Also the expected present value of services (im) generated by money equals $m_t + L_t$. That is,

$$m_t + L_t = \frac{1}{U_{c_t}} E_t \left(\int_t^{\infty} U_{c_T} \frac{i M_T}{p_T} \mathrm{d}T \right) = \frac{i m_t}{\rho}.$$

Therefore, $L_t = \frac{i-\rho}{\rho} m_t$. ■

C. Proof of Proposition 3.1:

Proof. For a European call written on the spot exchange rate with a striking price K that matures at time T, its nominal price at time $t \leq T$, $CC_t^D(p_t, T)$, is

$$CC_t^D(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left(\frac{1}{c_T} \frac{1}{p_T} \max(p_T - K, 0) \right).$$

Since $p = \frac{\alpha i}{1-\alpha} \frac{M}{c} = A \frac{M}{c}$, then

$$\begin{aligned} CC_t^D(p_t, T) &= p_t e^{-\rho\tau} c_t E_t \left(\max(\tfrac{1}{c_T} - \tfrac{K}{A} \tfrac{1}{M_T}, 0) \right) \\ &= p_t e^{-\rho\tau} c_t \int_{-\infty}^{\infty} \left(\int_{\frac{K}{A} c_T}^{\infty} (\tfrac{1}{c_T} - \tfrac{K}{A} \tfrac{1}{M_T}) g(M_T \mid M_t) \mathrm{d}M_T \right) h(c_T \mid c_t) \mathrm{d}c_T. \end{aligned}$$

Tedious exercises show that

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\int_{\frac{K}{A} c_T}^{\infty} \tfrac{1}{c_T} g(M_T \mid M_t) \mathrm{d}M_T \right) h(c_T \mid c_t) \mathrm{d}c_T \\ &= \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) e^{-\psi_\delta + \frac{1}{2}\Sigma_\delta} \int_{-\infty}^{\infty} z(v) \mathrm{d}v \int_{\underline{w1}}^{\infty} z(w) \mathrm{d}w, \end{aligned}$$

where $P(\cdot,\cdot)$ is defined in proposition 3.1, $z(\cdot)$ is the standard normal density and $\underline{w1} = \frac{-d_1^{\delta,m} - \varphi v}{\sqrt{1-\varphi^2}}$

with

$$d_1^{\delta,m} = \frac{\ln p_t/K + (r_\delta - i_m + \frac{1}{2}\sigma_{\delta,m}^2)\tau}{\sigma_{\delta,m}\sqrt{\tau}}, \qquad \varphi = -\sqrt{\frac{\Sigma_\delta}{\Sigma_\delta + \Sigma_m}} = -\frac{\sqrt{\Sigma_\delta}}{\sigma_{\delta,m}\sqrt{T-t}}.$$

$\sigma_{\delta,m}$ is defined in proposition 3.1. According to Abramowitz (1972), the probability function for bivariate normal with correlation φ is defined as

$$\int_a^\infty z(v)dv \int_{\frac{b-\varphi v}{\sqrt{1-\varphi^2}}}^\infty z(w)dw = L(a, b, \varphi).$$

Thus

$$\int_{-\infty}^\infty z(v)dv \int_{\underline{w1}}^\infty z(w)dw = L(-\infty, -d_1^{c,m}, \varphi) = N(d_1^{c,m}),$$

where $N(a) = \int_{-\infty}^a z(v)dv$. Therefore,

$$\int_{-\infty}^\infty \left(\int_{\frac{K}{A}c_T}^\infty \frac{1}{c_T} g(M_T \mid M_t)\mathrm{d}M_T\right) h(c_T \mid c_t)\mathrm{d}c_T = \sum_{n_\delta=0}^\infty \sum_{n_m=0}^\infty P(\lambda_\delta, \lambda_m) e^{-\psi_\delta + \frac{1}{2}\Sigma_\delta} N(d_1^{\delta,m}).$$

Similarly,

$$\int_{-\infty}^\infty \left(\int_{\frac{K}{A}c_T}^\infty \frac{K}{A}\frac{1}{M_T} g(M_T \mid M_t)\mathrm{d}M_T\right) h(c_T \mid c_t)\mathrm{d}c_T = \frac{K}{A}\sum_{n_\delta=0}^\infty \sum_{n_m=0}^\infty P(\lambda_\delta, \lambda_m) e^{-\psi_m + \frac{1}{2}\Sigma_m} N(d_2^{\delta,m}),$$

where $d_2^{\delta,m} = d_1^{\delta,m} - \sigma_{\delta,m}\sqrt{\tau}$. Rearrange terms, we have

$$CC_t^D(p_t, T) = \sum_{n_\delta=0}^\infty \sum_{n_m=0}^\infty P(\lambda_\delta, \lambda_m) C_{GK}(p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m}).$$

For a European currency put, we have

$$CP_t^D(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t\left(\max(\tfrac{K}{A}\tfrac{1}{M_T} - \tfrac{1}{c_T}, 0)\right)$$
$$= p_t e^{-\rho(T-t)} c_t \int_{-\infty}^\infty \left(\int_{-\infty}^{\frac{K}{A}c_T} (\tfrac{K}{A}\tfrac{1}{M_T} - \tfrac{1}{c_T}) g(M_T \mid M_t)\mathrm{d}M_T\right) h(c_T \mid c_t)\mathrm{d}c_T.$$

The same tedious exercises will give us

$$CP_t^D(p_t, T) = \sum_{n_\delta=0}^\infty \sum_{n_m=0}^\infty P(\lambda_\delta, \lambda_m) P_{GK}(p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m}). \blacksquare$$

D. Proof of Proposition 3.2:

Proof. For a European call written on the stock with a striking price k that matures at time T, its real price at time $t \le T$, $C_t(k, S_t, T)$, is $C_t(k, S_t, T) = e^{-\rho\tau} c_t E_t\left(\frac{1}{c_T}\max(S_T - k, 0)\right)$. Since $S_t = \frac{\delta_t}{\rho} = \frac{c_t - \rho f_t}{\rho}$ and $f_t = f_0 e^{(r-\rho)t}$, then

$$\begin{aligned} C_t(k, S_t, T) &= e^{-\rho\tau} c_t E_t\left(\frac{1}{c_T}\max(\frac{c_T}{\rho} - f_T - k, 0)\right) \\ &= e^{-\rho\tau} c_t \int_{-\infty}^{\infty} \max(\frac{1}{\rho} - \frac{f_t e^{(r-\rho)\tau} + k}{c_T}, 0) h(c_T \mid c_t) dc_T. \end{aligned}$$

Tedious exercises show that

$$C_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} C_{GK}(S_t + f_t, \tau; k + f_t e^{(r-\rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}).$$

Similarly, we have

$$P_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} P_{GK}(S_t + f_t, \tau; k + f_t e^{(r-\rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}). \blacksquare$$

E. Proof of Proposition 4.1:

Proof. Based on the risk-neutral process (4.4), the distribution of ω_T conditional on ω_t is:

$$G^*(\omega_T, T \mid \omega_t, t) = \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta^*, \lambda_m^*) \frac{1}{\sqrt{2\pi\Sigma^*}} e^{-\frac{(\ln \omega_T - \psi^*)^2}{2\Sigma^*}},$$

where

$$\psi^* = \ln \omega_t + (r - i - \tfrac{1}{2}\sigma_m^2 - \tfrac{1}{2}\sigma_\delta^2 - \lambda_m^* k_m^* - \lambda_\delta^* k_\delta^*)\tau + n_\delta \theta_\delta^* - n_m \theta_m^*,$$

$$\Sigma^* = (\sigma_m^2 + \sigma_\delta^2)\tau + n_m \phi_m^{*2} + n_\delta \phi_\delta^{*2}.$$

For the European put and call currency option from the perspective of the foreign agent, $CP_t^F(\omega_t, T)$ and $CC_t^F(\omega_t, T)$, we can compute according to the risk-neutral probability density. That is

$$\begin{aligned} CP_t^F(\omega_t, T) &= e^{-r(T-t)} E_t^F(\max(1 - \omega_T K, 0)), \\ CC_t^F(\omega_t, T) &= e^{-r(T-t)} E_t^F(\max(\omega_T K - 1, 0)). \end{aligned}$$

Then it is straightforward to prove proposition 4.1. $\blacksquare$