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The von Neumann Facet and the Turnpike Properties for a Neoclassical Optimal Growth Model with Many Capital Goods II

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Abstract

We will study a multi-sector discrete-time optimal growth model with a neoclassical non-joint technology and show the Neighborhood Turnpike; any optimal path will be trapped in the neighborhood of an associated optimal steady state and its neighborhood can be chosen as small as possible by taking the discount factor close enough to one and the full Turnpike; any optimal path converges to an associated optimal steady state path when discount factors are close enough to one. These two Turnpike properties will provide the firm theoretical background for an application of a neoclassical optimal growth model with heterogeneous capital goods to economic analyses.

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1 Introduction

In my previous paper[19] and [20], two turnpike properties are proved for a very general neoclassical optimal growth model where the capital goods are consumable and the social utility function depends on these particular consumables as well as on a pure-consumption good. The turnpike results obtained there, however, depend crucially on assumptions unlinked to capital intensities and other familiar parameters. Here we will consider a simpler model than that studied in [19] and [20]. The objective function is a discounted sum of a sequence of a pure-consumption good. This type of the model is studied originally in [18] and [23] for a two-sector continuous-time optimal growth model and in [1] and [6] for a multi-sector continuous-time case.

In this paper we demonstrate the two types of the Turnpike properties: the Neighborhood Turnpike and the full Turnpike. A Neighborhood Turnpike involves any optimal path being trapped in the neighborhood of an associated optimal steady state. The neighborhood can be chosen as small as possible by choosing the discount factor small as close to one. A full Turnpike involves any optimal path converging to an associated optimal steady state path when the discount factor is sufficiently close to one. With our above objective function, the discounted sum of sequence of pure-consumption good, we will prove the two Turnpike properties, which we did in [19] and [20], here with concrete assumptions based on a model structure. In fact, we will prove the Turnpike properties under the two generarized capital intensity conditions, which are intensively studied in [8]: The first one is a counterpart to the condition in a two-sector model and involves the consumption good sector using a more capital-intensive technology than the other sector. The second one is a counterpart to the condition in a two-sector case and involves the capital goods production sector using a more capital-intensive technology. In [19] and [20], the first condition was assumed. Our analysis makes essential use of *the von Neumann Facet* (VNF henceforth): an n-dimensional plane embraces a optimal steady state on today's and tomorrow's capital stock space. Long used a similar idea to show the global asymptotic stability in a two-sector continuous-time neoclassical optimal growth model in [10]. However, note that since his line segment is called the "Rybczynski line" and is defined on the output space, but not on today's and tomorrow's capital stock space, it is not the VNF. In contrast with [19] and [20], where any path on the von Neumann facet is explosive, we will show that under the second generalized capital intensity condition any path on the von Neumann facet converges to a corresponding optimal steady state path. This means that the n-dimensional von Neumann facet is actually an n-dimensional stable manifold. So if we can prove the Neighborhood Turnpike, then we can infer that an optimal path must jump to the von Neumann facet and converge to the corresponding optimal steady state. This means that the full Turnpike holds.

Section 2 presents the model and some basic properties of an optimal steady state (OSS henceforth). In Section 3, we study the VNF and review some relevant results obtained in [20]. We will show two types of the Turnpike properties in Section 4. Section 5 concludes.

2 The Model and Assumptions

Our model is an exact discrete-time version of the one studied by [6]:

$$\max imize \sum_{t=0}^{\infty} \rho^{-1} c_0(t)$$

$$subject\ k(0) = \overline{k}$$

$$y_i(t) + k_i(t) - \delta_i k_i(t) - (1+g) k_i(t+1) = 0 \tag{1}$$

$$c_0(t) = f^0(k_{10}(t), k_{20}(t), \cdots, k_{n0}(t), \ell_0(t)), \tag{2}$$

$$y_i(t) = f^i(k_{1i}(t), k_{2i}(t), \cdots, k_{ni}(t), \ell_i(t)), \tag{3}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1, \tag{4}$$

$$\sum_{j=0}^{n} k_{ij}(t) = k_i(t), \tag{5}$$

where i=1,2,...,n, t=0,1,2,...., and the notation is as follows:

g	$=$ *rate of population growth given as* $0 < g < 1$,
r	$=$ *subjective rate of discount,* $r \geq g$,
ρ	$= (1+g)/(1+r)$,
$c_0(t) \in R_+$	$=$ *per capita consumption goods consumed at* t,
$y_i(t) \in R_+$	$= t^{th}$ *period* i^{th} *per capita capital good output,*
$k_i(t) \in R_+$	$= t^{th}$ *period* i^{th} *per capita capital stock,*
$k_i(0) \in R_+$	$=$ *initial periodp* i^{th} *per capita capital stock,*
$f^j : R_+^{n+1} \longmapsto R_+$	$=$ *per capita production function of the* j^{th} *sector which is strictly quasi concave, homogeneous of degree one and continuously differentiable on the interior of* R_+^{n+1},
$k_{ij}(t)$	$= i^{th}$ *per capita capital good used in the* j^{th} *sector in the* t^{th} *period,*
δ_i	$=$ *depreciation rate of the* i^{th} *capital good, given as* $0 < \delta_i < 1$.

Due to [2], Eqs.(2)-(5) are summarized as the social transformation function $c_0(t) = T(\mathbf{y}(t), \mathbf{k}(t))$ where T is continuously differentiable on the interior R_+^{2n}, $\mathbf{y}(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ and $\mathbf{k}(t) = (k_1(t), k_2(t), \cdots, k_n(t))$. If $\mathbf{x}$ and $\mathbf{z}$ stand for initial and terminal capital stock vectors respectively, then the reduced form utility function $V(\mathbf{x}, \mathbf{z})$ and the feasible set D can be defined as follows:

$$V(x, z) = T[(1+g)\mathbf{z} - (\mathbf{I} - \Delta)\mathbf{x}, \mathbf{x}]$$

and

$$D = \{(\mathbf{x}, \mathbf{z}) \in R_+^n \times R_+^n : T[(1+g)\mathbf{z} - (\mathbf{I} - \mathbf{\Delta})\mathbf{x}, \mathbf{x}] \geq 0\}$$

where $\mathbf{x}$=$(x_1(t), x_2(t), \cdots, x_n(t))$, $\mathbf{z}$=$(k_1(t+1), k_2(t+1), \cdots, k_n(t+1))$, $\mathbf{\Delta}$ is a diagonal matrix

$$\mathbf{\Delta} = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_1 \end{pmatrix}$$

and $\mathbf{I}$ is an n-dimensional unit matrix.

Thus the above optimization problem can be summarized as the following standard reduced form problem, which is familiar in Turnpike Theory:

$$\max imize \sum_{t=0}^{\infty} \rho^t V(\mathbf{k}(t), \mathbf{k}(t+1))$$

$$subject\ to\ (\mathbf{k}(t), \mathbf{k}(t+1)) \in D\ and\ \mathbf{k}(0) = \overline{\mathbf{k}}.$$

Also note that any optimal path must satisfy the following Euler equations, indicating an intertemporal efficiency:

$$\rho \mathbf{V}_z(\mathbf{k}(t-1), \mathbf{k}(t)) + \mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = \mathbf{\Theta} \text{ for all t} \geq 0 \tag{6}$$

where the partial derivative vectors mean that $\mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = [\partial V(\mathbf{k}(t), \mathbf{k}(t+1))/\partial \mathbf{k}(t)]$, $\mathrm{V}_z(\mathbf{k}(t-1), \mathbf{k}(t)) = [\partial V(\mathbf{k}(t-1), \mathbf{k}(t))/\partial \mathbf{k}(t)]$ and $\mathbf{\Theta}$ means an n dimensional zero vector. So if $\mathbf{k}$ is an interior OSS with a given ρ then it must satisfy

$$\rho \mathbf{V}_z(\mathbf{k}, \mathbf{k}) + \mathbf{V}_x(\mathbf{k}, \mathbf{k}) = \mathbf{\Theta}. \tag{7}$$

Let us denote w_0 *and* $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ as the wage rate and other factor price vectors respectively. Then following [6], the following assumptions are made:

Assumption 1. For all positive factor vectors $(w^0, \mathbf{w})$, the non-negative input coefficient matrix

$$\mathbf{a} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is indecomposable and the row vector $(a_{00}, a_{01}, \cdots, a_{0n})$ is positive, where $a_{ij} = k_{ij}/y_i$ and $a_{0i} = \ell_i/y_i$ $(i = 0, 1, \cdots, n : j = 1, 2, \cdots, n)$.

Assumption 2. The technology is viable (see [6] or [5] for the definition of the viability).

Assumption 3. The exogenous rate of labor force growth $g \geq 0$ satisfies inequality $g \leq 1/\lambda^*$, where λ^* is the dominant characteristic root of the matrix $\overline{\mathbf{a}}^*(\mathbf{I} - \mathbf{\Delta}\overline{\mathbf{a}}^*)^{-1}$ where

$$\overline{\mathbf{a}}^* = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a^*_{10} & a^*_{11} & \cdots & a^*_{1n} \\ \vdots & \vdots & & \vdots \\ a^*_{n0} & a^*_{n1} & \cdots & a^*_{nn} \end{pmatrix}$$

and $\overline{\mathbf{a}}^*$ is uniquely chosen along an OSS with r=g (or equivalently $\rho = 1$) . Henceforth, we use the symbol $*$ to clarify that vectors and matrices are evaluated at $\mathbf{k}^*$.

Under these assumptions we can prove:

Lemma 1. When r=g, there exists a unique OSS $\mathbf{k}^*$ ($\gg \Theta$)[1] with the corresponding positive price vector p^* and positive factor price vector $(w_0^*, \mathbf{w}^*)$.

Proof. See Theorem1 of [6].□

Assumption 4. For all positive price vectors $(w_0^*, \mathbf{w}^*)$, the input coefficient matrix $\mathbf{A}^*$ has an inverse matrix $\mathbf{B}^*$ whose sign pattern is such that a diagonal element is negative ($b^*_{ii} < 0$) and an off-diagonal element is positive ($b^*_{ij} > 0,\ i \neq j$), where

$$\mathbf{A}^* = \begin{pmatrix} a^*_{00} & a^*_{01} & \cdots & a^*_{0n} \\ a^*_{10} & a^*_{11} & \cdots & a^*_{1n} \\ \vdots & \vdots & & \vdots \\ a^*_{n0} & a^*_{n1} & \cdots & a^*_{nn} \end{pmatrix} = \begin{pmatrix} a^*_{00} & \mathbf{a}^*_{0\cdot} \\ \mathbf{a}_{\cdot 0} & \mathbf{a}^* \end{pmatrix}$$

and

$$\mathbf{B}^* = (\mathbf{A}^*)^{-1} = \begin{pmatrix} b^*_{00} & \mathbf{b}^*_{0\cdot} \\ \mathbf{b}^*_{\cdot 0} & \mathbf{b}^* \end{pmatrix}.$$

Note that when n=1, this assumption is equivalent to the condition that the consumption goods sector is more capital intensive than the capital goods sector. See [8] for a detailed argument and [9] found a necessary and sufficient capital intensity condition for establishing this assumption for $n \geq 2$.

The following property can be proved:

[1] Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for all i, $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ for all i and at least one j, $x_i > y_i$ and $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for all i.

Lemma 2. Under Assumption 4, $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is negative for rows[2].

Proof. See Theorem 2 of [6].□

From the Euler equations Eq.7, its Jacobian $\mathbf{J}(\mathbf{k}, \rho)$ is

$$\mathbf{J}(\mathbf{k}, \rho) = \mathbf{V}_{xx}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \rho\mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \rho\mathbf{V}_{zz}(\mathbf{k}, \mathbf{k})$$

which at $\mathbf{k}^*$ is

$$\mathbf{J}(\mathbf{k}, 1) = \mathbf{V}_{xx}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{zz}(\mathbf{k}^*, \mathbf{k}^*)$$

wherein all matrices are evaluated at $\mathbf{k}^*$[3]. We will show the following important lemma, which is corresponding to Lemma 2.5 of [20].

Lemma 3. There exists a positive scalar $\overline{r}$ such that for $r \in [g, \overline{r}]$, the OSS $\mathbf{k}^r$ is unique and is a continuous vector function of r, namely $\mathbf{k}^r = \mathbf{k}(r)$.

Proof. If $\det\mathbf{J}(\mathbf{k}^r, 1) \neq 0$ then from the Implicit Function Theorem, the result follows. To show this we will use the following

fact derived in [1]:

$$\mathbf{T}_1 = [\partial T/\partial \mathbf{y}] = -\mathbf{p}, \ \mathbf{T}_2 = [\partial T/\partial \mathbf{k}] = \mathbf{w}$$

where $\mathbf{p}$ is an output price vector. Then differentiating again will yield the following second-order partial derivative matrices:

$$\mathbf{T}_{11} = [-\partial \mathbf{p}/\partial \mathbf{y}], \ \mathbf{T}_{12} = [-\partial \mathbf{p}/\partial \mathbf{k}], \ \mathbf{T}_{21} = [\partial \mathbf{w}/\partial \mathbf{y}] \text{ and } \mathbf{T}_{22} = [\partial \mathbf{p}/\partial \mathbf{k}].$$

Also note that if the matrices are evaluated at $\mathbf{k}^*$, then

[2]Suppose $\mathbf{A}$ is an $n \times n$ matrix and its diagonal elements are negative (positive). Let there exist a positive vector $\mathbf{h}$ such that $h_i \mid a_{ii} \mid > \sum_{j=1, j\neq i}^{n} h_j \mid a_{ij} \mid$, $i = 1, 2, \cdots, n$. Then $\mathbf{A}$ is said to have a quasi-dominant main diagonal that is negative (positive) for rows. See [12] and [15].

[3]We use the following notational convention for the partial derivative matrices: $\mathbf{V}_{xx} = [\partial^2\mathbf{V}(\mathbf{x}, \mathbf{z})/\partial\mathbf{x}^2], \mathbf{V}_{xz} = [\partial^2\mathbf{V}(\mathbf{x}, \mathbf{z})/\partial\mathbf{x}\partial\mathbf{z}]$ and $\mathbf{V}_{zz} = [\partial^2\mathbf{V}(\mathbf{x}, \mathbf{z})/\partial\mathbf{z}^2]$. Note that each matrix is an $n \times n$ matrix.

$$[\partial \mathbf{p}/\partial \mathbf{w}] = \mathbf{b}^*$$

and due to the symmetry of the Hessian matrix of $c_0(t) = T(\mathbf{y}(t), \mathbf{k}(t))$,

$$[\partial \mathbf{p}/\partial \mathbf{k}] = -[\partial \mathbf{w}/\partial \mathbf{y}]^T$$

where the suffix T means a transpose of a matrix. Utilizing this, all the partial derivative matrices at k^* can be expressed in terms of the matrix $\mathbf{b}^*$ and $\mathbf{T}_{22}$ as follows:

$$\mathbf{T}_{11} = \mathbf{b}^*\mathbf{T}_{22}^T b^* = \mathbf{b}^*\mathbf{T}_{22}\mathbf{b}^*, \ \mathbf{T}_{12} = -\mathbf{b}^*\mathbf{T}_{22}, \ \textit{and} \ \mathbf{T}_{21} = -\mathbf{T}_{22}\mathbf{b}^*.$$

Eliminating the first term of Eq.(2.22) of [20] and substituting $\partial u/\partial c_0 = 1$, $\mathbf{Y}_x = (g\mathbf{I} + \boldsymbol{\Delta})$ and $\mathbf{Y}_z = \mathbf{I}$ into the equation, the Jacobian can be expressed as follows:

$$\mathbf{J}(\mathbf{k}^*, 1) = [g\mathbf{I} + \boldsymbol{\Delta}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} g\mathbf{I} + \boldsymbol{\Delta} \\ \mathbf{I} \end{pmatrix}$$
$$= (g\mathbf{I} + \boldsymbol{\Delta})\mathbf{T}_{11}(g\mathbf{I} + \boldsymbol{\Delta}) + \mathbf{T}_{21}(g\mathbf{I} + \boldsymbol{\Delta}) + (g\mathbf{I} + \boldsymbol{\Delta})\mathbf{T}_{12}(g\mathbf{I} + \boldsymbol{\Delta}) + \mathbf{T}_{22}$$

Finally we have arrived at the following : if the matrix $\mathbf{b}^*$ is nonsingular,

$$\mathbf{J}(\mathbf{k}^*, 1) = -[\mathbf{b}^* - (g\mathbf{I} + \boldsymbol{\Delta})](\mathbf{b}^*)^{-1}\mathbf{T}_{22}.$$

The nonsingularity of $\mathbf{b}^*$ comes from the following observation: From [15], it follows that $\mathbf{b}^* = [\mathbf{a}^* - (1/a_{00}^*)\mathbf{a}_{\cdot 0}^*\mathbf{a}_{0\cdot}^*]^{-1}$. Furthermore, by [7] $\det\mathbf{A}^* = a_{00}^*\det[\mathbf{a}^* - (1/a_{00}^*)\mathbf{a}_{\cdot 0}^*\mathbf{a}_{0\cdot}^*]$. Combining both results and from Assumption 4 the result follows. Since the matrix $[\mathbf{b}^* - (g\mathbf{I} + \boldsymbol{\Delta})]$ has a quasi-dominant main diagonal that is negative it must be nonsingular. $\mathbf{T}_{22}$ is also non-singular due to the argument of [1] (see pp68-69). Thus the proof is completed.□

3 The Stable von Neumann Facet

Now we will introduce the von Neumann Facet (VNF), which takes important roles in stability arguments of neo-classical growth models as studied in [19] and [20] and has been intensively studied by L. McKenzie (see especially [13]). The VNF can be defined in the reduced form growth model as follows:

Definition 1. *The von Neumann Facet* $\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r)$ of the OSS $\mathbf{k}^r$ is defined as:

$$\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = \{(\mathbf{x}, \mathbf{z}) \in D : V(\mathbf{x}, \mathbf{z}) + \rho \mathbf{p}^r \mathbf{z} - \mathbf{p}^r \mathbf{x} = V(\mathbf{k}^r, \mathbf{k}^r) + \rho \mathbf{p}^r \mathbf{k}^r - \mathbf{p}^r \mathbf{k}^r\},$$

where $\mathbf{k}^r$ is an OSS and $\mathbf{p}^r$ is a supporting price of $\mathbf{k}^r$ when the subjective discount rate r is given and the price of the consumption good is normalized to one.

From the definition above, the VNF is the projection of a flat of the function V that is supported by the price vector $(-\mathbf{p}^r, \rho\mathbf{p}^r, 1)$ onto the $(\mathbf{x}, \mathbf{z})$ space. In [19] and [20], we considered the case of the objective function where n capital goods as well as a pure-consumption good were also consumable. Here, the capital goods are not consumable but the discounted sum of the sequence of the pure-consumption good is directly valued. Applying the same argument as that of [20] (pp14-15), the following equation can be derived by rewriting the definition of the VNF:

$$c_0 + \mathbf{p}^r \mathbf{y} - \mathbf{w}^r \mathbf{x} = c_0^r + \mathbf{p}^r \mathbf{y}^r - \mathbf{w}^r \mathbf{k}^r.$$

This implies that on the VNF, the same technology matrix as the corresponding OSS is chosen. This follows from the argument used in [20]. Note that in this case, the same consumption level as c_0^r need not be assigned. Therefore the VNF may be represented by the following form:

$\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = \{(\mathbf{k}(t), \mathbf{k}(t+1)) \in D :$ there exists $c_0(t) \geq 0$ and $\mathbf{y}(t) \geq 0$ such that
i) $1 = w_0^r a_{00}^r + \mathbf{w}^r \mathbf{a}_{.0}^r$, ii) $p^r = w_0^r a_{0.}^r + \mathbf{w}^r \mathbf{a}^r$,
iii) $1 = a_{00}^r c_0^r(t) + \mathbf{a}_{0.}^r \mathbf{y}(t)$, iv) $\mathbf{k}(t) = a_{.0}^r c_0(t) + \mathbf{a}^r \mathbf{y}(t)$ and
v) $\mathbf{k}(t+1) = 1/(1+g)[\mathbf{y}(t) + (\mathbf{I} - \mathbf{\Delta})\mathbf{k}(t)]\}$.

i) and ii) are cost-minimization conditions. From these conditions, it follows that $c_0(t) > 0$ and $\mathbf{y}(t) \gg \mathbf{0}$ for all t. iii) and iv) are market clearing conditions for labor and capital goods, respectively. v) are capital accumulation equations.

Now let us consider the VNF, $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$; the von Neumann Facet when r=g (or equivalently $\rho = 1$). Applying the same representation, from iii) and iv),

$$\mathbf{y}(t) = \mathbf{b}^*\mathbf{k}(t) + \mathbf{b}^*_{\cdot 0}.$$

Combining this equation with the accumulation equation v) yields

$$\mathbf{k}(t+1) = (1/(1+g))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta})\mathbf{k}(t) - ((1/(1+g))\mathbf{b}^*_{\cdot 0}.$$

Defining $\boldsymbol{\eta}(t) = \mathbf{k}(t) - \mathbf{k}^*$ yields,

$$\boldsymbol{\eta}(t+1) = ((1/(1+g))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta})\boldsymbol{\eta}(t). \tag{8}$$

This difference equation shows the motion on the VNF. So by studying this linear dynamical system, we can show the stability of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$.

Some important properties of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ concerned with a neoclassical optimal growth model have been studied in [19] and [20]. Adopting the same proofs, we can prove the following properties:

Lemma 4. dim $\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = n$.

Proof. See Lemma 3.2 of [20].□

When the pure-consumption good is directly evaluated in the reduced form utility function as the cases of [19] and [20], the same consumption level as c_0^r need be assigned. So the dimension of the VNF will loose one degree of freedom from the whole commodity dimension n+1. Furthermore due to the labor constraint, an extra one degree of freedom will be lost and the dimension of the VNF becomes n-1. On the other hand, in our model c_0 itself is a reduced form utility function and the same consumption level as c_0^r need not be assigned. So the VNF will loose only one degree of freedom from n+1due to the labor constraint and turn out to be n. Also note that the n-dimensional VNF clearly implies that the reduced form utility function V is never strictly concave, but just concave.

Lemma 5. There exists $\overline{r} > 0$ such that the VNF is a lower semi-continuous correspondence of $r \in (g, \overline{r})$.

Proof. See Lemma 3.3 of [20].□

Finally following [13] we define the stability of the VNF as follows:

Definition 2. The VNF is *stable* if there are no cyclic paths on it.

The stability of the VNF takes very important roles in proving the Turnpike properties as we will see soon. Under our capital intensity assumption, we actually show that any paths on the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ will converge to the corresponding OSS $\mathbf{k}^*$. To prove this we need the following lemma:

Lemma 6. Let us consider the following difference equation system with the equilibrium $x_e = 0$,

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t),$$

where $\mathbf{x}(t) \in R^n$ and $\mathbf{C}$ is an $n \times n$ matrix. If $\mathbf{C}$ has a quasi-dominant main diagonal that is negative for rows, $\mathbf{C} + \mathbf{I}$ is a contraction for $\mathbf{x}(t) \neq 0$ with the maximum norm $\| \cdot \|$ and the equation system is globally asymptotically stable and the Liapunov function is $\mathbf{V}(\mathbf{x})=$ $\|\mathbf{x}\|$, where $\| \cdot \|$ is defined as $\|\mathbf{x}\| = \max_i c_i \mid x_i \mid$ and c_i is a given set of positive numbers.

Proof. See pp.27-29 of [16].□

Note that if $\mathbf{C}$ has a quasi-dominant main diagonal that is positive for rows, $\mathbf{C}+\mathbf{I}$ has eigenvalues with their absolute values greater than one. This comes from the fact that if $\mathbf{C}$ has a quasi-dominant main diagonal that is positive for rows, then its eigenvalues have a positive real part. So the system is explosive; any path will diverges from the equilibrium.

Now we will show the stability of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$.

Lemma 7. The VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ is stable.

Proof. Since $\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta} = [\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})] + (1+g)\mathbf{I}$, it follows that $(1/(1{+}\mathrm{g}))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta}) = (1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})] + \mathbf{I}$. Defining $\mathbf{C}= (1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$, Eq.(8) can be rewritten as:

$$\boldsymbol{\eta}(t+1) = (\mathbf{C} + \mathbf{I})\boldsymbol{\eta}(t).$$

On the other hand, by Lemma 2 , $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is negative for rows. Thus applying Lemma 6, the result follows.□

4 Turnpike Properties

Since the lower semi-continuity and the stability of the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ have been proved, Mckenzie's Neighborhood Turnpike Theorem can be applicable as shown in [19] and [20], and finally we obtain the following theorem:

Theorem 1. *For any $\varepsilon > 0$, there exists a $\overline{r} > 0$ such that for $r \in [g, \overline{r}]$* and the corresponding $\varepsilon(\rho)$, *any optimal path $\{\mathbf{k}_t^r\}^\infty$ with a sufficient initial capital stock $\mathbf{k}(0)$*[4] *eventually lies in the $\varepsilon-$neighborhood of k^r.* Furthermore, as $\rho \rightarrow \infty$, $\varepsilon(\rho) \rightarrow 0$.

Proof. See the argument of Section 4 of [20].□

The Neighborhood Turnpike means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making ρ close enough to one. If we can show the local stability; there exists a stable manifold that will stretch out over the today's capital stock space (or equivalently along the $\mathbf{k}$(t)-plane), then combining between the Neighborhood Turnpike and the local stability implies that any optimal path must jump on the stable manifold, otherwise optimality will be violated due to the arguments by [11] and [17]. Note that in our case, the VNF itself is the n-dimensional stable manifold, because the dimension of the VNF is n and it stretched out over the $\mathbf{k}$(t)-plane. Furthermore, any path on the VNF will converge to the corresponding OSS. Thus the local stability is automatically satisfied. Therefore we have established the following full Turnpike property:

Theorem 2. *There is an $\overline{r} > 0$ such that any optimal path $\mathbf{k}^r(t)$ with a sufficient initial capital stock $\mathbf{k}(0)$ of the multi-sector neoclassical optimal growth model given by (1) through (5) must converge asymptotically to any OSS $\mathbf{k}^r$, i.e., $lim_{t\rightarrow\infty}\, \mathbf{k}^r(t) = \mathbf{k}^r$ when $r \in [g, \overline{r}]$.*

It is important to note that the stability of the VNF takes a very important role in establishing not only the Neighborhood Turnpike but also the full

[4] *A capital stock $\mathbf{x}$ is called sufficient if there is a finite sequence $(\mathbf{k}(0), \mathbf{k}(1), \cdots, \mathbf{k}(T))$ where $\mathbf{x} = \mathbf{k}(0)$, $(\mathbf{k}(t), \mathbf{k}(t+1)) \in$ D and $\mathbf{k}(T)$ is expansible. $\mathbf{k}(T)$ is expansible if there is $\mathbf{k}(T+1)$ such that $\mathbf{k}(T+1) \gg \mathbf{k}(T)$ and $(\mathbf{k}(T), \mathbf{k}(T+1)) \in$ D. Note that the sufficiency will be assured by assuming "Inada-type" condition on the production functions.*

Turnpike. Furthermore note that the capital intensity condition; Assumption 4 is an important assumption to establish these Turnpike properties. Then it is a natural question to ask whether we can establish the similar Turnpike properties under the opposite capital intensity condition to Assumption 4 assumed above. The answer to it is affirmative as we will show next.

Let assume the following opposite assumption to Assumption 4:

Assumption 4'. For all positive price vectors $(w_0^*, \mathbf{w}^*)$, the input coefficient matrix $\mathbf{A}^*$ has an inverse matrix $\mathbf{B}^*$ whose sign pattern is such that a diagonal element is negative $(b_{ii}^* > 0)$ and an off-diagonal element is positive $(b_{ij}^* < 0,\ i \neq j)$.

Note that when n=1, this implies that the capital good production sector is capital intensive than that of the pure-consumption good production sector. Under this assumption we can show the following lemma similar to Lemma 2.

Lemma 2'. Under Assumption 4, $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is positive for rows.

Proof. Since the OSS $\mathbf{k}^*$ belongs to $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$, it must satisfy Eq.(8). Then it follows

$$[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^* + \mathbf{b}_{.0} = 0.$$

Due to the fact that $\mathbf{b}_{.0} \ll 0$ from Assumption 4', we finally have

$$[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^* = -\mathbf{b}_{.0} \gg 0.$$

This clearly implies that $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^*$ has a quasi-dominant main diagonal that is positive for rows.□

Due to the fact that a quasi-dominant main diagonal matrix is nonsingular, we can show that the Jacobian $\mathbf{J}(\mathbf{k}^*, 1)$ is nonsingular. So we can establish Lemma 3. Furthermore Lemma 2' implies that any path on the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ is explosive as we will show in the next lemma.

Lemma 8. Under Assumption 4', any path on the VNF is explosive;it diverges from the corresponding OSS k*.

Proof. Defining $\mathbf{C}= (1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ and from Remark 2, $\mathbf{C}+\mathbf{I}$ has eigenvalues with their absolute values greater than one. This mens that any path on the VNF is explosive.□

Lemma 8 implies that the VNF is stable; no cyclic path on the VNF and the same Neighborhood Turnpike as Theorem 1 can be established. Moreover, under Assumption 4', since the VNF is an n-dimensional unstable manifold, we can not directly prove the local stability as the case studied before. Since an optimal path satisfies the Euler equations Eq.(6), a linear approximation of the Euler equation around (k^*, k^*) yields the following linear difference equation, provided that $\det\mathbf{V}_{xz}(\mathbf{k}^*,\mathbf{k}^*) \equiv \det \mathbf{V}^*_{xz} \neq 0$,

$$\mathbf{z}(t+1) = -(\mathbf{V}^*_{xz})^{-1}(\mathbf{V}^*_{xx} + \mathbf{V}^*_{zz})\mathbf{z}(t) - (\mathbf{V}^*_{xz})^{-1}\mathbf{V}^*_{zx}\mathbf{z}(t-1) \qquad (9)$$

where $\mathbf{z}(t) = \mathbf{k}(t) - \mathbf{k}^*$ and all the matrices are evaluated at k^*. Furthermore the characteristic equation of Eq.(10) is the following:

$$\mid \mathbf{V}^*_{xz}\lambda^2 + (\mathbf{V}^*_{xx} + \mathbf{V}^*_{zz})\lambda + \mathbf{V}^*_{zx} \mid = 0. \qquad (10)$$

To show the full Turnpike property, we need to utilize the following well-known lemma in [?]:

Lemma 9. Provided that $\det\mathbf{V}^*_{xz} \neq 0$, if the characteristic equation Eq.(11) has λ as a root of the equation then it also has$1/\lambda$ as its root.

The following lemma will establish the condition of Lemma 9 in our case.

Lemma 10. $\det \mathbf{V}^*_{xz} \neq 0$.

Proof. As we have done in Lemma 3, eliminating the first term of Eq.(2.16) of [20], substituting $\partial u/\partial c_0 = 1,\ \mathbf{Y}_x = -(\mathbf{I} - \mathbf{\Delta})$ *and* $\mathbf{Y}_z = (1+g)\mathbf{I}$ into the equation yields

$$\mathbf{V}^*_{xz} = [-(\mathbf{I} - \mathbf{\Delta}), \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{I} \\ \mathbf{\Theta} \end{pmatrix}$$

Substituting the result on the partial derivative matrices obtained in p.6 into the equation gives

$$\mathbf{V}^*_{xz} = -(1+g)(\mathbf{I}-\mathbf{\Delta})[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]\mathbf{T}_{22}\mathbf{b}^{*T}.$$

Note that $\mathbf{T}_{22}$ is non-singular as we discussed before. Since $[\mathbf{b}^*+(\mathbf{I}-\mathbf{\Delta})]$ has a quasi-dominant main diagonal that is positive for rows, $[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]$ should have also a quasi-dominant main diagonal that is positive for rows due to the fact that $(\mathbf{I}-\mathbf{\Delta})^{-1} \gg (\mathbf{I}-\mathbf{\Delta})$. This implies that $[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]$ is non-singular and thus $\mathbf{V}^*_{xz}$ is non-singular.□

Remark. Applying the similar argument as above, we can show that

$$\mathbf{V}^*_{zz} = [(1+g)\mathbf{I}, \mathbf{\Theta}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{I} \\ \mathbf{\Theta} \end{pmatrix}$$

or

$$\mathbf{V}^*_{zz} = [(1+g)\mathbf{I}]\mathbf{T}_{11}[(1+g)\mathbf{I}]^T = [(1+g)\mathbf{I}]\mathbf{b}^*\mathbf{T}_{22}\mathbf{b}^{*T}[(1+g)\mathbf{I}]^T.$$

.

Since $\mathbf{T}_{22}$ is negative definite, $\mathbf{V}^*_{zz}$ is also negative definite.

From the fact that the VNF is n-dimensional unstable manifold it follows that along the VNF there are n eigenvalues whose absolute values are greater one. Applying Lemma 10, there are n corresponding eigenvalues whose absolute values are less than one. This means that there exists an n dimensional stable manifold near the OSS. Furthermore that $\mathbf{V}^*_{zz}$ is also negative definite guarantees that the stable manifold will stretch out over the k(t)-plane due to the argument in Lemma 5.1 of [20]. Thus we have proved the following theorem:

Theorem 3. Under Assumption 4', the Neighborhood Turnpike and the full Turnpike hold.

5 Concluding Remarks

We have proved the Turnpike properties under the two types of the generalized capital intensity condition. The sharp contrast to [20] is that we need not make any direct assumptions in the sense that all the assumptions are based on the model structure, whereas in [20] we need some assumptions, especially Assumptions 6 and 8 concerned with the reduced form utility function V. And actually the assumptions made here are familiar in Optimal Growth Theory and can be found in the textbook, say [5].

Note that the Turnpike properties proved here have no contradiction to the recent arguments on cycles and chaos. Here only the control parameter is a subjective discount rate r and the other parameters are fixed. So if we would make other parameters than r change, an optimal path might take a cyclic or even a chaotic behavior. For example in[22], I have shown that for a two-sector version of our model studied in this paper, there is a certain combination among a depreciation rate, a subjective discount rate and a technology parameter, under which an optimal path converges to a cyclic path of period two (see also [4]). Therefore in a more general case studied in this paper anything could happen to a behavior of an optimal path including cycles and chaos.

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Discussion Paper #921

The von Neumann Facet and the Turnpike Properties for a Neoclassical Optimal Growth Model with Many Capital Goods II

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The von Neumann Facet and the Turnpike Properties for a Neoclassical Optimal Growth Model with Many Capital Goods II

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Abstract

We will study a multi-sector discrete-time optimal growth model with a neoclassical non-joint technology and show the Neighborhood Turnpike; any optimal path will be trapped in the neighborhood of an associated optimal steady state and its neighborhood can be chosen as small as possible by taking the discount factor close enough to one and the full Turnpike; any optimal path converges to an associated optimal steady state path when discount factors are close enough to one. These two Turnpike properties will provide the firm theoretical background for an application of a neoclassical optimal growth model with heterogeneous capital goods to economic analyses.

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1 Introduction

In my previous paper[19] and [20], two turnpike properties are proved for a very general neoclassical optimal growth model where the capital goods are consumable and the social utility function depends on these particular consumables as well as on a pure-consumption good. The turnpike results obtained there, however, depend crucially on assumptions unlinked to capital intensities and other familiar parameters. Here we will consider a simpler model than that studied in [19] and [20]. The objective function is a discounted sum of a sequence of a pure-consumption good. This type of the model is studied originally in [18] and [23] for a two-sector continuous-time optimal growth model and in [1] and [6] for a multi-sector continuous-time case.

In this paper we demonstrate the two types of the Turnpike properties: the Neighborhood Turnpike and the full Turnpike. A Neighborhood Turnpike involves any optimal path being trapped in the neighborhood of an associated optimal steady state. The neighborhood can be chosen as small as possible by choosing the discount factor small as close to one. A full Turnpike involves any optimal path converging to an associated optimal steady state path when the discount factor is sufficiently close to one. With our above objective function, the discounted sum of sequence of pure-consumption good, we will prove the two Turnpike properties, which we did in [19] and [20], here with concrete assumptions based on a model structure. In fact, we will prove the Turnpike properties under the two generarized capital intensity conditions, which are intensively studied in [8]: The first one is a counterpart to the condition in a two-sector model and involves the consumption good sector using a more capital-intensive technology than the other sector. The second one is a counterpart to the condition in a two-sector case and involves the capital goods production sector using a more capital-intensive technology. In [19] and [20], the first condition was assumed. Our analysis makes essential use of *the von Neumann Facet* (VNF henceforth): an n-dimensional plane embraces a optimal steady state on today's and tomorrow's capital stock space. Long used a similar idea to show the global asymptotic stability in a two-sector continuous-time neoclassical optimal growth model in [10]. However, note that since his line segment is called the "Rybczynski line" and is defined on the output space, but not on today's and tomorrow's capital stock space, it is not the VNF. In contrast with [19] and [20], where any path on the von Neumann facet is explosive, we will show that under the second generalized capital intensity condition any path on the von Neumann facet converges to a corresponding optimal steady state path. This means that the n-dimensional von Neumann facet is actually an n-dimensional stable manifold. So if we can prove the Neighborhood Turnpike, then we can infer that an optimal path must jump to the von Neumann facet and converge to the corresponding optimal steady state. This means that the full Turnpike holds.

Section 2 presents the model and some basic properties of an optimal steady state (OSS henceforth). In Section 3, we study the VNF and review some relevant results obtained in [20]. We will show two types of the Turnpike properties in Section 4. Section 5 concludes.

2 The Model and Assumptions

Our model is an exact discrete-time version of the one studied by [6]:

$$\max imize \sum_{t=0}^{\infty} \rho^{-1} c_0(t)$$

$$subject\ k(0) = \overline{k}$$

$$y_i(t) + k_i(t) - \delta_i k_i(t) - (1+g)k_i(t+1) = 0 \tag{1}$$

$$c_0(t) = f^0(k_{10}(t), k_{20}(t), \cdots, k_{n0}(t), \ell_0(t)), \tag{2}$$

$$y_i(t) = f^i(k_{1i}(t), k_{2i}(t), \cdots, k_{ni}(t), \ell_i(t)), \tag{3}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1, \tag{4}$$

$$\sum_{j=0}^{n} k_{ij}(t) = k_i(t), \tag{5}$$

where i=1,2,...,n, t=0,1,2,...., and the notation is as follows:

g	$=$ *rate of population growth given as* $0 < g < 1$,
r	$=$ *subjective rate of discount*, $r \geq g$,
ρ	$= (1+g)/(1+r)$,
$c_0(t) \in R_+$	$=$ *per capita consumption goods consumed at* t,
$y_i(t) \in R_+$	$= t^{th}$ *period* i^{th} *per capita capital good output*,
$k_i(t) \in R_+$	$= t^{th}$ *period* i^{th} *per capita capital stock*,
$k_i(0) \in R_+$	$=$ *initial periodp* i^{th} *per capita capital stock*,
$f^j : R_+^{n+1} \longmapsto R_+$	$=$ *per capita production function of the* j^{th} *sector which is strictly quasi concave, homogeneous of degree one and continuously differentiable on the interior of* R_+^{n+1},
$k_{ij}(t)$	$= i^{th}$ *per capita capital good used in the* j^{th} *sector in the* t^{th} *period*,
δ_i	$=$ *depreciation rate of the* i^{th} *capital good, given as* $0 < \delta_i < 1$.

Due to [2], Eqs.(2)-(5) are summarized as the social transformation function $c_0(t) = T(\mathbf{y}(t), \mathbf{k}(t))$ where T is continuously differentiable on the interior R_+^{2n}, $\mathbf{y}(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ and $\mathbf{k}(t) = (k_1(t), k_2(t), \cdots, k_n(t))$. If $\mathbf{x}$ and $\mathbf{z}$ stand for initial and terminal capital stock vectors respectively, then the reduced form utility function $V(\mathbf{x}, \mathbf{z})$ and the feasible set D can be defined as follows:

$$V(x, z) = T[(1+g)\mathbf{z} - (\mathbf{I} - \Delta)\mathbf{x}, \mathbf{x}]$$

and

$$D = \{(\mathbf{x}, \mathbf{z}) \in R_+^n \times R_+^n : T[(1+g)\mathbf{z} - (\mathbf{I} - \mathbf{\Delta})\mathbf{x}, \mathbf{x}] \geq 0\}$$

where $\mathbf{x}=(x_1(t), x_2(t), \cdots, x_n(t))$, $\mathbf{z}=(k_1(t+1), k_2(t+1), \cdots, k_n(t+1))$, $\mathbf{\Delta}$ is a diagonal matrix

$$\mathbf{\Delta} = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_1 \end{pmatrix}$$

and $\mathbf{I}$ is an n-dimensional unit matrix.

Thus the above optimization problem can be summarized as the following standard reduced form problem, which is familiar in Turnpike Theory:

$$\max imize \sum_{t=0}^{\infty} \rho^t V(\mathbf{k}(t), \mathbf{k}(t+1))$$

$$subject\ to\ (\mathbf{k}(t), \mathbf{k}(t+1)) \in D\ and\ \mathbf{k}(0) = \overline{\mathbf{k}}.$$

Also note that any optimal path must satisfy the following Euler equations, indicating an intertemporal efficiency:

$$\rho \mathbf{V}_z(\mathbf{k}(t-1), \mathbf{k}(t)) + \mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = \mathbf{\Theta} \text{ for all t} \geq 0 \tag{6}$$

where the partial derivative vectors mean that $\mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = [\partial V(\mathbf{k}(t), \mathbf{k}(t+1))/\partial \mathbf{k}(t)]$, $\mathrm{V}_z(\mathbf{k}(t-1), \mathbf{k}(t)) = [\partial V(\mathbf{k}(t-1), \mathbf{k}(t))/\partial \mathbf{k}(t)]$ and $\mathbf{\Theta}$ means an n dimensional zero vector. So if $\mathbf{k}$ is an interior OSS with a given ρ then it must satisfy

$$\rho \mathbf{V}_z(\mathbf{k}, \mathbf{k}) + \mathbf{V}_x(\mathbf{k}, \mathbf{k}) = \mathbf{\Theta}. \tag{7}$$

Let us denote w_0 *and* $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ as the wage rate and other factor price vectors respectively. Then following [6], the following assumptions are made:

Assumption 1. For all positive factor vectors $(w^0, \mathbf{w})$, the non-negative input coefficient matrix

$$\mathbf{a} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is indecomposable and the row vector $(a_{00}, a_{01}, \cdots, a_{0n})$ is positive, where $a_{ij} = k_{ij}/y_i$ and $a_{0i} = \ell_i/y_i$ $(i = 0, 1, \cdots, n : j = 1, 2, \cdots, n)$.

Assumption 2. The technology is viable (see [6] or [5] for the definition of the viability).

Assumption 3. The exogenous rate of labor force growth $g \geq 0$ satisfies inequality $g \leq 1/\lambda^*$, where λ^* is the dominant characteristic root of the matrix $\overline{\mathbf{a}}^*(\mathbf{I} - \mathbf{\Delta}\overline{\mathbf{a}}^*)^{-1}$ where

$$\bar{\mathbf{a}}^* = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{10}^* & a_{11}^* & \cdots & a_{1n}^* \\ \vdots & \vdots & & \vdots \\ a_{n0}^* & a_{n1}^* & \cdots & a_{nn}^* \end{pmatrix}$$

and $\bar{\mathbf{a}}^*$ is uniquely chosen along an OSS with r=g (or equivalently $\rho = 1$) . Henceforth, we use the symbol $*$ to clarify that vectors and matrices are evaluated at $\mathbf{k}^*$.

Under these assumptions we can prove:

Lemma 1. When r=g, there exists a unique OSS $\mathbf{k}^*$ ($\gg \Theta$)[1] with the corresponding positive price vector p^* and positive factor price vector $(w_0^*, \mathbf{w}^*)$.

Proof. See Theorem1 of [6].□

Assumption 4. For all positive price vectors $(w_0^*, \mathbf{w}^*)$, the input coefficient matrix $\mathbf{A}^*$ has an inverse matrix $\mathbf{B}^*$ whose sign pattern is such that a diagonal element is negative ($b_{ii}^* < 0$) and an off-diagonal element is positive ($b_{ij}^* > 0,\ i \neq j$), where

$$\mathbf{A}^* = \begin{pmatrix} a_{00}^* & a_{01}^* & \cdots & a_{0n}^* \\ a_{10}^* & a_{11}^* & \cdots & a_{1n}^* \\ \vdots & \vdots & & \vdots \\ a_{n0}^* & a_{n1}^* & \cdots & a_{nn}^* \end{pmatrix} = \begin{pmatrix} a_{00}^* & \mathbf{a}_{0\cdot}^* \\ \mathbf{a}_{\cdot 0} & \mathbf{a}^* \end{pmatrix}$$

and

$$\mathbf{B}^* = (\mathbf{A}^*)^{-1} = \begin{pmatrix} b_{00}^* & \mathbf{b}_{0\cdot}^* \\ \mathbf{b}_{\cdot 0}^* & \mathbf{b}^* \end{pmatrix}.$$

Note that when n=1, this assumption is equivalent to the condition that the consumption goods sector is more capital intensive than the capital goods sector. See [8] for a detailed argument and [9] found a necessary and sufficient capital intensity condition for establishing this assumption for $n \geq 2$.

The following property can be proved:

[1]Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for all i, $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ for all i and at least one j, $x_i > y_i$ and $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for all i.

Lemma 2. Under Assumption 4, $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is negative for rows[2].

Proof. See Theorem 2 of [6].□

From the Euler equations Eq.7, its Jacobian $\mathbf{J}(\mathbf{k}, \rho)$ is

$$\mathbf{J}(\mathbf{k}, \rho) = \mathbf{V}_{xx}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \rho\mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \rho\mathbf{V}_{zz}(\mathbf{k}, \mathbf{k})$$

which at $\mathbf{k}^*$ is

$$\mathbf{J}(\mathbf{k}, 1) = \mathbf{V}_{xx}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{zz}(\mathbf{k}^*, \mathbf{k}^*)$$

wherein all matrices are evaluated at $\mathbf{k}^*$[3]. We will show the following important lemma, which is corresponding to Lemma 2.5 of [20].

Lemma 3. There exists a positive scalar $\overline{r}$ such that for $r \in [g, \overline{r}]$, the OSS $\mathbf{k}^r$ is unique and is a continuous vector function of r, namely $\mathbf{k}^r = \mathbf{k}(r)$.

Proof. If $\det\mathbf{J}(\mathbf{k}^r, 1) \neq 0$ then from the Implicit Function Theorem, the result follows. To show this we will use the following

fact derived in [1]:

$$\mathbf{T}_1 = [\partial T/\partial \mathbf{y}] = -\mathbf{p}, \ \mathbf{T}_2 = [\partial T/\partial \mathbf{k}] = \mathbf{w}$$

where $\mathbf{p}$ is an output price vector. Then differentiating again will yield the following second-order partial derivative matrices:

$$\mathbf{T}_{11} = [-\partial \mathbf{p}/\partial \mathbf{y}], \ \mathbf{T}_{12} = [-\partial \mathbf{p}/\partial \mathbf{k}], \ \mathbf{T}_{21} = [\partial \mathbf{w}/\partial \mathbf{y}] \text{ and } \mathbf{T}_{22} = [\partial \mathbf{p}/\partial \mathbf{k}].$$

Also note that if the matrices are evaluated at $\mathbf{k}^*$, then

---

[2]Suppose $\mathbf{A}$ is an $n \times n$ matrix and its diagonal elements are negative (positive). Let there exist a positive vector $\mathbf{h}$ such that $h_i \mid a_{ii} \mid > \sum_{j=1, j\neq i}^{n} h_j \mid a_{ij} \mid$, $i = 1, 2, \cdots, n$. Then $\mathbf{A}$ is said to have a quasi-dominant main diagonal that is negative (positive) for rows. See [12] and [15].

[3]We use the following notational convention for the partial derivative matrices: $\mathbf{V}_{xx} = [\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})/\partial\mathbf{x}^2]$, $\mathbf{V}_{xz} = [\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})/\partial\mathbf{x}\partial\mathbf{z}]$ and $\mathbf{V}_{zz} = [\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})/\partial\mathbf{z}^2]$. Note that each matrix is an $n \times n$ matrix.

$$[\partial \mathbf{p}/\partial \mathbf{w}] = \mathbf{b}^*$$

and due to the symmetry of the Hessian matrix of $c_0(t) = T(\mathbf{y}(t), \mathbf{k}(t))$,

$$[\partial \mathbf{p}/\partial \mathbf{k}] = -[\partial \mathbf{w}/\partial \mathbf{y}]^T$$

where the suffix T means a transpose of a matrix. Utilizing this, all the partial derivative matrices at k^* can be expressed in terms of the matrix $\mathbf{b}^*$ and $\mathbf{T}_{22}$ as follows:

$$\mathbf{T}_{11} = \mathbf{b}^*\mathbf{T}_{22}^T b^* = \mathbf{b}^*\mathbf{T}_{22}\mathbf{b}^*, \ \mathbf{T}_{12} = -\mathbf{b}^*\mathbf{T}_{22}, \ \textit{and} \ \mathbf{T}_{21} = -\mathbf{T}_{22}\mathbf{b}^*.$$

Eliminating the first term of Eq.(2.22) of [20] and substituting $\partial u/\partial c_0 = 1$, $\mathbf{Y}_x = (g\mathbf{I} + \mathbf{\Delta})$ and $\mathbf{Y}_z = \mathbf{I}$ into the equation, the Jacobian can be expressed as follows:

$$\mathbf{J}(\mathbf{k}^*, 1) = [g\mathbf{I} + \mathbf{\Delta}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} g\mathbf{I} + \mathbf{\Delta} \\ \mathbf{I} \end{pmatrix}$$
$$= (g\mathbf{I} + \mathbf{\Delta})\mathbf{T}_{11}(g\mathbf{I} + \mathbf{\Delta}) + \mathbf{T}_{21}(g\mathbf{I} + \mathbf{\Delta}) + (g\mathbf{I} + \mathbf{\Delta})\mathbf{T}_{12}(g\mathbf{I} + \mathbf{\Delta}) + \mathbf{T}_{22}$$

Finally we have arrived at the following : if the matrix $\mathbf{b}^*$ is nonsingular,

$$\mathbf{J}(\mathbf{k}^*, 1) = -[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})](\mathbf{b}^*)^{-1}\mathbf{T}_{22}.$$

The nonsingularity of $\mathbf{b}^*$ comes from the following observation: From [15], it follows that $\mathbf{b}^* = [\mathbf{a}^* - (1/a_{00}^*)\mathbf{a}_{\cdot 0}^*\mathbf{a}_{0\cdot}^*]^{-1}$. Furthermore, by [7] $\det \mathbf{A}^* = a_{00}^* \det[\mathbf{a}^* - (1/a_{00}^*)\mathbf{a}_{\cdot 0}^*\mathbf{a}_{0\cdot}^*]$. Combining both results and from Assumption 4 the result follows. Since the matrix $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is negative it must be nonsingular. $\mathbf{T}_{22}$ is also non-singular due to the argument of [1] (see pp68-69). Thus the proof is completed.□

3 The Stable von Neumann Facet

Now we will introduce the von Neumann Facet (VNF), which takes important roles in stability arguments of neo-classical growth models as studied in [19] and [20] and has been intensively studied by L. McKenzie (see especially [13]). The VNF can be defined in the reduced form growth model as follows:

Definition 1. *The von Neumann Facet* $\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r)$ of the OSS $\mathbf{k}^r$ is defined as:

$$\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = \{(\mathbf{x}, \mathbf{z}) \in D : V(\mathbf{x}, \mathbf{z}) + \rho \mathbf{p}^r \mathbf{z} - \mathbf{p}^r \mathbf{x} = V(\mathbf{k}^r, \mathbf{k}^r) + \rho \mathbf{p}^r \mathbf{k}^r - \mathbf{p}^r \mathbf{k}^r\},$$

where $\mathbf{k}^r$ is an OSS and $\mathbf{p}^r$ is a supporting price of $\mathbf{k}^r$ when the subjective discount rate r is given and the price of the consumption good is normalized to one.

From the definition above, the VNF is the projection of a flat of the function V that is supported by the price vector $(-\mathbf{p}^r, \rho \mathbf{p}^r, 1)$ onto the $(\mathbf{x}, \mathbf{z})$ space. In [19] and [20], we considered the case of the objective function where n capital goods as well as a pure-consumption good were also consumable. Here, the capital goods are not consumable but the discounted sum of the sequence of the pure-consumption good is directly valued. Applying the same argument as that of [20] (pp14-15), the following equation can be derived by rewriting the definition of the VNF:

$$c_0 + \mathbf{p}^r \mathbf{y} - \mathbf{w}^r \mathbf{x} = c_0^r + \mathbf{p}^r \mathbf{y}^r - \mathbf{w}^r \mathbf{k}^r.$$

This implies that on the VNF, the same technology matrix as the corresponding OSS is chosen. This follows from the argument used in [20]. Note that in this case, the same consumption level as c_0^r need not be assigned. Therefore the VNF may be represented by the following form:

$\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = \{(\mathbf{k}(t), \mathbf{k}(t+1)) \in D :$ there exists $c_0(t) \geq 0$ and $\mathbf{y}(t) \geq 0$ such that
i) $1 = w_0^r a_{00}^r + \mathbf{w}^r \mathbf{a}_{\cdot 0}^r$, ii) $p^r = w_0^r a_{0\cdot}^r + \mathbf{w}^r \mathbf{a}^r$,
iii) $1 = a_{00}^r c_0^r(t) + \mathbf{a}_{0\cdot}^r \mathbf{y}(t)$, iv) $\mathbf{k}(t) = a_{\cdot 0}^r c_0(t) + \mathbf{a}^r \mathbf{y}(t)$ and
v) $\mathbf{k}(t+1) = 1/(1+g)[\mathbf{y}(t) + (\mathbf{I} - \mathbf{\Delta})\mathbf{k}(t)]\}$.

i) and ii) are cost-minimization conditions. From these conditions, it follows that $c_0(t) > 0$ and $\mathbf{y}(t) \gg \mathbf{0}$ for all t. iii) and iv) are market clearing conditions for labor and capital goods, respectively. v) are capital accumulation equations.

Now let us consider the VNF, $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$; the von Neumann Facet when r=g (or equivalently $\rho = 1$). Applying the same representation, from iii) and iv),

$$\mathbf{y}(t) = \mathbf{b}^*\mathbf{k}(t) + \mathbf{b}^*_{\cdot 0}.$$

Combining this equation with the accumulation equation v) yields

$$\mathbf{k}(t+1) = (1/(1+g))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta})\mathbf{k}(t) - ((1/(1+g))\mathbf{b}^*_{\cdot 0}.$$

Defining $\boldsymbol{\eta}(t) = \mathbf{k}(t) - \mathbf{k}^*$ yields,

$$\boldsymbol{\eta}(t+1) = ((1/(1+g))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta})\boldsymbol{\eta}(t). \tag{8}$$

This difference equation shows the motion on the VNF. So by studying this linear dynamical system, we can show the stability of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$.

Some important properties of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ concerned with a neoclassical optimal growth model have been studied in [19] and [20]. Adopting the same proofs, we can prove the following properties:

Lemma 4. dim $\mathbf{F}(\mathbf{k}^r, \mathbf{k}^r) = n$.

Proof. See Lemma 3.2 of [20].□

When the pure-consumption good is directly evaluated in the reduced form utility function as the cases of [19] and [20], the same consumption level as c_0^r need be assigned. So the dimension of the VNF will loose one degree of freedom from the whole commodity dimension n+1. Furthermore due to the labor constraint, an extra one degree of freedom will be lost and the dimension of the VNF becomes n-1. On the other hand, in our model c_0 itself is a reduced form utility function and the same consumption level as c_0^r need not be assigned. So the VNF will loose only one degree of freedom from n+1due to the labor constraint and turn out to be n. Also note that the n-dimensional VNF clearly implies that the reduced form utility function V is never strictly concave, but just concave.

Lemma 5. There exists $\bar{r} > 0$ such that the VNF is a lower semi-continuous correspondence of $r \in (g, \bar{r})$.

Proof. See Lemma 3.3 of [20].□

Finally following [13] we define the stability of the VNF as follows:

Definition 2. The VNF is *stable* if there are no cyclic paths on it.

The stability of the VNF takes very important roles in proving the Turnpike properties as we will see soon. Under our capital intensity assumption, we actually show that any paths on the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ will converge to the corresponding OSS $\mathbf{k}^*$. To prove this we need the following lemma:

Lemma 6. Let us consider the following difference equation system with the equilibrium $x_e = 0$,

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t),$$

where $\mathbf{x}(t) \in R^n$ and $\mathbf{C}$ is an $n \times n$ matrix. If $\mathbf{C}$ has a quasi-dominant main diagonal that is negative for rows, $\mathbf{C} + \mathbf{I}$ is a contraction for $\mathbf{x}(t) \neq 0$ with the maximum norm $\|\cdot\|$ and the equation system is globally asymptotically stable and the Liapunov function is $\mathbf{V}(\mathbf{x}) = \|\mathbf{x}\|$, where $\|\cdot\|$ is defined as $\|\mathbf{x}\| = \max_i c_i \mid x_i \mid$ and c_i is a given set of positive numbers.

Proof. See pp.27-29 of [16].□

Note that if $\mathbf{C}$ has a quasi-dominant main diagonal that is positive for rows, $\mathbf{C}+\mathbf{I}$ has eigenvalues with their absolute values greater than one. This comes from the fact that if $\mathbf{C}$ has a quasi-dominant main diagonal that is positive for rows, then its eigenvalues have a positive real part. So the system is explosive; any path will diverges from the equilibrium.

Now we will show the stability of $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$.

Lemma 7. The VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ is stable.

Proof. Since $\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta} = [\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})] + (1+g)\mathbf{I}$, it follows that $(1/(1+g))(\mathbf{b}^* + \mathbf{I} - \mathbf{\Delta}) = (1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})] + \mathbf{I}$. Defining $\mathbf{C} = (1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$, Eq.(8) can be rewritten as:

$$\boldsymbol{\eta}(t+1) = (\mathbf{C} + \mathbf{I})\boldsymbol{\eta}(t).$$

On the other hand, by Lemma 2 , $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is negative for rows. Thus applying Lemma 6, the result follows.□

4 Turnpike Properties

Since the lower semi-continuity and the stability of the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ have been proved, Mckenzie's Neighborhood Turnpike Theorem can be applicable as shown in [19] and [20], and finally we obtain the following theorem:

Theorem 1. *For any* $\varepsilon > 0$*, there exists a* $\bar{r} > 0$ *such that for* $r \in [g, \bar{r}]$ and the corresponding $\varepsilon(\rho)$, *any optimal path* $\{\mathbf{k}_t^r\}^\infty$ *with a sufficient initial capital stock* $\mathbf{k}(0)$[4] *eventually lies in the* $\varepsilon-$*neighborhood of* k^r. Furthermore, as $\rho \to \infty$, $\varepsilon(\rho) \to 0$.

Proof. See the argument of Section 4 of [20].□

The Neighborhood Turnpike means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making ρ close enough to one. If we can show the local stability; there exists a stable manifold that will stretch out over the today's capital stock space (or equivalently along the $\mathbf{k}$(t)-plane), then combining between the Neighborhood Turnpike and the local stability implies that any optimal path must jump on the stable manifold, otherwise optimality will be violated due to the arguments by [11] and [17]. Note that in our case, the VNF itself is the n-dimensional stable manifold, because the dimension of the VNF is n and it stretched out over the $\mathbf{k}$(t)-plane. Furthermore, any path on the VNF will converge to the corresponding OSS. Thus the local stability is automatically satisfied. Therefore we have established the following full Turnpike property:

Theorem 2. *There is an* $\bar{r} > 0$ *such that any optimal path* $\mathbf{k}^r(t)$ *with a sufficient initial capital stock* $\mathbf{k}(0)$ *of the multi-sector neoclassical optimal growth model given by (1) through (5) must converge asymptotically to any OSS* $\mathbf{k}^r$*, i.e.,* $\lim_{t\to\infty} \mathbf{k}^r(t) = \mathbf{k}^r$ *when* $r \in [g, \bar{r}]$.

It is important to note that the stability of the VNF takes a very important role in establishing not only the Neighborhood Turnpike but also the full

[4] *A capital stock* $\mathbf{x}$ *is called sufficient if there is a finite sequence* $(\mathbf{k}(0), \mathbf{k}(1), \cdots, \mathbf{k}(T))$ *where* $\mathbf{x} = \mathbf{k}(0)$, $(\mathbf{k}(t), \mathbf{k}(t+1)) \in$ D *and* $\mathbf{k}(T)$ *is expansible.* $\mathbf{k}(T)$ *is expansible if there is* $\mathbf{k}(T+1)$ *such that* $\mathbf{k}(T+1) \gg \mathbf{k}(T)$ *and* $(\mathbf{k}(T), \mathbf{k}(T+1)) \in$ D. *Note that the sufficiency will be assured by assuming "Inada-type" condition on the production functions.*

Turnpike. Furthermore note that the capital intensity condition; Assumption 4 is an important assumption to establish these Turnpike properties. Then it is a natural question to ask whether we can establish the similar Turnpike properties under the opposite capital intensity condition to Assumption 4 assumed above. The answer to it is affirmative as we will show next.

Let assume the following opposite assumption to Assumption 4:

Assumption 4'. For all positive price vectors $(w_0^*, \mathbf{w}^*)$, the input coefficient matrix $\mathbf{A}^*$ has an inverse matrix $\mathbf{B}^*$ whose sign pattern is such that a diagonal element is negative $(b_{ii}^* > 0)$ and an off-diagonal element is positive $(b_{ij}^* < 0,\ i \neq j)$.

Note that when n=1, this implies that the capital good production sector is capital intensive than that of the pure-consumption good production sector. Under this assumption we can show the following lemma similar to Lemma 2.

Lemma 2'. Under Assumption 4, $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ has a quasi-dominant main diagonal that is positive for rows.

Proof. Since the OSS $\mathbf{k}^*$ belongs to $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$, it must satisfy Eq.(8). Then it follows

$$[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^* + \mathbf{b}_{.0} = 0.$$

Due to the fact that $\mathbf{b}_{.0} \ll 0$ from Assumption 4', we finally have

$$[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^* = -\mathbf{b}_{.0} \gg 0.$$

This clearly implies that $[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]\mathbf{k}^*$ has a quasi-dominant main diagonal that is positive for rows.□

Due to the fact that a quasi-dominant main diagonal matrix is nonsingular, we can show that the Jacobian $\mathbf{J}(\mathbf{k}^*, 1)$ is nonsingular. So we can establish Lemma 3. Furthermore Lemma 2' implies that any path on the VNF $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ is explosive as we will show in the next lemma.

Lemma 8. Under Assumption 4', any path on the VNF is explosive;it diverges from the corresponding OSS k*.

Proof. Defining **C**= $(1/(1+g))[\mathbf{b}^* - (g\mathbf{I} + \mathbf{\Delta})]$ and from Remark 2, **C**+**I** has eigenvalues with their absolute values greater than one. This mens that any path on the VNF is explosive.□

Lemma 8 implies that the VNF is stable; no cyclic path on the VNF and the same Neighborhood Turnpike as Theorem 1 can be established. Moreover, under Assumption 4', since the VNF is an n-dimensional unstable manifold, we can not directly prove the local stability as the case studied before. Since an optimal path satisfies the Euler equations Eq.(6), a linear approximation of the Euler equation around (k^*, k^*) yields the following linear difference equation, provided that $\det \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) \equiv \det \mathbf{V}^*_{xz} \neq 0$,

$$\mathbf{z}(t+1) = -(\mathbf{V}^*_{xz})^{-1}(\mathbf{V}^*_{xx} + \mathbf{V}^*_{zz})\mathbf{z}(t) - (\mathbf{V}^*_{xz})^{-1}\mathbf{V}^*_{zx}\mathbf{z}(t-1) \qquad (9)$$

where $\mathbf{z}(t) = \mathbf{k}(t) - \mathbf{k}^*$ and all the matrices are evaluated at k^*. Furthermore the characteristic equation of Eq.(10) is the following:

$$\mid \mathbf{V}^*_{xz}\lambda^2 + (\mathbf{V}^*_{xx} + \mathbf{V}^*_{zz})\lambda + \mathbf{V}^*_{zx} \mid = 0. \qquad (10)$$

To show the full Turnpike property, we need to utilize the following well-known lemma in [?]:

Lemma 9. Provided that $\det \mathbf{V}^*_{xz} \neq 0$, if the characteristic equation Eq.(11) has λ as a root of the equation then it also has$1/\lambda$ as its root.

The following lemma will establish the condition of Lemma 9 in our case.

Lemma 10. $\det \mathbf{V}^*_{xz} \neq 0$.

Proof. As we have done in Lemma 3, eliminating the first term of Eq.(2.16) of [20], substituting $\partial u/\partial c_0 = 1$, $\mathbf{Y}_x = -(\mathbf{I} - \mathbf{\Delta})$ *and* $\mathbf{Y}_z = (1+g)\mathbf{I}$ into the equation yields

$$\mathbf{V}^*_{xz} = [-(\mathbf{I} - \mathbf{\Delta}), \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{I} \\ \mathbf{\Theta} \end{pmatrix}$$

Substituting the result on the partial derivative matrices obtained in p.6 into the equation gives

$$\mathbf{V}_{xz}^* = -(1+g)(\mathbf{I}-\mathbf{\Delta})[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]\mathbf{T}_{22}\mathbf{b}^{*T}.$$

Note that $\mathbf{T}_{22}$ is non-singular as we discussed before. Since $[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})]$ has a quasi-dominant main diagonal that is positive for rows, $[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]$ should have also a quasi-dominant main diagonal that is positive for rows due to the fact that $(\mathbf{I}-\mathbf{\Delta})^{-1} \gg (\mathbf{I}-\mathbf{\Delta})$. This implies that $[\mathbf{b}^* + (\mathbf{I}-\mathbf{\Delta})^{-1}]$ is non-singular and thus $\mathbf{V}_{xz}^*$ is non-singular.□

Remark. Applying the similar argument as above, we can show that

$$\mathbf{V}_{zz}^* = [(1+g)\mathbf{I}, \mathbf{\Theta}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{I} \\ \mathbf{\Theta} \end{pmatrix}$$

or

$$\mathbf{V}_{zz}^* = [(1+g)\mathbf{I}]\mathbf{T}_{11}[(1+g)\mathbf{I}]^T = [(1+g)\mathbf{I}]\mathbf{b}^*\mathbf{T}_{22}\mathbf{b}^{*^T}[(1+g)\mathbf{I}]^T.$$

.

Since $\mathbf{T}_{22}$ is negative definite, $\mathbf{V}_{zz}^*$ is also negative definite.

From the fact that the VNF is n-dimensional unstable manifold it follows that along the VNF there are n eigenvalues whose absolute values are greater one. Applying Lemma 10, there are n corresponding eigenvalues whose absolute values are less than one. This means that there exists an n dimensional stable manifold near the OSS. Furthermore that $\mathbf{V}_{zz}^*$ is also negative definite guarantees that the stable manifold will stretch out over the k(t)-plane due to the argument in Lemma 5.1 of [20]. Thus we have proved the following theorem:

Theorem 3. Under Assumption 4', the Neighborhood Turnpike and the full Turnpike hold.

5 Concluding Remarks

We have proved the Turnpike properties under the two types of the generalized capital intensity condition. The sharp contrast to [20] is that we need not make any direct assumptions in the sense that all the assumptions are based on the model structure, whereas in [20] we need some assumptions, especially Assumptions 6 and 8 concerned with the reduced form utility function V. And actually the assumptions made here are familiar in Optimal Growth Theory and can be found in the textbook, say [5].

Note that the Turnpike properties proved here have no contradiction to the recent arguments on cycles and chaos. Here only the control parameter is a subjective discount rate r and the other parameters are fixed. So if we would make other parameters than r change, an optimal path might take a cyclic or even a chaotic behavior. For example in[22], I have shown that for a two-sector version of our model studied in this paper, there is a certain combination among a depreciation rate, a subjective discount rate and a technology parameter, under which an optimal path converges to a cyclic path of period two (see also [4]). Therefore in a more general case studied in this paper anything could happen to a behavior of an optimal path including cycles and chaos.

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