Queen's Economics Department Working Paper No. 907

Standardized Variables, Risks and Preference

Frank Milne Edwin H. Neave

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

7-1994

Discussion Paper #907

Standardized Variables, Risks, and Preference

by

Frank Milne
Department of Economics
Queen's University

and

Edwin H. Neave
School of Business
Queen's University

July 1994

Standardized Variables, Risks, and Preference

Frank Milne and Edwin H. Neave
Queen's University
Kingston, Ontario K7M 3N6

Latest revision: July 22, 1994

Abstract

This paper examines the expected utility effects of adding one risk to another. In comparison to related works, it places fewer restrictions on utilities and more structure on risky asset returns. The paper, entailing little loss of generality, uses discrete variables defined on a common domain (hereafter standardized variables) to find sufficient conditions for either of two (dependent or independent) variables to dominate their sum in the second degree. It then finds (higher order) sufficient conditions for either of the variables to dominate their sum in the third degree. While utilities are only restricted to be increasing concave, the expected utility differences for the respective risk positions are the same as if the investors were respectively proper or standard risk averse (Pratt-Zeckhauser [1987], Kimball [1993]).

Presented to the 1994 Summer Meeting of the Econometric Society, Quebec City. This is the second of two companion papers.

Standardized Variables, Risks, and Preference

Frank Milne and Edwin H. Neave

1. Introduction

1.1 Review of Literature

Properties of risky investment demand are usually established by restricting utilities. For example, Pratt-Zeckhauser [1987] (hereafter PZ) and Kimball [1993] define classes of utilities permitting them to characterize expected utility relations between a given variable and the sum of two independent variables.

The PZ and Kimball results, like those to be presented below, are useful in insurance and other applications. For example, suppose a client has insured against a given risk. If she must then face an additional (independent or dependent) risk, will that increase the insurance premium she is willing to pay? As a second example, if an exporter gains a second export contract yielding the same rate of return as a first, will his unhedged risk, relative to the contracts' return, be viewed as greater or lesser?

Previously established results can be viewed as research aimed at explaining such choices:

> "Proper risk aversion is a descriptive property of a utility function with important implications for behavior. If its normative appeal is accepted, it should be useful in refining the strategy that began by developing the von Neumann-Morgenstern utility from axiomatic restrictions...." (PZ [1987, p. 153].)

However, Milne-Neave [1994b] observe that existing research has not yet examined how investors might respond to further restriction of the variables themselves, and the present paper addresses this question by examining how fewer restrictions on utilities, along with additional structure on risks, can imply behavior similar to that in PZ and in Kimball. In so doing, the paper both complements existing results and raises anew an old question:[1] is observed behavior attributable to preferences or to probabilities?

1.2 Present Approach

This paper uses a method of standardized variables to obtain its results with only minimal sacrifice of generality. The paper first finds new relations between dominance criteria and conditions on random variables' moments. Next, the paper analyzes the expected utility effects of adding one variable to another, finding conditions sufficient for either of two (dependent or independent) variables to be preferred to their sum by second degree

---

[1]See, for example, Ward Edwards [1953].

stochastic dominance. This result establishes a closer connection between dominance and proper risk aversion, and also finds results like those of PZ [1988], but for dependent as well as independent variables.[2] In addition, the paper finds higher order sufficiency conditions for either of two variables, dependent or independent, to be preferred to their sum by third degree dominance. This result establishes a closer connection between standard risk aversion and third degree dominance, and also extends some of Kimball's [1993] results to dependent as well as independent variables.

The paper is organized as follows. Section 2, after reviewing some of the properties of standardized variables established in Milne-Neave ([1994a], [1994b]), offers new interpretations of dominance criteria in terms of conditional moments. Section 3 finds conditions on risks sufficient for any risk averter to regard them as if she were properly risk averse, while Section 4 finds (higher order) conditions on risks sufficient for any decreasing absolute risk averter to regard them as if his utility satisfied standard risk aversion. Section 5 concludes.

2. Interpreting Dominance Criteria

This section interprets dominance criteria in terms of conditional moments.

2.1 Preliminaries

If U is the class of strictly increasing strictly concave utilities, a random variable A is said to be preferred to B by second degree stochastic dominance iff

$$E\{\ u(A)\ \} \geq E\{\ u(B)\ \},\ u \in U. \tag{2.1}$$

If A and B are discrete variables, A stochastically dominates B in the second degree if and only if[3]

$$\sum_{j=-k}^{m} F_A(j)\epsilon_j \leq \sum_{j=-k}^{m} F_B(j)\epsilon_j\ ;\ m \in J_k\ , \tag{2.2}$$

where $F_X(j)$ is the distribution function of X; cf. Hadar-Russell [1969].

Next, consider a family of discrete random variables, generically denoted X, with the common domain $J_k \equiv \{\ -k, -k+1, \ldots, k\ \}$. In this case,

---

[2]PZ discuss extensions to dependent variables and to stochastic dominance, but do not obtain the conditions for such extensions.

[3]To eliminate trivialities, (2.2) is assumed to hold strictly for at least one $m \in J_k$.

$$\epsilon_j \equiv j - (j-1) = 1,$$

a property that simplifies the rest of the paper's analysis. In particular, X can be described by its **probability vector** x, where the components of x are indexed according to outcome values,

$$x \equiv (x_{-k}, \ldots, x_k)' \geq 0. \tag{2.3}$$

In (2.3), $'$ denotes transpose, and $e'x = 1$, where e is a $(2k+1)$-dimensional vector of ones. Then

$$x_j \equiv Pr\{X = j\}; j \in J_k, \tag{2.4}$$

where Pr means probability. Different variables need not have exactly the same outcomes: if j is not a realizable value of X, $x_j \equiv 0$.

Assuming henceforth that A and B are both defined on J_k, let

$$d \equiv b - a. \tag{2.5}$$

Then

$$d_j = Pr\{B = j\} - Pr\{A = j\}; j \in J_k,$$

and $e'd = 0$.

2.2 Second Degree Dominance

Dominance relations among variables defined on J_k can be expressed by rewriting (2.1) to show that A stochastically dominates B in the second degree if and only if (cf. Milne-Neave [1994a], [1994b])

$$\beta \equiv S^2 d \equiv S(Sd) \geq 0, \tag{2.6}$$

where S is a $(2k+1)$ x $(2k+1)$ matrix with ones on and below its main diagonal, zeroes above, while $S^2 \equiv SS$. Examples of S and S^2 are

$$S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

and

$$S^2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} . \tag{2.7}$$

Inequality (2.6) means $\beta_j \geq 0$ for all $j \in J_k$, and $\beta_j > 0$ for at least one $j \in J_k$.

The system (2.6) can be solved to express d in terms of β:[4]

$$d = S^{-2}\beta = (\beta_{-k} , \beta_{-k+1} - 2\beta_{-k} , \beta_{-k+2} - 2\beta_{-k+1} + \beta_{-k} , \ldots, \beta_k - 2\beta_{k-1} + \beta_{k-2})'$$

where $S^{-2} \equiv (SS)^{-1}$. Whether or not (2.6) is satisfied,

$$\beta_{k-1} = \beta_k = E(A) - E(B), \tag{2.8}$$

cf. Milne-Neave [1994a]. Moreover since $\beta_{k-1} = \beta_k$, $d_k = - \beta_{k-1} + \beta_{k-2}$.

For any probability vector x, S^2x is a vector whose j'th component is:

$$(S^2x)_j = \sum_{i=-k}^{j} (j+1-i)x_i = [(j+1) - E \{X \leq j\}] \, Pr \{X \leq j\},$$

where $E\{ X \leq j \}$ means expectation conditional on X achieving no outcome greater than j, and $Pr\{ X \leq j \}$ is the probability that X achieves no outcome greater than j. Thus,

---

[4]The numbers d_j, $j \in J_k$, need not all be positive, but since both a and b are probabilities $a - d = b \geq 0$, a condition satisfied by (2.9).

recalling (2.5), A dominates B in the second degree iff

$$(S^2d)_j = [\,(j+1) - E\{B \le j\}\,]\,Pr\{B \le j\} \\ -[\,(j+1) - E\{A \le j\}\,]\,Pr\{A \le j\} \ge 0;\ j \in J_k\,. \tag{2.9}$$

Conditions (2.9) show that ranking by second degree dominance imposes restrictions on successive conditional means. The applications of Section 3 and 4, considering dominance relations between X, Y and $Z \equiv X + Y$, find that related restrictions on the successive conditional means of X, and further conditional on fixed values of Y, is sufficient for Z to be dominated by Y in the second degree and in the third degree respectively.

2.3 Third Degree Dominance

Suppose R *is* the class of non-decreasing concave utilities whose third difference is non-negative; i.e.,

$$u_{j+3} - 3u_{j+2} + 3u_{j+1} - u_j \ge 0, \tag{2.10}$$

$j \in J_{k-3} \sim \{\, j_{k-2}\,, j_{k-1}\,, j_k \,\}$, where $\sim$ means does not include. Then continuing to study variables defined on J_k, A dominates B by TSD iff

$$E\{\ u(A)\ \} \ge E\{\ u(B)\ \}$$

for any $u \in R$. Analogous to (2.6), A stochastically dominates B in the third degree if and only if:

$$\gamma \equiv S^3 d \ge 0$$

and[5] (2.11)

$$[E(A) - E(B)] \ge 0;$$

see Milne-Neave [1994a]. In (2.11) $d \equiv b - a$ as before. An example of S^3 is:

[5]Whitmore [1970] shows $E(A) \ge E(B)$ is necessary if A is to dominate B by TSD.

$$S^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 6 & 3 & 1 & 0 \\ 10 & 6 & 3 & 1 \end{vmatrix} . \tag{2.12}$$

More generally, the elements of S^3 can be written in the form $q(k, 2) \equiv k!/(k-2)!2!$ and $q(j, k) \equiv 0$ if $j < k$. Also as in Section 2.2,

$$d = S^{-3}\gamma = (\gamma_0, -3\gamma_0+\gamma_1, 3\gamma_0-3\gamma_1+\gamma_2 , -\gamma_0+3\gamma_1-3\gamma_2+\gamma_3 , \ldots, -\gamma_{k-3}+3\gamma_{k-2}-3\gamma_{k-1}+\gamma_k)'.$$

Moreover,

$$\gamma_{k-1} = \gamma_{k-2} + \beta_{k-1} , \gamma_k = \gamma_{k-1} + \beta_k , \tag{2.13}$$

and since by (2.8) $\beta_k = \beta_{k-1}$,

$$\gamma_{k-2} - 2\gamma_{k-1} + \gamma_k = 0.$$

Finally,

$$(S^3x)_j = \sum_{i=-k}^{j} [(j+2-i)(j+1-i)/2]x_i = \tag{2.14}$$

$$[((j^2+3j+2)/2) - ((2j+3)/2)E \{X \leq j\} + (1/2)E\{X^2 \leq j\}]Pr\{X \leq j\}.$$

where $E\{ X^2 \leq j \}$ is the second moment about zero, conditional on X achieving no outcome greater than j.

Conditions (2.14) interpret the necessary and sufficient conditions for ranking by third degree dominance as restrictions on the first and second conditional moments. The application of Section 4, considering dominance relations between X, Y and $Z \equiv X + Y$, finds a restriction on the conditional means of X that is sufficient for Z to be dominated by Y in the third degree. As would be expected upon noting that the utility assumptions for third degree dominance are more restrictive than for second degree dominance, the new restriction on risks is similar to but less stringent than those sufficient for second degree dominance.

3. Proper Risk Aversion and Second Degree Dominance

This section reviews the notion of proper risk aversion, develops notation for jointly distributed random variables, and then finds sufficiency conditions for dominance of the sum of two variables by one of its summands.

3.1 Preliminaries

Given a von Neumann - Morgenstern utility u, deterministic wealth w and independent random variables X and Y, u is said to be fixed wealth[6] proper (cf. PZ [1988]) if

$$E\{\ u(w + Z)\ \} \leq E\{\ u(w + X)\ \}$$

where $Z \equiv X + Y$, whenever

$$E\{\ u(w + X)\ \} \leq E\{\ u(w)\ \} \tag{3.1}$$

and

$$E\{\ u(w + Y)\ \} \leq E\{\ u(w)\ \}. \tag{3.2}$$

The PZ results for proper risk aversion are based on restricting utilities. This paper's results are based on further restrictions of discrete variables defined on J_k.

A necessary condition for (3.1) is $Pr\ \{X = j\} > 0$ for at least one outcome $j < 0$. Similarly, a necessary condition for (3.2) is $Pr\{\ Y = j\ \} > 0$ for at least one outcome $j < 0$. If (3.1) and (3.2) are required to hold for all $u \in U$, it is also necessary that $E(X) \leq 0$ and $E(Y) \leq 0$.

Lemma 3.1 next shows that if X and Y are independent random variables which satisfy (3.1) and (3.2) for all $u \in U$, then X dominates Z by SSD.

Lemma 3.1: Let X and Y be independent random variables such that $E(X) \leq 0$. Then Y stochastically dominates Z in the second degree.

---

[6]Since the signs of derivatives and therefore such properties as concavity are preserved under expectation, it suffices here to consider fixed rather than random wealth (cf. Kimball [1993]). Some properties such as decreasing absolute risk aversion are also preserved under both maximization and expectation, a characteristic Kimball refers to as heritability. Although he does not use the term, Neave [1971] obtains heritability results for the Arrow-Pratt indices, preceding the sources cited by Kimball.

Proof: Let $u \in U$. By the definition of Z, the independence of X and Y, and by concavity,

$$E\{\, u(w + Z | Y = j) \,\} = E\{\, u(\, w + X + j) \,\} \leq u(w + E(X) + j).$$

Then employing the law of iterated expectations

$$E_j\{E\{u(w+Z | Y=j)\}\} \leq E_j\{u(w+E(X)+j)\} \leq E_j\{u(w+j)\} = E\{u(w + Y)\}$$

for any $u \in U$.■

Corollary: If X and Y are independent random variables such that $E(Y) \leq 0$, then X stochastically dominates Z in the second degree.

Lemma 3.2: When they hold for any $u \in U$, (3.1) and (3.2) are necessary as well as sufficient for SSD.

Proof: See Hadar-Russell [1969]; Milne-Neave [1994a].

3.2 Standardizations for Two Random Variables

This section's principal theorem will be established using the standardized variables developed in Section 2. To represent sums as standardized random variables, note that for any X and Y defined on J_k, Z is defined on J_{2k}. Thus it is convenient to define X and Y on J_{2k} by assigning a probability of zero to unattainable outcomes. For example,

$$y = (\, 0_{-2k}, \ldots, 0_{-k-1}, y_{-k}, y_{-k+1}, \ldots, y_k, 0_{k+1}, \ldots, 0_{2k})'$$

is now regarded as a probability vector of dimension $4k+1$, with the subscripts on the zeroes indicating outcomes which always have probability zero.

Continuing to assume that all vectors are defined to have dimension $4k + 1$, a necessary and sufficient condition for Y to dominate Z by SSD is, by (2.6),

$$S^2[\, z - y \,] \geq 0, \tag{3.3}$$

where z is the probability vector of Z and S is understood to have dimension $(4k+1) \times (4k+1)$. A representation using joint probabilities will make it easier to interpret (3.3). Let

$$M \equiv |z_{ij}|;\ i, j \in J_k;$$

where

$$z_{ij} \equiv Pr\{\, X = i,\ Y = j \,\}.$$

Then

$$Pr\{ Z = m \} = z_{i,m-i} + z_{i-1,m-i+1} + \ldots + z_{-i,m+i} \equiv z_m ; \tag{3.4}$$

$$m = i + j, i \in J_k , j \in J_k .$$

Next, let

$$Pr\{ Z = m \} \equiv e'[D_m \otimes M]e.$$

where D_m is a matrix of zeroes except for ones on the positively sloped diagonal whose outcomes add to $m \in J_{2k}$. The symbol $\otimes$ indicates element-by-element multiplication of matrices. To illustrate, if $k = 3$ and $j = -2$ then

$$D_{-2} = \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. \end{matrix}$$

Similarly,

$$Pr\{ Y = j \} \equiv e'[C_j \otimes M]e$$

where C_j is a matrix of zeroes, with ones in the column corresponding to outcome j. If $j \in J_{2k} \sim J_k$, $C_j \equiv 0$. Again for illustration, still taking $k = 3$ but now letting $j = 1$,

$$C_1 = \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0. \end{matrix}$$

To denote weighted conditional means, let

$$E\{ X | Y = j \} Pr\{ Y = j \} \equiv e'K \cdot C_j \otimes Me \tag{3.5}$$

where K is a matrix with the vector $(-k, \ldots, k)'$ composing its main diagonal, and zeroes everywhere else.

Next rewrite the dominance condition (3.3) in terms of individual rows; i.e.,

$$\{ S^2[z - y] \}_j \geq 0; j \in J_{2k} . \tag{3.6}$$

To find sufficiency conditions which satisfy inequalities (3.6), rewrite them as

$$\{ S^2[z - y] \}_j = e'[(T_j - U_j)\otimes M]e; j \in J_{2k}.$$

where

$$T_j \equiv \sum_{i=-2k}^{j} (j+1-i)D_i$$

and

$$U_j \equiv \sum_{i=-2k}^{j-k} (j+1-k-i)C_i .$$

For example taking $k = 3$ and $j = -2$ as before,

$$T_{-2} - U_{-2} = \begin{matrix} 5 & 4 & 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} - \begin{matrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0. \end{matrix}$$

Next, define the cumulative sums of conditional means as

$$F_j \equiv K \sum_{i=-2k}^{j} C_i .$$

Define matrices T_j^- using the following two steps. First, let the upper left hand corner element equal $j + 2k + 1$, the elements on the adjacent second diagonal (elements (1,2) and (2,1)) equal $j + 2k$, and so on. Second, set all the elements in columns $j + 1, \ldots, k$ equal to zero. By construction, $T_j \geq max\{ T_j^-, 0 \}$, where the maximization is understood as being taken on an element by element basis. Continuing the previous example for $k = 3, j = -2$,

$$T_{-2}^- - U_{-2} = \begin{matrix} 5 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 \end{matrix} \quad - \quad \begin{matrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0. \end{matrix}$$

It follows immediately that $T_{-2}^- - U_{-2} = -F_{-2}$, and in a similar fashion $T_j^- - U_j = -F_j, j \in J_{2k}$.

We can now establish

Theorem 3.3: If

$$\Sigma_j E\{ X|Y = j \}Pr\{Y = j\} = e'[F_j \otimes M]e \leq 0; j \in J_{2k},$$

then

$$\{ S^2[z - y] \}_j = e'[(T_j - U_j) \otimes M]e \geq 0; j \in J_{2k}.$$

Proof: By construction of T_j^-,

$$e'[(T_j - U_j) \otimes M]e \geq e'[(T_j^- - U_j) \otimes M]e = -e'[F_j \otimes M]e.$$

Since by hypothesis

$$-e'[F_j \otimes M]e \geq 0; j \in J_{2k},$$

the desired conclusion follows.■

Remark: Appendix I shows how combined matrices can be set up to prove the Theorem for all j simultaneously.

Remark: The sufficiency conditions of Theorem 3.3 rule out certain forms of negative correlation between X and Y. To see this, let Y be such that $Pr\{Y = -j\} = Pr\{Y = j\} > 0$ for all $j \in J_k$, thus implying that $E\{ Y \} = 0$. Take $X \equiv -Y$. Then X and Y are perfectly negative correlated, $Z \equiv 0$ and Z dominates Y in the second degree. There is, however, no contradiction of the theorem: the example frustrates the conditions of Theorem 3.3 since, for example, Y can be specified so that $E\{ X|Y = -k \} > 0$.

Finally, while the conditions of Theorem 3.3 are only sufficient, they do have an intuitive appeal. Moreover, the examples of Appendices I and II suggest that finding necessary conditions is not a straightforward task.

4. Standard Risk Aversion and Third Degree Dominance

This section first reviews the concept of standard risk aversion, then considers relations between standard risk aversion and third degree dominance for either dependent or independent variables.

4.1 Preliminaries

Given a von Neumann - Morgenstern utility u, deterministic wealth w and independent random variables X and Y, u is said to be standard risk averse (Kimball [1993]) if

$$E\{ u(w + Z) \} \leq E\{ u(w + X) \} \tag{4.1}$$

whenever

$$E\{ u'(w + X) \} \geq E\{ u'(w) \} \tag{4.2}$$

and

$$E\{ u(w + Y) \} \leq E\{ u(w) \}. \tag{4.3}$$

A necessary condition for (4.2) is that $Pr\{ X = j \} > 0$ for some outcome $j < 0$. Similarly it is necessary for (4.3) that $Pr\{ Y = j \} > 0$ for some outcome $j < 0$. As before, $E(Y) \leq 0$ is also necessary if (4.3), the same condition as (3.2), is to hold for all $u \in U$.

The rest of this section considers utility functions $u \in U^*$, the set of strictly increasing strictly concave utilities with $u''' \geq 0$. As is well known, $u \in U^*$ implies that u exhibits decreasing absolute risk aversion.

Lemma 4.1: Condition (4.2) holds for every $u \in U^*$ iff $E(X) \leq 0$.

Proof (Sufficiency): Suppose $E(X) \leq 0$. For any $u \in U^*$, convexity of u' implies that

$$E\{ u'(w + X) \} \geq u'(w + E(X)) \geq u'(w).$$

Proof (Necessity): Suppose i) $E(X) > 0$ and ii) $E\{ (u'(w+X) \} \geq u'(w)$ for some $u \in U^*$. Let $u'(t) = 1 + \xi, t < w, u'(w)=1,$ and $u'(t) = 1 - \xi, t > w$. But then i) and the form of u imply that $u'\{ w + E(X) \} < u'(w)$, contradicting ii).■

If X satisfies (4.2) and the conditions of Lemma 4.1, it also satisfies (3.1). Moreover, if Y satisfies (4.3), the same condition as (3.2), then Lemmas 4.1 and 3.1 together imply that X dominates Z by SSD, and Y dominates Z by SSD. That is, when X and Y are independent the risks analogous to standard risk aversion are also analogous to proper risk aversion; in the setting of this paper the only role played by restricting u to U^* is to permit using (4.2) instead of (3.1).

4.2 Third Degree Dominance

Theorem 4.2 establishes that for $u \in U^*$ we can find variables which, when summed, are regarded as if the investors assuming the combined risks were standard risk averse. That is, Theorem 4.2 establishes sufficiency conditions such that X and Y dominate Z by TSD. Since TSD can rank some variables that cannot be ranked by SSD, the sufficiency conditions of Theorem 3.3 imply the sufficiency conditions of Theorem 4.2, but the converse is not true.

Theorem 4.2: Let X and Y be random variables such that $E(X) \geq E(Y)$. Then $E\{ u(w + X) \} \geq E\{ u(w + Y) \}$ for every $u \in U^*$ iff $S^3(y - x) \geq 0$.

Proof: See Milne-Neave [1994a]. ■

Using Theorem 4.2, we now establish conditions for TSD similar to the SSD conditions of Theorem 3.3. Let

$$W_{-2k} \equiv T_{-2k} \; , \; W_j \equiv W_{j-1} + T_{j-1} \; , j \in J_{2k} \sim \{ -2k \}.$$

In this case W_{-2k} has a one in the upper left hand corner and zeroes everywhere else, W_{-2k+1} has a three in the top left corner, 1's in the two positions on the adjacent second diagonal, and so on. Finally, W_{2k} has the element $q(4k+2, 2)$ in the upper left-hand corner, the elements $q(4k+1, 2)$ along the second diagonal adjacent to the upper left hand corner, ... , $q(2, 2)$ along the last of the successive second diagonals. The binomial coefficients $q(j, k)$ are defined immediately following (2.12).

Next, let

$$Q_{-2k} \equiv U_{-2k} \; , \; Q_j \equiv Q_{j-1} + U_{j-1} \; , j \in J_{2k} \sim \{ -2k \}.$$

Then

$$Q_j \equiv 0, j \in \{-2k, \ldots, -k-1\}.$$

For $j \in \{-k, \ldots, 2k\}$, the Q_j have the vectors

$$q(k+j+2,\, 2) \cdot (-1, \ldots, 1)',$$

$$q(k+j+1,\, 2) \cdot (-1, \ldots, 1)',$$

$$\ldots$$

$$q(j+2,\, 2)(-1, \ldots, 1)'$$

forming the first $j + k + 1$ columns, and zeroes in any remaining columns. In particular, the first column of Q_{2k} is a constant $(2k+1)$-vector whose components all equal $q(3k+2, 2)$, the second column a vector with components $q(3k+1, 2)$, and the last column a vector with components $q(k+2, 2)$.

For example taking $k = 3, j = 0$,

$$W_0 - Q_0 \quad = \quad \begin{matrix} 28 & 21 & 15 & 10 & 6 & 3 & 1 \\ 21 & 15 & 10 & 6 & 3 & 1 & 0 \\ 15 & 10 & 6 & 3 & 1 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \quad - \quad \begin{matrix} 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0. \end{matrix}$$

Finally, let

$$G_{-2k} \equiv F_{-2k}\,,\; G_j \equiv G_{j-1} + F_{j-1}\,,\; j \in J_{2k} \sim \{\,-2k\,\}.$$

Next, define the matrices W_j^- according to the following two steps. First, subtract the vector

$$(\, q(k+1,2),\, q(k,\, 2),\, \ldots,\, q(2,2),\, 0,\, 0,\, q(2,2),\, \ldots,\, q(k,2))'$$

from each of the first $k + j + 1$ columns of W_j. Second, set all the elements in columns $j + 1, \ldots, k$ equal to zero. By construction, $W_j \geq W_j^-$, and it follows that $W_j^- - Q_j \geq -G_j$. Continuing the previous example for $k = 3, j = 0$,

$$W_0^- - Q_0 \quad = \quad \begin{matrix} 22 & 15 & 9 & 10 & 0 & 0 & 0 \\ 18 & 12 & 7 & 6 & 0 & 0 & 0 \\ 14 & 9 & 5 & 3 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ -2 & -3 & -3 & -3 & 0 & 0 & 0 \end{matrix} \quad - \quad \begin{matrix} 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0. \end{matrix}$$

Noting that

$$\sum_{i=-k}^{j} (j+1-i)E\{ X|Y=i \}Pr\{Y=i\} \ ; \ j\varepsilon J_k$$

$$= e'[(G_j)\otimes M]e; \ j \in J_k ,$$

we can establish

Theorem 4.3: If

$$\sum_{i=-k}^{j} (j+1-i)E\{ X|Y=i \}Pr\{Y=i\} \leq 0; \ j\varepsilon J_k \ . \qquad (2.16)$$

then

$$\{ S^3[z - y] \}_j = e'[(W_j - Q_j)\otimes M]e \geq 0; j \in J_{2k} .$$

Proof: By construction of W_j^-,

$$e'[(W_j - Q_j)\otimes M]e \geq e'[(W_j^- - Q_j)\otimes M]e \geq -e'[G_j \otimes M]e.$$

Since by hypothesis

$$-e'[G_j \otimes M]e \geq 0; j \in J_{2k},$$

the desired conclusion follows. ■

Remark: Appendix II shows how combined matrices can be set up to prove the Theorem for all j simultaneously.

While the conditions of Theorem 4.3 are only sufficient, they have an intuitive appeal similar to those of Theorem 3.3. Moreover, they permit comparing analogues to proper and to standard risk aversion in a manner similar to the comparisons of Kimball [1993].

Since TSD requires signing the utilities' first three derivatives, its preference assumptions are more restrictive than those for SSD, which requires signing only the first two. As a result, TSD permits comparing a larger class of random variables than SSD. Accordingly, the sufficiency conditions established in Theorem 4.3 are less restrictive than those established in Theorem 3.3.

5. Conclusions

This paper used standardized variables to establish new portfolio theoretic results. The paper extends both PZ and Kimball by finding classes of returns, for either dependent or independent risky assets, to which investor attitudes are similar to those implied by proper or by standard risk aversion. To do so, it defines one class of variables which can be ranked in relation to their sums by second degree dominance, and a second class which can be ranked in relation to their sums by third degree dominance.

By showing that restrictions on asset returns can imply investor attitudes similar to those implied by restricting utilities, the paper raises a familiar question in descriptive economics. Whenever the behaviors discussed here are encountered in practical contexts, they may ultimately prove to be consequences of either preferences or probabilities.

References

Edwards, Ward, "Probability Preferences in Gambling," American Journal of Psychology 66, 349-364, 1953.

Hadar, Joseph, and W. R. Russell (1969), "Rules for Ordering Uncertain Prospects," American Economic Review 59, 25 - 34.

Kimball, Miles S., (1993), "Standard Risk Aversion," Econometrica 61, 589-612.

Milne, Frank, and Neave, Edwin H. (1994a), "Dominance Relations on a Standardized Domain," Management Science, forthcoming.

Milne, Frank, and Neave, Edwin H. (1994b), "Standardized Variables and Optimal Risky Investment" Queen's University Working Paper.

Neave, Edwin H., "Multiperiod Consumption - Investment Decisions and Risk Preference", Journal of Economic Theory 2, (1971), 40-53.

Pratt, John W., (1964), "Risk Aversion in the Small and in the Large," Econometrica 32, 122 - 136.

Pratt, John W., and Richard J. Zeckhauser (1987), "Proper Risk Aversion," Econometrica 55, 143 - 154.

Appendix I: Combined Coefficient Matrices for Theorem 3.3

Example for $k = 3$.

T

13	12	11	10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0
12	11	10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0
11	10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0
8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0
7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0

U

10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0

F

-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	0	0	0	0	0	0	0	0	0
-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	0
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	3	0	0	0	0	0	0	0	0	0

$T-U+F$

0	0	0	0	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0

The last $2k+1$ columns of the T matrix form T_{-2k} , the $2k+1$ columns beginning one to the left of that form T_{-2k+1} , and so on for all remaining j. The construction is exactly the same for the matrices U_j and F_j from U and F respectively.

Appendix II: Combined Coefficient Matrices for Theorem 4.4

Example for $k = 3$.

W

91	78	66	55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0
78	66	55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0
66	55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0	0
36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0	0	0
28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0	0	0	0

Q

55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0
55	45	36	28	21	15	10	6	3	1	0	0	0	0	0	0	0	0	0

G

-30	-27	-24	-21	-18	-15	-12	-9	-6	-3	0	0	0	0	0	0	0	0	0
-20	-18	-16	-14	-12	-10	-8	-6	-4	-2	0	0	0	0	0	0	0	0	0
-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	9	8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0	0
20	18	16	14	12	10	8	6	4	2	0	0	0	0	0	0	0	0	0
30	27	24	21	18	15	12	9	6	3	0	0	0	0	0	0	0	0	0

$W-Q+G$

6	6	6	6	6	6	6	6	6	6	6	3	1	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	3	3	1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	2	0	0	0	0	0	0	0	0	0

The last $2k+1$ columns of the W matrix form W_{-2k} , the $2k+1$ columns beginning one to the left of that form W_{-2k+1} , and so on for all remaining j. The construction is exactly the same for the matrices Q_j and G_j from Q and G respectively.