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Standardized Variables and Optimal Risky Investment

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Abstract

This paper studies the optimal risky investment problem with fewer restrictions on utilities, and more structure on risks, than does the current literature. It uses discrete random variables defined on a common domain, hereafter called standardized variables, to obtain new results without important loss of generality.

The optimal amount of investment in a single risky asset does not always decrease as risk increases in the Rothschild-Stiglitz ([1970, 1971]; hereafter RS) sense. However, by using standardized variables to define wealth dependent measures of risk and return, the paper finds necessary and sufficient conditions on risks such that an increase in risk does cause decreasing optimal risky investment. The paper thus complements the RS results. For investment in two risky assets, the paper uses standardized variables to find conditions on risks such that the riskier asset's demand to decrease (increase) as the Arrow-Pratt absolute risk aversion index increases (decreases), and thereby complements Ross' [1981] results.

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1. Introduction

1.1 Review of Literature

Properties of risky investment demand functions are usually established by imposing restrictions on utilities. Rothschild and Stiglitz [1970, 1971] (hereafter RS) restrict the class of concave utilities[1] to find conditions sufficient for increasing risk in the RS sense to imply decreasing optimal investment in a single risky asset. Diamond and Stiglitz [1974] extend RS to utility - compensated increases in risk. Ross [1981] considers optimal investment in two risky assets. After showing that risky investment need not decrease with an increase in Arrow-Pratt risk aversion, Ross develops a more restrictive measure of risk aversion to characterize asset demands.

The foregoing research program thus aims to define preferences with observable portfolio choice implications. Since this program has expended relatively less effort examining how investors might respond to further restriction of the risks themselves, the present paper examines this approach. By placing fewer restrictions on utilities and additional structure on risks, the paper obtains behavioral implications similar to those obtained earlier. Thus, it both complements existing results and raises anew an old question:[2] is observed behavior attributable to preferences or to probabilities?

1.2 Present Approach

This paper employs discrete random variables with a common domain, called standardized variables, to develop new relations between dominance criteria, wealth dependent risk - return tradeoffs, and conditional means. These properties are then applied to characterizing optimal risky investment decisions.

Unless either utilities or risks are appropriately restricted, optimal risky investment need not decrease as risk increases in the RS sense. Nevertheless, the paper finds necessary and sufficient conditions on risks such that any risk averse investor will decrease risky investment in response to an increase in risk. Risks must satisfy a dominance related restriction, but utilities must only be concave.

For optimal investment in two risky assets, the paper finds sufficient conditions for the riskier asset's demand to decrease (increase) as the Arrow-Pratt absolute risk aversion

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[1]The RS conditions are i) decreasing absolute risk aversion, ii) increasing relative risk aversion, and iii) relative risk aversion less than unity.

[2]See, for example, Ward Edwards [1953].

index increases (decreases). The analysis shows both why Ross' [1981] counterexample frustrates a decrease in demand for the riskier investment and why demand for it will decrease if the random variables satisfy a dominance related restriction.

In summary, the paper establishes:

i) new interpretations of second degree dominance comparisons involving wealth dependent risk and return effects;

ii) new interpretations of second degree dominance comparisons involving differences between conditional means;

iii) necessary and sufficient conditions on a single risky asset such that any risk averter will invest less in the asset as its risk increases;

iv) sufficient conditions on two risky assets such that an increase in absolute risk aversion implies the investor will buy less of the riskier asset;

v) Given i) and ii), conditions iii) through and iv) can be expressed using any one of: dominance criteria, risk - return relations or differences between conditional means.

In a companion paper (Milne-Neave [1994b]), we use the same technology to analyze the expected utility effects of adding one risk to another, when the risks have fixed size.

The paper is organized as follows. Section 2 defines standardized variables and relates dominance, risk - return measures, and conditional means. The findings of Section 2 are then used to establish the paper's portfolio theoretic results. Section 3 finds necessary and sufficient conditions under which all risk averters decrease their optimal risky investment whenever asset risk increases. For investment in two risky assets, Section 4 finds sufficient conditions implying that more risk averse investors (in the Arrow-Pratt sense) purchase less of the riskier asset. Section 5 concludes.

2. Tools of Analysis

After defining standardized random variables, this section relates second degree dominance criteria to risk - return effects and to conditional means.

2.1 Interpreting Dominance Criteria

Consider a family of discrete random variables, generically denoted X, with the common domain[3] $J_k \equiv \{ -k, -k+1, \ldots, k \}$. If A and B are members of this family, A stochastically dominates B in the second degree if and only if[4]

$$\sum_{j=-k}^{m} F_A(j)\varepsilon_j \leq \sum_{j=-k}^{m} F_B(j)\varepsilon_j \ ; \ m \in J_k \ , \tag{2.1}$$

where $F_X(j)$ is the distribution function of X. The standardized domain J_k implies

$$\epsilon_j \equiv j - (j-1) = 1,$$

and this simplification is used throughout the paper. If U is the class of strictly increasing strictly concave utilities, (2.1) is equivalent to

$$E\{ u(A) \} > E\{ u(B) \}, u \in U;$$

cf. Hadar-Russell [1969]. When $E(A) = E(B)$, RS show that (2.1) implies:

$$B = A + \Delta, \tag{2.2}$$

where the equality in (2.2) refers to equality in distribution[5] and

$$E(\Delta | A = j) = 0; j \in J_k. \tag{2.3a}$$

If B satisfies (2.2) and (2.3a) it is termed riskier than A in the sense of RS. In the sequel it will often prove convenient to think of the distributional equality in (2.2) as involving a sum

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[3]Variables need not all have the same outcomes, since for a given variable an outcome may obtain only with probability zero.

[4]To eliminate trivialities, (2.1) is assumed to hold strictly for at least one $m \in J_k$.

[5]Two random variables are equal in distribution if they have the same outcomes with the same probabilities, while two random variables are equal if all possible conditional distributions have the same outcomes with the same probabilities; cf. Ross [1979].

of terms:

$$B = A + \Sigma_j \Delta_j, j \in J_k, \tag{2.3b}$$

where for each j, Δ_j represents a conditional variable with an expectation as defined in (2.3a). We later express the individual conditional variables using sums of the probability vectors defined next.

Given J_k, X can be described by its **probability vector** x. The components of x will be indexed according to outcome values

$$x \equiv (x_{-k}, \ldots, x_k)' \geq 0,$$

where $'$ denotes transpose, and

$$x_j \equiv Pr\{ X = j \}; j \in J_k . \tag{2.4}$$

where Pr means probability. Whenever j is not a realizable value of X, $x_j = 0$. Moreover, $e'x = 1$, where e is a $(2k+1)$-dimensional vector of ones.

At later points in the paper, we employ multiplicative transformations of variables, e.g. ηA, $\eta \in (0, 1)$. If A is defined on J_k, then ηA is defined on $J_{\eta k}$, and, while the corresponding probability vector ηa has exactly the same component values as a, the outcomes to which ηa refers are scaled outcomes. Moreover, the realizations of A and ηA are perfectly positively correlated, and

$$Pr\{ A = j, \eta A = \eta j \} = Pr\{ A = j \} = Pr\{ \eta A = \eta j \}; j \in J_k .$$

Given any two variables A and B defined on J_k, let

$$d \equiv b - a. \tag{2.5}$$

Then

$$d_j = Pr\{ B = j \} - Pr\{ A = j \}; j \in J_k ,$$

and $e'd = 0$.

Dominance relations can now be expressed as linear transformations of probability vectors. Given the standardized domain J_k, (2.1) can be rewritten to show that A stochastically dominates B in the second degree if and only if (cf. Milne-Neave [1994a])

$$\beta \equiv S^2 d \equiv S(Sd) \geq 0, \tag{2.6}$$

where S is a $(2k+1) \times (2k+1)$ matrix with ones on and below its main diagonal, zeroes above. In addition $S^2 \equiv SS$. Examples of S and S^2 are

$$S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

and

$$S^2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}.$$

Inequality (2.6) means $\beta_j \geq 0$ for all $j \in J_k$, and $\beta_j > 0$ for at least one $j \in J_k$. Whether or not (2.6) is satisfied,

$$\beta_{k-1} = \beta_k = E(A) - E(B). \tag{2.8}$$

Properties (2.6), (2.7) and (2.8) will be used frequently below.

Next, (2.6) can be rewritten to express the probability differences d as linear transformations of β. Equations (2.6) have a solution[6]

$$d = S^{-2}\beta$$

(where $S^{-2} \equiv (SS)^{-1}$); i.e.:

[6]The numbers d_j, $j \in J_k$, need not all be positive, but since both a and b are probabilities $a - d = b \geq 0$, a condition satisfied by (2.9).

$$d_{-k} = \beta_{-k}$$

$$d_{-k+1} = \beta_{-k+1} - 2\beta_{-k}$$

$$d_{-k+2} = \beta_{-k+2} - 2\beta_{-k+1} + \beta_{-k} \tag{2.9}$$

$$\dots$$

$$d_{k-1} = \beta_{k-1} - 2\beta_{k-2} + \beta_{k-3}$$

$$d_k = \beta_k - 2\beta_{k-1} + \beta_{k-2} = - \beta_{k-1} + \beta_{k-2} .$$

The second equality in the last line of (2.9) follows from (2.8). An example of S^{-2} is:

$$S^{-2} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{vmatrix} .$$

2.2 Dominance and Risk - Return Tradeoffs

Conditions (2.9) further permit interpreting second degree dominance relations as risk (noise) and return (mean shift) effects. To see this, rewrite (2.9) as

$$d = S^{-2}\beta = \delta_{-k} + \dots + \delta_{k-2} + \delta_{k-1}, \tag{2.10a}$$

where

$$\delta_j = (0_{-k}, \dots, 0_{j-1}, \beta_j , -2\beta_j , \beta_j , 0_{j+3}, \dots, 0_k)'; \tag{2.10b}$$

$$j \in \{-k, \dots, k-2\}$$

and

$$\delta_{k-1} = (0_{-k}, \dots, 0_{k-2} , \beta_{k-1} , -\beta_{k-1})'. \tag{2.10c}$$

(The subscripts on the zeroes indicate which outcome probabilities are affected by a given noise or mean shift term.) If $\beta_j > 0, j \in \{-k, \dots, k-2\}$, the corresponding δ_j show that each risk effect is the product of a scale factor β_j and a unit measure of increasing risk conditional on the outcome $A = j + 1$. If $\beta_j < 0, A$ will no longer dominate B by second degree stochastic dominance, but β_j can formally be interpreted as a risk reduction term.[7]

The effects of the term β_{k-1} are next discussed, beginning with $\beta_{k-1} = 0$. If $\beta_{k-1} = 0$, $E(A) = E(B)$ and, because B is riskier in the RS sense, A dominates B by SSD. This case is closely related to the mean - variance criteria for portfolio selection, since $\sigma^2(B) - \sigma^2(A)$ is a linear function of β.

Lemma 2.1: $\sigma^2(B) - \sigma^2(A) = 2e'\beta - \beta_{k-1}\{ [2k + 3] - [E(A) + E(B)] \}$.

Proof: For any A and B,

$$\sigma^2(B) - \sigma^2(A) = e'K^2d - [(e'Kb)^2 - (e'Ka)^2]$$

where K is a diagonal matrix with elements $-k, -k+1, \ldots, k$. Then by (2.10b) and (2.8)

$$\sigma^2(B) - \sigma^2(A) =$$

$$e'K^2(\delta_{-k} + \ldots + \delta_{k-2} + \delta_{k-1}) + \beta_{k-1}[E(B) + E(A)]$$

Noting that each of the first $2k-1$ terms $e'K^2\delta_j$ in (2.11) has the form

$$j^2\beta_j - 2(j+1)^2\beta_j + (j+2)^2\beta_j = 2\beta_j ,$$

we obtain

$$\sigma^2(B) - \sigma^2(A) =$$

$$2e'\beta - \beta_{k-1}\{ [2k+3] - [E(B) + E(A)] \}. \blacksquare$$

Corollary: If $E(A) = E(B)$, then

$$\sigma^2(B) - \sigma^2(A) = 2e'\beta.$$

Proof: The result follows immediately from the fact that $E(A) = E(B)$ implies $\beta_{k-1} = 0$; cf. Milne-Neave[1994a]. ■

Lemma 2.1 and its corollarly show the mean-variance criterion in effect suppresses information related to differing levels of wealth (location information), since by definition a variance weights all the terms of β_j equally. Thus, the mean-variance criterion does not

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[7]Any term β_j can also be interpreted as two shift effects, but in the applications we have developed so far, interpretations as risk effects have been more useful.

allow for the differing marginal utilities associated with the risk effects reflected in $\beta_j = \beta_\ell$ when $j \neq \ell$. In contrast, this paper exploits these distinctions below.

Now consider $\beta_{k-1} > 0$, supposing also that $\beta_{-k} \geq 0, \ldots, \beta_{k-2} \geq 0$ and that strict inequality holds for at least some $j \in \{-k, \ldots, k-2\}$. In this case, $E(A) > E(B)$. Since A is also less risky than B, A dominates B by SSD.

For the case $\beta_{k-1} < 0$, suppose as before that $\beta_{-k} \geq 0, \ldots, \beta_{k-2} \geq 0$ and that inequality holds in at least one case. Now A cannot dominate B because $E(A) < E(B)$, but at the same time B cannot dominate[8] A because B is riskier in the RS sense.

2.3 Illustrations

Suppose A and B are defined on J_2 and respectively have the probability vectors

$$a = (0, 1/2, 0, 1/2, 0)'; \; b = (1/4, 0, 1/4, 1/2, 0)'.$$

Then

$$d = (1/4, -1/2, 1/4, 0, 0)'; \; S^2 d = (1/4, 0, 0, 0, 0)';$$

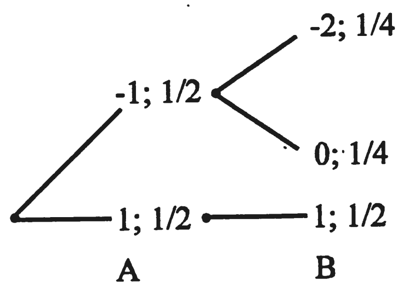
A dominates B in the second degree. In this case $B = A + \{\Delta_1 | A = -1\}$ (where the relation is equality in distribution),

$$\{\Delta_1 \mid A = -1\} = \begin{cases} -1; & \textit{with prob } 1/2 \\ 0; & \textit{with prob } 0 \\ 1; & \textit{with prob } 1/2 \end{cases}$$

and $\delta_{-1} = (1/4, -1/2, 1/4, 0, 0)'$. For comparison's sake, a probability tree relating A and B is shown in Table 1.

[8]A risk averse investor who places considerable weight on mean returns would prefer B, an investor who weighed risks more heavily would prefer A.

Table 1: Probability Tree Relating A and B



Numbers in each cell indicate outcome and probability of outcome.

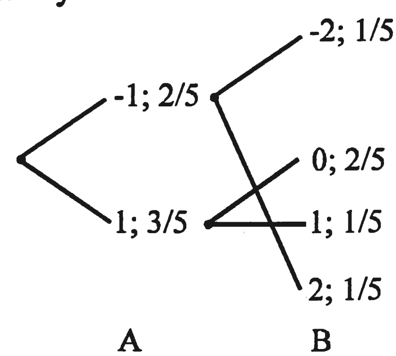
Table 2 displays two ways of examining differences between two random variables comparable by second degree dominance. Since the two examples assume that the variables differ by the same probability vector *d*, we have that $B = A + \Delta$ and $B^* = A + \Delta^*$ are such that $B = B^*$, where the equality is interpreted as equality in distribution. The difference between the two sets of circumstances lies in the second case's standardized interpretation of risk differences. Note from this second interpretation that the marginal utility of the second risk effect is less than the marginal utility of the first for any strictly concave utility, and the two will thus have different impacts on portfolio choice, although the RS measures of risk do not recognize these differences.

Table 2: Interpreting Second Degree Dominance

Non-Standard Interpretation

J_k	-2	-1	0	1	2
na	0	2	0	3	0
Δ_{-1}	1	-2	0	0	1
Δ_1	0	0	2	-2	0
nb	1	0	2	1	1
$nS^2(b - a)$	1	0	1	0	0

Associated Probability Tree



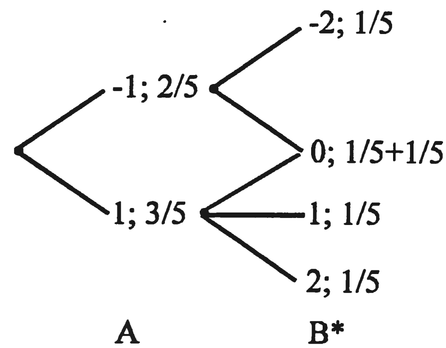
In this and the following examples it is often convenient to convenient to express the values of x_j as rational numbers of the form ℓ_j/n, and multiply by n so as to express probabilities using integers ℓ_j. In the present example, $n = 5$.

Standardized Interpretation

(Assumes equality in distribution can be employed)

J_k	-2	-1	0	1	2
na	0	2	0	3	0
Δ_{-1}	1	-2	1	0	0
Δ_1	0	0	1	-2	1
nb^*	1	0	2	1	1

Associated Probability Tree



As above, n = 5 in this example.

2.3 Dominance and Conditional Means

Dominance criteria are also related to conditional means. First, as is well known, the first order dominance criterion is interpreted in terms of probability distribution functions:

$$S_j x = Pr\{ X \leq j \}; j \in J_k ,$$

where S_j is the j'th row of S (indexed according to the outcomes of X), and $Pr\{ X \leq j \}$ is the cumulative probability that X assumes a value no greater than j. Next, let K be defined as in Section 2.2. Then

Lemma 2.3: $(SK)_j x = S_j(Kx) = E\{ X \leq j \} Pr\{ X \leq j \}, j \in J_k ,$ (2.12)

where $(SK)_j$ refers to the j'th row of SK, and $E\{ X \leq j \}$ is the expectation of X conditional on $X \leq j$.

Proof: Follows immediately from the definitions of S and K. ■

Note that when $E(A) = E(B)$, applying (2.12) to d and examining the *k'th* row shows $e'(Kd) = 0$.

A similar relation can be established for the second order criterion:

Lemma 2.4: $(S^2)_j x = [(j + 1) - E\{ X \leq j \}] Pr\{ X \leq j \}, j \in J_k .$ (2.13)

Proof: It can readily be verified that

$$S^2 + SK = (K + 1)S,$$

and hence on a component-by-component basis

$$(S^2)_j x + (SK)_j x = (j + 1)S_j x, \; j \in J_k . \blacksquare \qquad (2.14)$$

Note that applying (2.13) to d and examining the k-1'st and k'th rows shows $\beta_{k-1} = \beta_k$.

The paper's portfolio theoretic results are obtained using variants of the foregoing conditions. The conditions $S^2 d \geq 0$ and $SKd \leq 0$ are invoked to extend RS, and

$$nS^2 d = (1, 1, \ldots, 1, -1, -1)'$$

is used to extend Ross.

3. Optimal Risky Investment and Changes in Risk: The Single Risky Asset Case

This section considers allocating funds between cash[9] and a risky asset with future payoffs C. For any $u \in U$, the problem is

$$max_\eta \; E\{ \; u(w - \eta + \eta C) \; \} \equiv max_\eta \; E\{ \; u(w + \eta[C-1]) \; \},$$

where w is initial, deterministic wealth.

3.1 Optimal Risky Investment

Substituting A for C-1, suppose (3.1) has a positive, unique interior solution[10] defined by:

$$E\{ \; u'(w + \eta_A A)A \; \} = 0, \qquad (3.1)$$

where $'$ indicates a first derivative and $\eta_A > 0$ indicates the solution.[11] Similarly, let $\eta_B > 0$ be the optimal interior solution defined by

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[9]Making the interest rate on cash positive requires no essential modifications.

[10]Two necessary conditions for an optimal interior solution are that the risky investment must have at least one outcome inferior to investing in cash, and that the expected return on risky investment exceed the (deterministic) return on cash. These restrictions imply that neither asset can dominate the other by SSD; cf. Lemma 5.1 below.

[11]The paper assumes throughout that the appropriate derivatives of u exist.

$$E\{u'(w + \eta_B B)B\} = 0. \tag{3.2}$$

We assume:

i) A dominates B by SSD; and

ii) $E(A) = E(B)$.

Thus, in terms of the conventions of Section 2:

$$S^2 d = (\beta_{-k}, \ldots, \beta_{k-2}, 0, 0)' \geq 0. \tag{3.3}$$

Next, we seek additional conditions on the risks such that

$$E\{u'(w + \eta_A B)B\} < 0; \tag{3.4}$$

i.e., $\eta_B < \eta_A$. Conditions (2.2), (2.5), (3.1) and (3.2) show that (3.4) is satisfied iff

$$\begin{aligned} &E\{u'(w+\eta_A B)B\} - E\{u'(w+\eta_A A)A\} \\ &= \sum_{t=-k}^{k} [u'(w+\eta_A t)\cdot tb_t - u'(w+\eta_A t)\cdot ta_t] \\ &= \sum_{t=-k}^{k} u'(w + \eta_A t)\cdot td_t < 0. \end{aligned} \tag{3.5}$$

(Note that since (3.5) compares $\eta_A A$ and $\eta_A B$, the variables are both defined on the domain $J_\eta k$ and are therefore related by equality in distribution as in section 2.) Section 3.2 restricts changes in risk to find necessary and sufficient conditions for (3.5). Before obtaining these results formally, it is instructive to note from (2.9) that (3.5) can be rewritten as

$$\sum_{t=-k}^{k} u'(w+\eta_A t)\cdot t\cdot(\beta_{t-2} - 2\beta_{t-1} + \beta_t) < 0, \tag{3.6}$$

where $\beta_{-k-1} \equiv \beta_{-k-2} \equiv 0$. Rearranging (3.6) gives

$$\sum_{s=-k+1}^{k-1} \{u'(w+\eta_A[s-1])(s-1) - 2u'(w+\eta_A s)s + u'(w+\eta_A[s+1])(s+1)\}\beta_{s-1} < 0. \tag{3.7}$$

The RS comparative statics result is obtained from sufficiency conditions for $u'(w + \eta_A z)z$ to be a strictly concave function of z, where z refers to any outcome of A or

B. In effect, the RS conditions are sufficient to sign each term of the summation in (3.7). But to find plausible comparative statics it is only necessary to sign the sum of terms, as discussed next.

3.2 Changes in Risk

Since marginal utility decreases monotonically for $u \in U$, both the last line of (3.5) and (3.6) suggest that conditions reflecting both noise terms and wealth levels are needed to capture the tradeoff between the marginal disutility of increasing risk and the marginal utility of increasing mean return. The second degree dominance criterion cannot do so because it gives the same weighting to noise terms conditional on different asset realizations. However, the next Theorem shows that a suitable restriction is:

$$SKd = [(K + I)S - S^2]d \leq 0,$$

where S, S^2 , K, and d were defined in Section 2, and where I is an identity matrix.

Since the first k diagonal elements of K are negative and since $S^2d \geq 0$ when A dominates B by SSD, the last condition will hold for at least some choices of A and B. The condition can be interpreted either as an outcome-weighted first degree dominance condition (the expression on the left of the equality) or as a vector of differences between conditional means and the second degree dominance criterion (the expression on the right of the equality). Intuitively, the condition assures that an increase in RS risk will occur at levels of wealth low enough to ensure that the marginal disutility of the risk increase will be greater, at the originally chosen optimum than the marginal utility of the mean increase which accompanies it as optimal risky investment is varied. This interpretation is further elaborated below.

Theorem 3.1: Let $u \in U$, and suppose that

$$S^2d = (\beta_{-k}, \beta_{-k+1}, \ldots , \beta_{k-2}, 0, 0)' \geq 0.$$

Then $SKd \leq 0$ is a necessary and sufficient condition for (3.5) to be negative.

Proof: (Sufficiency). Let $SKd \equiv Sv \equiv \alpha \leq 0$, and consider

$$v = S^{-1}\alpha,$$

whose solution has the explicit form:

$$v_{-k} = \alpha_{-k}$$
$$v_{-k+1} = \alpha_{-k+1} - \alpha_{-k}$$
$$v_{-k+2} = \alpha_{-k+2} - \alpha_{-k+1} \tag{3.9}$$
$$\ldots$$
$$v_{k-1} = \alpha_{k-1} - \alpha_{k-2}$$
$$v_k = \alpha_k - \alpha_{k-1} = 0 - \alpha_{k-1};$$

cf. Milne-Neave [1994]. Using (3.3), (3.5) can be rewritten, first as

$$\sum_{t=-k}^{k} u'(\,w+\eta_A t\,) \cdot v_t = \sum_{t=-k}^{k} u'(\,w+\eta_A t\,) \cdot (\alpha_t - \alpha_{t-1}) \tag{3.10}$$

where $\alpha_{-k-1} \equiv 0$, and then as

$$\sum_{t=-k}^{k} [\, u'(\,w + \eta_A t\,) - u'(\,w + \eta_A(t+1)\,)\,] \cdot \alpha_t < 0, \tag{3.11}$$

where $u'(w + \eta_A[k+1]) \equiv 0$. The inequality in (3.5) then follows from $\eta_A > 0$, u' strictly decreasing, $\alpha \leq 0$, and $\alpha_j < 0$ for some $j \in J_k$.

(Necessity): Suppose the theorem's hypotheses are not satisfied; i.e. suppose $\alpha_m > 0$ for some $m \in J_k$. In this case (3.5) can be contradicted as next shown, establishing necessity.

Choose a piecewise linear, strictly increasing utility function y. Even though $y \notin U$, it can be used to find $y^* \in U$ by altering y while keeping the sign of (3.5) unchanged. As a first step choose $y, y \notin U$ such that

$$y'(w + \eta_A i) = 2;\ i = -k, -k+1, \ldots, m,$$

and (3.12)

$$y'(w + \eta_A i) = 1;\ i = m+1, \ldots, k.$$

Substituting (3.12) in (3.11) shows that term m equals $\alpha_m > 0$, and that all other terms are zero. To find $y^* \in U$, alter conditions (3.12) so that

$$y^{*\prime}(w + \eta_A i) = 2 + (m-i)\epsilon;\ i = -k, -k+1, \ldots, m,$$

and (3.13)

$$y^{*\prime}(w + \eta_A i) = 1 + (m+1-i)\epsilon;\ i = m+1, \ldots, k,$$

where

$$0 < \epsilon < 1/2k.$$

The resulting piecewise linear function y^* can be regarded as a member of U, since it is strictly increasing and strictly concave over more than an single outcome.[12] Moreover, using y^* (3.5) can be rewritten as

$$\sum_{t=-k}^{m-1} \varepsilon \cdot \alpha_t + \alpha_m + \sum_{t=m+1}^{k} \varepsilon \cdot \alpha_t = (1-\varepsilon)\alpha_m + \varepsilon \sum_{t=-k}^{k} \alpha_t = (1-\varepsilon)\alpha_m > 0, \qquad (3.14)$$

where the summation term after the first equality in (3.14) is zero because $E(A) = E(B)$. Hence, (3.14) gives the desired contradiction.■

3.3 Examples

This section's first example satisfies the conditions of Theorem 3.1 and implies decreasing risky investment. The second violates the conditions and implies increasing risky investment for some risk averters. Let[13]

$$k = 21, J = \{ -1, \ldots, 21 \}, u(w) = -w^{-1}, w = 10,$$

and consider the variables A, B, and B^* distributed as in Table 3:

[12]The further modifications necessary to create a strictly decreasing marginal utility with continuous derivatives are of formal interest, but can be ignored here.

[13]The change in domain is for notational simplicity.

Table 3: Examples

Outcomes:	-1	0	1	...	19	20	21
na'	1	2	0	...	0	10	0
nb'	2	0	1	...	0	10	0
$nb^{*}{}'$	1	2	0	...	1	8	1

Note that $E(A) = E(B) = E(B^*) = 199/13$. In these examples, $n = 13$.

Consider first the optimal investment in A, for which $\eta_A = 3.849243$ satisfies (3.2). That is, after multiplying by 13 (3.2) becomes

$$-(w - \eta_A)^{-2} + 200 \cdot (w+20\eta_A)^{-2} = 0.000000.$$

Now, B is riskier than A in the RS sense and $d = b - a$ satisfies the conditions of Theorem 3.1:

$$13(SKd)' = (-1, -1, 0, \ldots, 0).$$

At η_A the marginal disutility of increasing risk is exactly balanced against the marginal utility of increasing return. Moreover, η_B will be less than η_A whenever the marginal disutility at η_A of increased risk (weighted by outcomes) exceeds the marginal utility of increased return. Evaluating (3.4) now gives (again after multiplying by 13)

$$-2 \cdot (w - \eta_A)^{-2} + (w + \eta_A)^{-2} + 200 \cdot (w+20\eta_A)^{-2} = -0.021219$$

confirming that $\eta_B < \eta_A$.

Finally, B^* presents the same increase in RS risk as B, but the noise term is conditional on a higher outcome and should therefore have a smaller marginal disutility. Moreover, $d^* = b^* - a$ does not satisfy the conditions of Theorem 3.1:

$$13(SKd^*)' = (0, 0, 0, \ldots, 0, 19, -21, 0).$$

Evaluating (3.4) gives

$$-(w - \eta_A)^{-2} + 19 \cdot (w+19\eta_A)^{-2} + 160 \cdot (w+20\eta_A)^{-2} + 21 \cdot (w+21\eta_A)^{-2}$$

$$= 0.000008,$$

implying $\eta_{B^*} > \eta_A$ for the chosen utility function.

4. Two risky investments

This section considers investing in two risky assets, A and and a riskier $B = A + \Delta$. The problem to be studied is

$$max_\eta\ E\{\ v(w + (1\text{-}\eta)A + \eta B)\ \} \equiv max_\eta\ E\{\ v(w + A + \eta\Delta)\ \} \tag{4.1}$$

where $v \in U$. In effect, the investor has an initial random wealth position $w + A$, and by choosing η optimally equates the marginal utility of increasing mean return with the marginal disutility of increasing risk. If the optimal decision is to involve positive investments in both A and B, neither A nor B can dominate the other in the second degree. To rule out this possibility of second degree dominance we assume i) B is riskier than A in the RS sense and ii) $E(B) > E(A)$.

4.1 Relations to Ross

Ross [1981] provides an example showing that for risks A and $B^* = A + \Delta^*$ (the asterisks indicate a difference in approach to be elaborated shortly) there are utilities $u, v \in U$, v more risk averse than u, such that an investor with utility v purchases more of a riskier asset than one with utility u. We seek more restrictive conditions on risks, such that an investor with utility v will purchase less of the riskier asset B than will an investor with utility u.

Our results are based on the restrictions illustrated in Table 4. Beginning with A and $B^* = A + \Delta^*$ very similar to the variables used by Ross, we choose A and $B = A + \Delta$, and require $\Delta = \Delta^*$ in distribution.[14] Table 4 uses Ross' variable A, while Δ^* represents combinations of noise and mean shift terms similar, but not identical to those used by Ross. The changes in the example permit a direct comparison with the present Δ.

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[14]Since (4.1) shows the optimal choices of A and B depend on the form of Δ which relates them, and since we use standardized forms of Δ, the optima we obtain differ from Ross'.

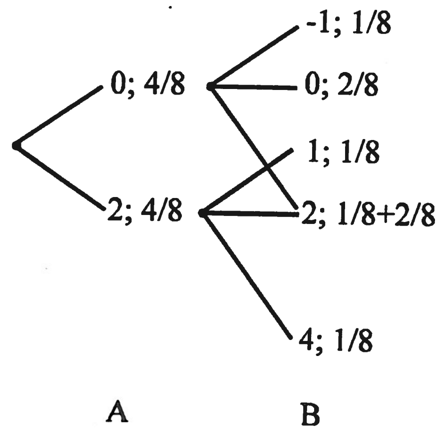
Table 4: Relations to Ross [1981]

TWO RISKY INVESTMENTS

Amended Ross Example

Outcomes[15]	-1	0	1	2	3	4
$8a$	0	4	0	4	0	0
$8\Delta^{*}\|A=0$	1	-2	0	1	0	0
$8\Delta^{*}\|A=1$	0	0	1	-2	0	1
$8b^{*}$	1	2	1	3	0	1
$8d^{*}$	1	-2	1	-1	0	1
$8S^2d$	1	0	0	-1	-2	-2

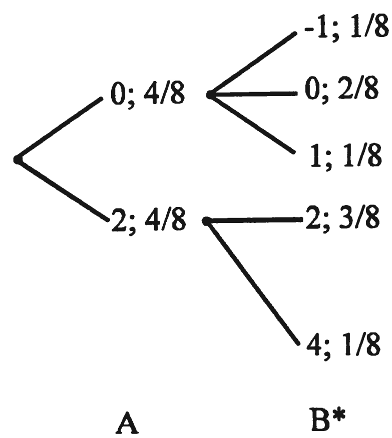
Associated Probability Tree



Variables for Present Approach

Outcomes	-1	0	1	2	3	4
$8a$	0	4	0	4	0	0
$8\Delta\|A=0$	1	-2	1	0	0	0
$8\Delta_m\|A=1$	0	0	0	-1	0	1
$8b$	1	2	1	3	0	1
$8d$	1	-2	1	-1	0	1
$8S^2d$	1	0	0	-1	-2	-2

Associated Probability Tree



$E(A) = 8/8,\ E(B) = E(B^{*}) = 10/8$

As shown in the next section, the relations between A and B^{*} do not satisfy the conditions of Theorem 4.1, but those between A and B satisfy a generalization of them.[16]

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[15]Since the outcomes -3 and -2 have zero probabilities, they are omitted for brevity.

[16]It is easier to establish Theorem 4.1 for $S^2d = (1, 1, \ldots, 1, -1, -1)'$ first, then show how these conditions can be generalized to take care of examples like the one just examined.

4.2 Optimal investment for a given class of risks

To proceed, consider variables A and $B \equiv A + \Delta$ related by the unit effects:[17]

$$\{ \Delta_j \mid A = j \} = \begin{cases} -1; & \textit{with prob } p \\ 0; & \textit{with prob } 1-2p \\ 1; & \textit{with prob } p \end{cases} \tag{4.2a}$$

for $j \in \{ -k+1, \dots, k-1 \}$ and

$$\{ \Delta_m \mid A = k - 1 \} = \begin{cases} 0; & \textit{with prob } 1-p \\ 1; & \textit{with prob } p\,. \end{cases} \tag{4.2b}$$

In (4.2a) the subscripts j are used to indicate risk terms Δ_j conditional on the outcomes $j \in \{ -k+1, \dots, k-1 \}$. In (4.2b) the subscript m is used to refer to the mean shift term Δ_m, which is also conditional on outcome $k-1$. Conditions (4.2a) and (4.2b) are sufficient but not necessary for:

$$nd = (1, -1, 0, \dots, 0, -2, 2)$$

and (4.2c)

$$nS^2d = (1, 1, \dots 1, -1, -1)'.$$

Given the foregoing, problem (4.1) can now be written:

$$\max_{\eta} \{ v(w-k)a_{-k} + \sum_{j=-k+1}^{k-1} [v(w+j-\eta)p + v(w+j)(1-2p) + v(w+j+\eta)p]a_j + [v(w + k)(1 - p) + v(w + k + \eta) p] a_k \}, \tag{4.3}$$

where

$$a_j \equiv Prob \{ A = j \}$$

[17] The results generalize in a straightforward way; the main cost of allowing the β_j to differ from unity is additional notation. We assume each of the unconditional probabilities a_j is large enough that the conditional changes in (4.2a) and (4.2b) do not violate the condition $b \geq 0$.

and p is a unit change in probability as defined in (4.2a) and (4.2b) and illustrated in the second example of Table 4. Assuming the appropriate derivatives exist, the optimal solution η_v is given by

$$\sum_{j=-k+1}^{k-1} [-v'(w+j-\eta_v) + v'(w+j+\eta_v)]p\, a_j + v'(w + k + \eta_v)\, p\, a_k = 0. \tag{4.4}$$

Note that without the last, positive term in (4.4) there would be no interior solution: the investor will not purchase both A and the riskier $A + \Delta$ unless the additional risk of the latter is offset by a higher mean.

Next, let

$$v(w) \equiv G((u(w)) \tag{4.5}$$

where $G(\cdot)$ is a strictly increasing strictly concave function. The function $v(w)$ is said to be strictly more risk averse (in the Arrow - Pratt sense) than the function $u(w)$; cf. Pratt [1964], Huang-Litzenberger [1988].

Theorem 4.1: Suppose (4.2) is satisfied and that Δ_j and Δ_m are defined as in (4.2a) and (4.2b). Let $\eta_u > 0$ and $\eta_v > 0$ be the optimal solutions to (4.4) for $u(w)$ and $v(w)$ respectively. Then if $v(w) = G(u(w))$ and G is a strictly increasing strictly concave function as in (4.5), $\eta_u > \eta_v$.

Proof: Rewrite (4.4), using $\eta_v \equiv \eta$ to minimize notation, as

$$\sum_{j=-k}^{k-1} [-G'(u(w+j-\eta))u'(w+j-\eta) + G'(u(w+j+\eta))u'(w+j+\eta)]p\, a_j + G'(u(w+k+\eta))u'(w+k+\eta)\, p\, a_k = 0. \tag{4.6}$$

Since the bracketed terms in the first line of (4.6) are negative by diminishing marginal utility,

$$\sum_{j=-k}^{k-1} G'(u(w + j)) [-u'(w + j - \eta) + u'(w + j + \eta)]p\, a_j + G'(u(w + k +\eta))u'(w + k + \eta)\, p\, a_k = H > 0. \tag{4.7}$$

But the last line of (4.7) is positive and $\eta > 0$ as well, so that

$$G'(u(w + k))\sum_{j=-k}^{k-1}[-u'(w + j - \eta) + u'(w + j + \eta)]p\, a_j$$
$$+ G'(u(w + k))u'(w + k + \eta)\, p\, a_k = I > H > 0, \tag{4.8}$$

from which it follows that

$$\sum_{j=-k}^{k-1}[-u'(w+j-\eta) + u'(w+j+\eta)]p\, a_j + u'(w+k+\eta)\, p\, a_k > 0. \tag{4.9}$$

Thus, finally, $\eta_u > \eta_v$. ■

Remark: The converse, that an increase in absolute risk aversion implies a decrease in optimal risky investment, follows by similar reasoning. ■

Remark: Conditions (4.2a), (4.2b) and (4.3) can be relaxed so long as any mean shift effect is conditional on an outcome at least as great as the largest outcome on which any risk increases are conditioned; cf. (4.6) through (4.8). Moreover there can be more than a single mean shift effect, neither mean shift terms nor risk terms need be of unit value, and the conditional probabilities p can also differ.

5. Conclusions

This paper used standardized variables to characterize risks and to establish new results for optimal risky investment decisions. The paper first extends RS, showing that a dominance related criterion is necessary and sufficient for increasing risk to imply decreasing risky investment for every risk averter. Second, the paper extends Ross, finding a class for which a riskier asset's demand decreases if and only if the Arrow-Pratt absolute risk aversion index increases (decreases).

By showing that restrictions on risks can have the same behavioral implications as the literature's preference restrictions, the paper raises a familiar question in descriptive economics. Whenever the behaviors discussed here are encountered in practical contexts, they may ultimately prove to be consequences of either preferences or probabilities.

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