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OPTIMUM FIRM LOCATION AND THE THEORY OF PRODUCTION

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DISCUSSION PAPER NO. 90

OPTIMUM FIRM LOCATION AND THE THEORY OF PRODUCTION[1]

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Introduction

Theories of firm location have, over the last century, fallen into two basic streams: location in homogeneous space and location in heterogeneous space. The former stream normally builds on the notion that consumers or demanders or a firm's product are evenly distributed over geographic space so that demand considerations become intrinsically enmeshed with market area considerations. Location in heterogeneous space incorporates the notion that c.i.f. prices faced by the locator vary continuously over space either because inputs and outputs must be transported at positive cost or because market determined values for the rights to alternative locations are not identical. The emphasis in this paper will be on locational problems of the latter sort.

The pioneering work in this context was carried out by Alfred Weber[2] whose analysis involved a firm with fixed coefficients technology attempting to determine the profit-maximizing location with respect to the sources of the inputs and the location of the market. With all prices parametric at source the solution to this problem was found to be equivalent to the solution to the minimization of the total transport bill associated with assembling the product and shipping it to

[1]This paper benefits greatly from discussions with John M. Hartwick of Queen's University and from comments from a number of participants in the Ph.D. thesis seminar -- Frank Flatters, R. G. Lipsey, Trent Bertrand, P. M. Meiszkowski, and D. Kam.

The research was done at Queen's University with financial support from the Transport Development Agency of Canada. The author is presently an economist with the Economic Council of Canada.

[2]Alfred Weber, *Alfred Weber's Theory of the Location of Industries*, (translated by C. J. Friedrich) Chicago: University of Chicago Press (1929).

market. In the case of two inputs located at points 1 and 2 and one market located at M the optimum location was determined to lie somewhere within the triangle formed by linking the market point with the two input points and the two input points with each other.

In the early 50's Isard established the compatability of much of spatial theory with the substitution principle of general economics. That is, rates of substitution among "spatial inputs" would in equilibrium be set equal to the ratio of the corresponding prices.[1]

For economists long trained in a theory of the firm dominated by the "law of variable proportions" the first substantial breakthrough came in 1958. Leon Moses[2] at that time published a geometric analysis elucidating the location principles of a Weber-type problem when production technology is described by a variable proportions relationship with variable returns to scale. Moses' main conclusions were that the determination of the profit maximizing location of the firm requires a proper adjustment of output, input combination, location and price and that the optimum location probably would not correspond to the point of minimum transport costs.

In 1967 Noboru Sakashita[3] attempted to approach the Moses problem analytically by restricting his analysis to include only linearly homogeneous production functions. He then derives a number of results

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[1]Walter Isard, *Location and the Space Economy*, Chap. 4, pp. 77-90. The MIT Press, Cambridge, Mass.(1956).

[2]Leon N. Moses, "Location and the Theory of Production", *The Quarterly Journal of Economics*, 72(1958) pp. 259-72.

[3]Noboru Sakashita, "Production Function, Demand Function and Location Theory of the Firm", Regional Science Association: Papers, XX; Hague Congress (1967).

the most startling being the exclusion of intermediate locations in all cases considered.

The first part of this paper will be to demonstrate that Sakashita erred in setting up a structural framework within which to study the Moses problem as such and it will be shown that a correct formulation of the model negates the strong no-intermediate-locations result. In addition, it will be shown that the Moses framework *can* be used to analyse Sakashita's *special case* and that use of this technique permits the derivation of a stronger result than Sakashita's. The Moses technique will also be employed to show an important class of problems in which homogeneity (to any degree) of the production function will not assure us of continuous site preference for all levels of output. Still within the world of geometry it will be demonstrated that there is another "locational iso-outlay curve" than the Moses envelope curve and that this one, while yielding identical solution results, loans itself to a more intuitive interpretation and permits ready comparison of production losses from being at suboptimal locations.

The second part of the paper is a revolt against the unnecessarily lengthy manipulations required to get any mileage out of the analytical technique of Sakashita. A much simpler method is developed and used to explore the location problem in general as well as a number of specific cases. Particular attention will be paid to a partial equilibrium model of firm location in an urban field from which a number of interesting propositions can be derived. Also included in the second part of the paper will be some formalization of the sense in which firms can be said to minimize costs as a consequence of seeking greatest profits. This will shed important light on the question of whether firms will, in any way, minimize transport costs as a natural consequence of their assumed inclination to maximize profits.

PART I

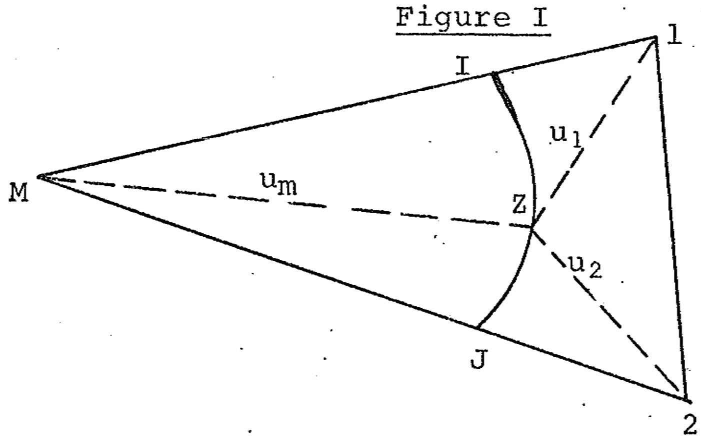
(i) *A Summary of the Moses Framework*

To gain a perspective of where the Sakashita paper fits into the development of a "neoclassical" theory of firm location one must outline the Moses technique in a cursory way. For the purposes of this paper there is no loss in using a linearly homogeneous production function even though Moses did not impose such a restriction. Efficient production possibilities will then be expressed by:

$$(1) \quad Q = F(V_1, V_2) \; ; \; \frac{\partial Q}{\partial V_i} > 0 \; ; \; \frac{\partial^2 Q}{\partial V_i^{\,2}} < 0 \quad (i = 1,2)$$

The space in which the firm seeks to maximize profit will be as depicted in Figure I where the points 1 and 2 represent the locations of inputs one and two respectively and the point M represents the location of the only demand point for the final product. The distances u_1, u_2, and u_M symbolize the respective distances from some arbitrary location Z to the input source or consumption point indicated by the subscript.

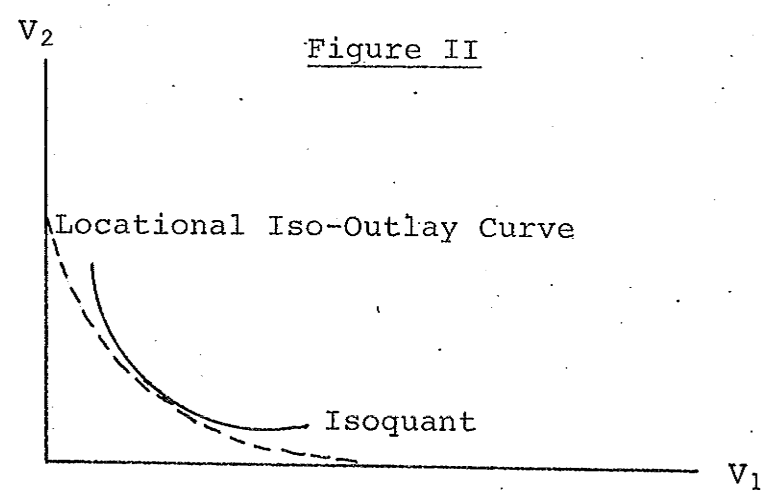
Figure I



The Weber-Moses Locational Environment

In the case where input and product prices are fixed at source and market the firm's objective is to select that constellation of average revenue and input prices at the plant such that maximum production is obtained for a given outlay. To clarify this process Moses breaks the problem up into two parts: finding the optimum location along *any given* arc IJ such that the distance to the consumption point is held constant and, finding the best arc IJ or best u_M. The relationship between input substitution and location is elaborated with the aid of a "locational iso-outlay curve". This is the locus of input combinations represented by the envelope of dominant points when the infinite number of c.i.f. isocost lines are considered for all locations along arc IJ. The best location along IJ would then correspond to the tangency of the locational iso-outlay curve to the highest attainable isoquant. There being a locational iso-outlay curve for any and all such arcs the firm must ultimately select that iso-outlay curve which permits maximum production for that level of outlay. Indeed

Figure II



(although this point was not made by Moses) there will exist a "locational iso-outlay frontier" consisting of all undominated (V_1, V_2) combinations from all possible locational iso-outlay curves and the full locational equilibrium will correspond to the highest isoquant tangency to this frontier.

(ii) *The Structure of the Sakashita Model*

Sakashita simplifies the first stage of the Moses problem by examining it in the context of the firm seeking a best location on the straight line connecting the two input sources. Rather than locate the firm along an arc IJ (so as to hold the distance to market constant) Sakashita dismisses the demand side entirely thereby redefining the problem to be one of merely selecting the cost minimizing location on the straight line. This, it will be shown, plays a critical role in the results obtained and is hence not particularly relevant to the two-space case which Moses was usually concerned with.

(iii) *Location along an Arc vs. Location on a Line*

The natural query at this juncture concerns the difference between locating on an arc or a straight line. Figure III(a) sets out the two confining spaces in a comparable way and III(b) brings

Figure III (a)

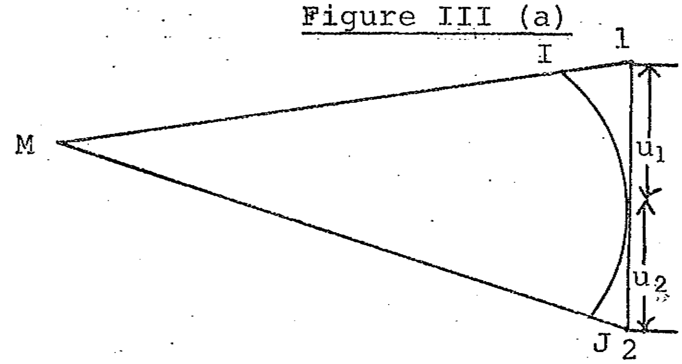
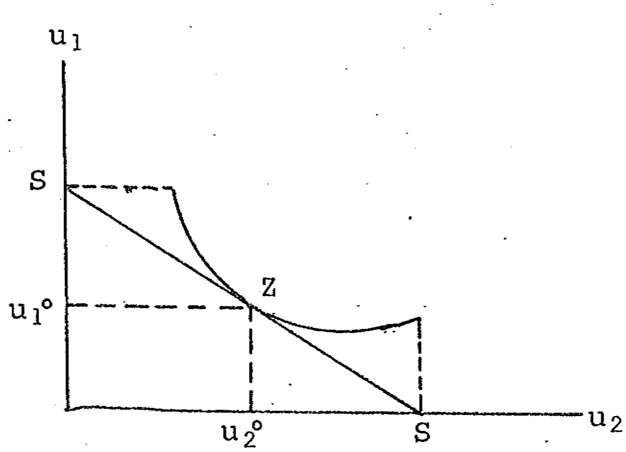


Figure III (b)



out the essential point -- in the Sakashita case the trade-off (s-s) between distance to input 1 and distance to input 2 is linear while in the Moses case (m-m) it is strictly convex from below. To see that the shape of the $u_1 - u_2$ trade-off is an important difference the model will be reworked incorporating a more general trade-off function with the Sakashita and Moses location problems emerging as special cases.

The basic technique is to explore the topological properties of the firm's total cost curve in the distance plane.[1] If it is convex from below then intermediate locations are a distinct possibility while concavity would restrict the set of possible optimal locations to end points.

Total costs are given by:

$$(2)\quad C = [\{r_1 + m_1 u_1 (u_2)\}v + \{r_2 + m_2 u_2\}]\ V_2$$

where:

r_i = parametric price of factor i at source.

u_i = distance from some arbitrary location to the source of input i.

m_i = transport rate per unit of i per unit of distance.

v = ratio of factor one to factor two in production.

$f(v)$ = productivity function with properties $f'>0$ and $f''<0$.

For the firm at any *given* u_1 costs will be minimized in the usual way so that the first order conditions are:

[1]This is so because the objective reduces to minimizing costs for any level of output. If the C curve is convex from below in the distance plane then an extremum, if it exists, will be a minimum. If C is concave from below the extremum would be a maximum and cost minimization would lead to a corner and a minimum minimoram.

(3) $$\frac{f(v)^* - f'(v^*)}{f'(v^*)} = \frac{r_2 + m_2u_2}{r_1 + m_1u_1(u_2)}$$

(4) $$\bar{Q} = f(v^*)V_2$$

These equations will determine the optimal v^* and V^*_2 at any location but now one must consider that the optimized total cost function C^* is related to u_2. Applying the Wong-Viner theorem:[1]

(5) $$\frac{dC^*}{du_2} = \frac{\partial C^*}{\partial u_2} = m_1 \left(\frac{\partial u_1}{\partial u_2} v^* + m_2 \right) V^*_2$$

Concavity or convexity of C^* in the distance plane will be determined by the sign of the second derivative with respect to u_2. Some algebraic manipulation shows:[2]

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[1]P. A. Samuelson, *Foundations of Economic Analysis*, Cambridge University Press (1963), pp. 36-38.

[2] $$\frac{d^2C^*}{du^2_2} = m_1 \left(\frac{\partial^2 u_1}{\partial u^2_2} v^* + m_1 \frac{\partial u_1}{\partial u_2} \frac{dv^*}{du_2} \right) V_2^* + \left(m_1 \frac{\partial u_1}{\partial u_2} v^* + m_2 \right) \frac{dV^*}{du_2}$$

where $$\frac{dV^*_2}{du_2} = \frac{dV^*_2}{dv^*} \frac{dv^*}{du_2} \quad \text{and} \quad \frac{dV^*}{dv^*} = \frac{-f'(v^*)}{f(v^*)} V^*_2$$

Substituting and collecting terms yields:

$$\frac{d^2C^*}{du^2_2} = \left[m_1 \frac{\partial^2 u_1}{\partial u^2_2} v^* + \left\{ m_1 \frac{\partial u_1}{\partial u_2} + \left(m_1 \frac{\partial u_1}{\partial u_2} v^* \; m_2 \right) \left(\frac{-f'(v^*)}{f(v^*)} \right) \right\} \frac{dv^*}{du_2} \right] V^*_2$$

$\frac{dv^*}{du_2}$ is found by taking the total differential of (3) and rearranging to get:

$$\frac{dv^*}{du_2} = \frac{\frac{\partial u_1}{\partial u_2} m_1 [f(v^*) - v^* f'(v^*)] - f'(v^*) m_2}{[r_2 + m_2u_2 + v^*\{r_1 + m_1u_1(u_2)\}] \; f''(v^*)}$$

so that substitution and minor manipulation yields (6).

$$(6) \quad \frac{d^2C^*}{du^2_2} = V^*_2 \left[m_1 \left(\frac{\partial^2 u_1}{\partial u^2_2} \right) v^* + \left\{ \frac{\left[m_1 \frac{\partial u_1}{\partial u_2} f(v^*) - m_1 \frac{\partial u_1}{\partial u_2} v^* f'(v^*) - m_2 f'(v^*) \right]^2}{f(v^*) f''(v^*) \ [(r_2 + m_2 u_2) + v^*(r_1 + m_1 u_1)]} \right\} \right]$$

Now when the Sakashita locational space prevails the term $m_1 \left(\frac{\partial^2 u_1}{\partial u^2_2} \right) v^*$ drops out and (6) is strictly negative so that the least-cost location will necessarily be at one end or the other of s-s. If, however, the term $m_1 \left(\frac{\partial^2 u_1}{\partial u_2} \right) v^*$ is positive -- as it must be in a properly formulated extension of Moses' -- then the result becomes entirely ambiguous.

Two further points should be made. First, had Sakashita used any other straight line approximation to the arc IJ than that joining the two input sources his result would not have obtained either. Second, the basic reason for Sakashita's result is that the locational space is such that both c.i.f. factor prices are linear in the distance index u_2. The Moses space destroys this linearity and hence destroys the intriguing result.

It can be similarly shown that the profit maximizing model with the downward-sloping demand curve at the market is complicated in exactly the same way when the more relevant case of location along an arc is treated. There is, however, nothing to be gained by developing this case here as it can be shown[1] that when the production function

[1] Sakashita (op. cit. pp. 119-120) made the point that demand conditions are irrelevant when the production function is linearly homogeneous. A very short and simple discussion of this point is also contained in a later section of this paper.

is linearly homogeneous the demand conditions for the final product will not influence the location decision.

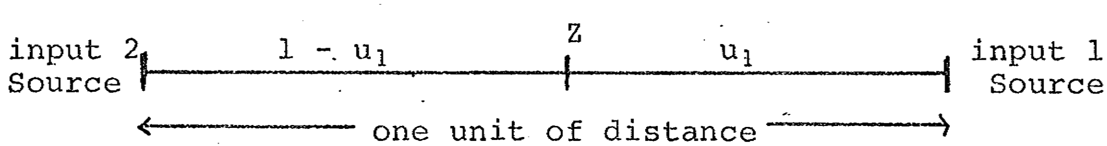
(iv) *On the Adequacy of the Moses Framework for Analyzing Location on a Line*

The raison d'être of the analytical excercise carried out by Sakashita was purportedly to show location theorists the ease with which they could renounce their familiar geometric paraphernalia and replace it with the 'tighter' paradigm which he offered. The strategy, of course, was to select a special case (location on a line) and to generate some results which Moses' model did not appear to handle well. Here it will be shown that use of a little ingenuity and the Moses technique can actually *strengthen* the no-intermediate-locations result for location on a line.

PROPOSITION 1 A sufficient condition for the exclusion of cost minimizing intermediate locations on the straight line joining the two input sources at which prices are taken as given is that there be nonzero variability in factor proportions in production.

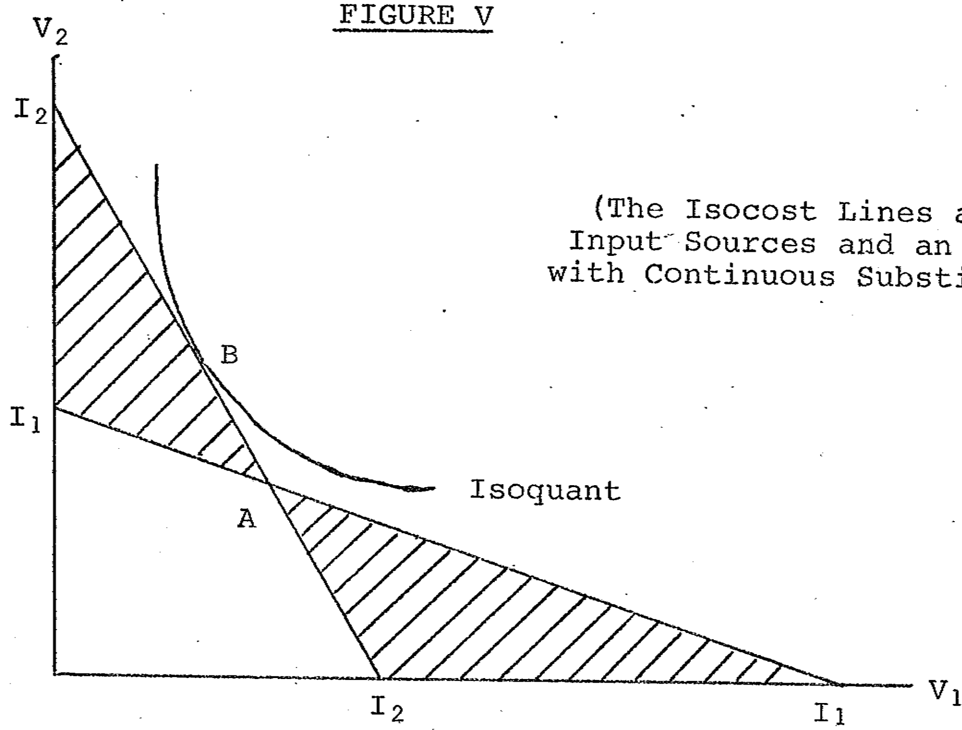
Consider the location space in Figure IV. At each input source one obtains an isocost line which together appear in Figure V.

FIGURE IV



A Simplified Locational Space

FIGURE V



(The Isocost Lines at the Input Sources and an Isoquant with Continuous Substitutability)

$I_2 - I_2$ represents the isocost line at the source of input 2 and $I_1 - I_1$ the isocost line at the source of input 1. (A) is the simultaneous solution to the equations defining the isocost lines. The proof of proposition 1 rests on the truth of the following lemma:

LEMMA #1: The solution to any pair of c.i.f. isocost lines corresponding to any pair of locations along the line connecting the source of input 1 with the source of input 2 is independent of the pair of locations selected.

Consider two arbitrary locations u'_1 and u_1 units of distance from input 1. We thus obtain two isocost lines defined by:

(7) $V_1(r_1 + m_1u_1) + V_2(r_2 + m_2(1 - u_1)) = k$

(8) $V_1(r_1 + m_1u'_1) + V_2(r_2 + m_2(1 - u'_1)) = k$

Using Cramer's Rule one can solve for V_1 and V_2. The solution for V_1 is:

$$V_1 = \frac{km_2}{r_1m_2 + r_2m_1 + m_1m_2}$$

In exactly the same way one can solve for V_2 and in both cases it is found that the solution is independent of the distance indices. It thus follows that all points which are candidates for the locational iso-outlay curve lie in the cross-hatched area in Figure V. The stated lemma therefore assures us that the locational iso-outlay curve will be the kinked line I_2AI_2 so that only a tangency at *A* could possibly correspond to an optimum intermediate location. Clearly nonzero substitutability in production is sufficient to exclude A as a possible tangency. Indeed even if by some remarkable coincidence we had a *fixed coefficient* production function with a tangency at A the end points would be *just as preferred* from a cost minimization standpoint and for this reason nonzero substitutability is not a necessary condition. Thus there is no technology capable of generating a *strong* preference for an intermediate site when location is on a line.

It is interesting to note that a locational proposition demonstrated by Bradfield[1] can be shown to hold for the case of location on a line but that it will be negated by a fairly relevant form of freight rate structure. The point made by Bradfield was that a firm would have continuous preference for one site over another for all levels of output as long as the production function were homogeneous to any degree. Figure VI reproduces the location result of Figure V (optimum location at the source of input two represented by the tangency at B) except now it is explicitly assumed that the production function

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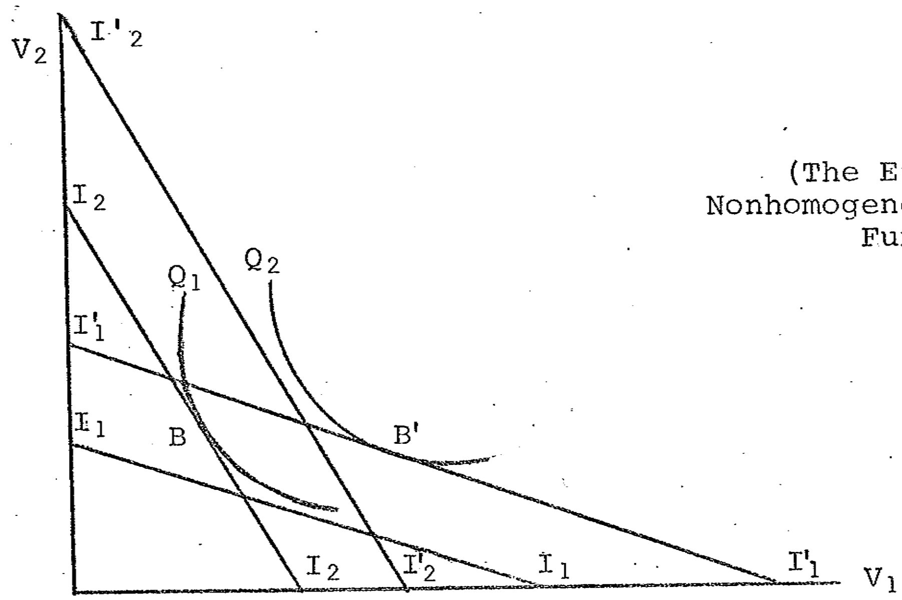
[1]Bradfield, M. "A Note on Location and the Theory of Production", *Journal of Regional Science*, Vol. II, No. 2 (1971).

is not homogeneous. The effects of varying the level of output can be analyzed by considering parallel shifts in the locational iso-outlay

FIGURE VI

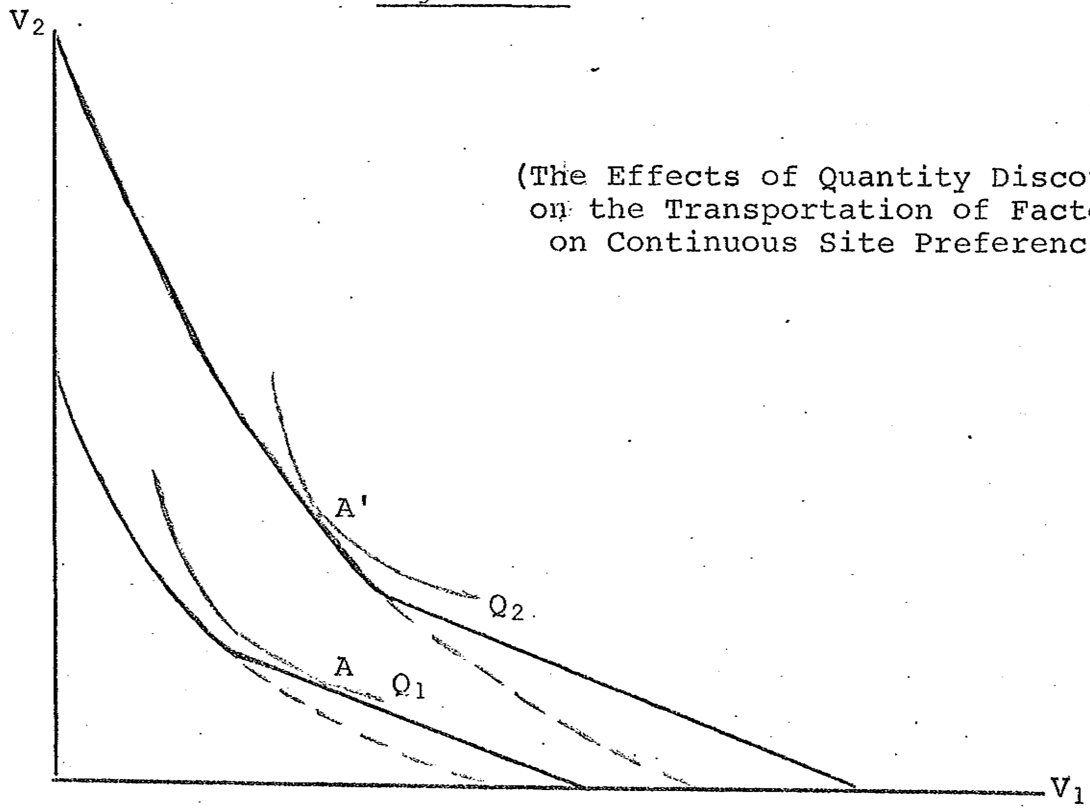


(The Effect of a Nonhomogeneous Production Function)

curve. If isoquant slopes were constant along a ray then tangency's would always be on the same segment of the locational iso-outlay curve and continuous preference for the sight determined to be optimal for the original level of output (Q_1 in the diagram) would hold for all levels of output. If expansion paths were nonlinear then it is possible to have 'location switching' from the source of input two, in this case, to the source of input one. In Figure VI, for example, we have the optimum location switching from the source of input two when the output level is Q_1 to the source of input one when the output level is Q_2.

PROPOSITION 2 Homogeneity to any degree (including the first) will not assure continuous site preference when there are quantity discounts on the transportation of factors.

Figure VII



In Figure VII it is assumed that there is a quantity discount only on the transportation of factor 1. As a matter of clarification this will be taken to mean that the transport rate m_1 is a decreasing function of the quantity of input 1 (V_1) transported per time period. From the standard theory-of-the-firm textbooks we know that quantity discounts on factor purchases have the effect of generating isocost lines which are strictly convex to the origin. Considering the isocost lines corresponding to the input sources one would now find the isocost line at the source of input 2 to be strictly convex to the origin so that the locational iso-outlay curve now has a smooth convex section, a kink and a linear section. The truth of Proposition 2 then hinges on the fact that the new locational iso-outlay curve need not, indeed it probably will not, have a constant slope along a ray. This slope is given by:

$$\frac{dV_2}{dV_1} = \frac{r_1 + u_1 m_1 + V_1 \frac{dm_1}{dV_1}}{r_2 + u_2 m_2} \quad (-1)$$

Movement outward along a ray necessarily increases V_1 and this affects the slope defined above by its effect on m_1, $\frac{dm_1}{dV_1}$ and the direct effect on the slope of the change in V_1 itself. It therefore follows that the slope of the locational iso-outlay curve need not be constant along a ray and this means that we could have a tangency with the linear segment of the locational iso-outlay (at output level Q_1 in Figure VII for example) and, due to the changing slope of the curve, have a tangency on the curved segment at Q_2. This of course implies a switch from a preference for the site at input one source to a preference for the site at input two source.

(v) *An Alternative Locational Iso-Outlay Curve*

In analyzing the location problem, Moses makes use of the envelope of dominant points from all isocost lines pertinent to an arc of constant distance to market. Of course this procedure is legitimate and does yield the appropriate solution to the location problem. There is, however, another spatial outlay curve which has two useful characteristics: it can be interpreted more readily in the context of the theory of the firm and, it explicitly shows the production losses resulting from the firm selecting an economically suboptimal location. The latter could become relevant in the context of the influence of locational factors which are "external" to the production relationships as normally conceived of by economists. Thus, it may be that production and spatial factors combine to generate situations where some firms suffer very minimal "production losses" when locating at any of a large number of suboptimal locations. This being the case, certain types of amenities may attract firms to locations which are, strictly speaking, suboptimal.

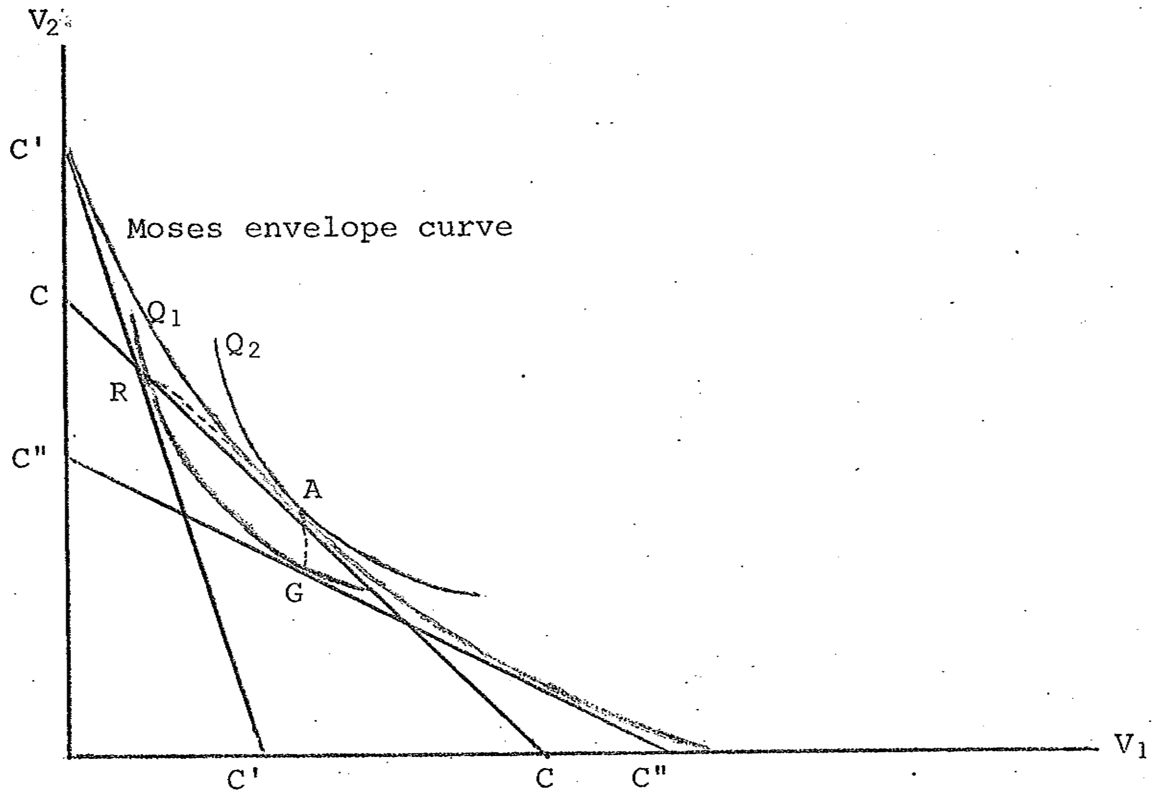
The location problem facing the firm might be construed as follows. For a given level of outlay on productive factors and transportation the firm will wish to maximize its output. At any particular location this would be done in the normal spaceless way since factor prices at a given location would be parametric. Every conceivable location corresponds to a solution to this problem with the price parameters being the relevant c.i.f. prices at that point in space. Hence, there will be a set of attainable output levels for a given outlay which can be subscripted by location indices. The best location(s) is then the location(s) corresponding to the maximum element(s) of this set.

The alternative outlay curve -- term it the 'optimized outlay curve' -- will be the locus of solutions to the infinite number of location-given-optimization for a given level of outlay. Whereas the Moses envelope curve will necessarily be convex from below, the optimized curve need not be.

In Figure VIII we have depicted 3 isocost lines c-c, c'-c' and c"-c" corresponding to three possible locations on an arc IJ. The points A, R and G would be the corresponding points on the optimized outlay curve and it is obvious that, this being the case, the curve cannot be convex from below. The production loss from being at either of the more extreme locations would be I-II.[1]

[1]It is worth noting that numerical simulations suggest that when the elasticity of substitution is relatively high (say greater than unity) in a CES production function that the production loss within a relatively large area of the optimum is extremely small.

FIGURE VIII



Points on the 'Optimized' Outlay Curve

While the Moses outlay curve depends only on the relationship between the two factor prices as we move along an arc the optimized outlay curve synthesizes the relevant space-technology information into the same curve.

PART II AN ALTERNATIVE ANALYTICAL FORMULATION

(i) *Some Simple Models*

The previous sections have been geared to showing that traditional Moses-type geometry can still be very useful, particularly when compared to the algebraic gymnastics involved in using the Sakashita analytical model. There are, however, limitations to the use of geometry that have probably become painfully obvious to anyone who has attempted to use the geometric approach to derive comparative static results. It is simply too messy to clearly assess the geometric changes emanating from certain parametric perturbations. This means there is a need for a simple and efficient analytical paradigm with which to attack certain locational problems. Part II will present an apparatus that appears to be an effective combination of simplicity and efficiency.

When dealing with linearly homogeneous production functions the level of output at a given location is irrelevant and one can select the location at which *unit costs* are minimized. Since unit costs are invariant with respect to the level of output but may vary over space we can write:

$$(9) \quad AC(u_1) = a_{1j}r_1(u_1) + a_{2j}r_2(u_2)$$

where:

u_i = a distance or spatial index

a_{ij} = quantity of input i required to produce a unit of j

r_i = c.i.f. price of factor i

Using this simple relationship a number of locational models can be analyzed.

MODEL #1 Sakashita Linear Space -- Fixed Coefficients in Production

In this case unit costs become:

$$(10) \quad AC(u_1) = a_{1j}(\bar{r}_1 + u_1m_1) + a_{2j}(\bar{r}_2 + u_2(u_1)m_2)$$

where:

$$\frac{\partial u_2}{\partial u_1} = -1 \;;\; \frac{\partial^2 u_2}{\partial u^2{}_1} = 0.$$

m_i = transport rate per unit i per unit distance

$\bar{r}_i$ = parametric price of factor i at source

The slope of $AC(u_1)$ in the distance plane is then:

$$(11) \quad \frac{dAC(u_1)}{du_1} = a_{1_j}m_1 + a_{2_j}\frac{\partial u_2}{\partial u_1}m_2 = a_{1_j}m_1 - a_{2_j}m_2.$$

Observe that the slope of $AC(u_1)$ in the distance plane is independent of the distance variable and will be positive, negative or zero as:

$$a_{1_j}m_1 - a_{2_j}m_2 \gtreqless 0.$$

Thus the unit cost curve is linear in the distance plane and the least-cost location will be at either end of the location line. In Figure IX AC^1 is an illustrative case.

MODEL #2 Moses Space -- Fixed Coefficients in Production

Here the $AC(u_1)$ specification would be:

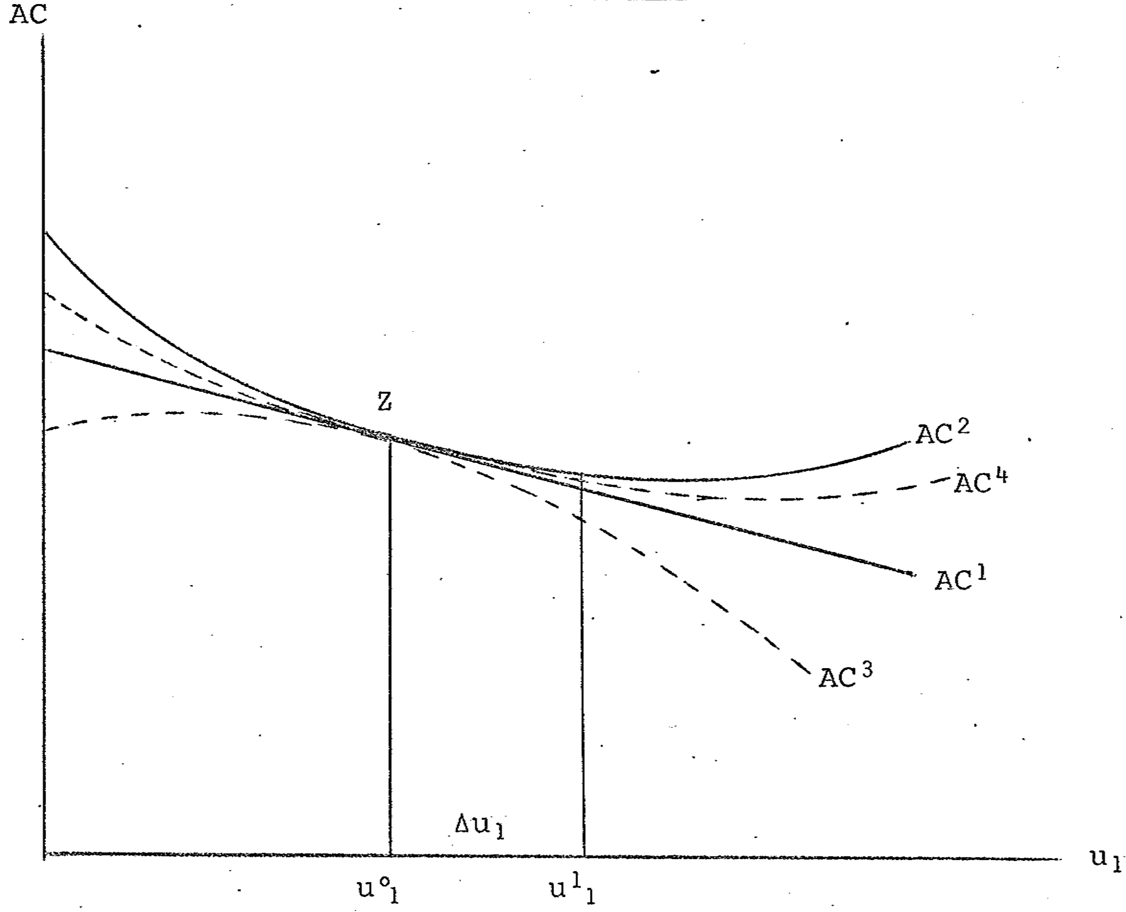
$$(12) \quad AC(u_1) = a_{1_j}(\bar{r}_1 + u_1m_1) + a_{2_j}(\bar{r}_2 + u_2(u_1)m_2)$$

$$\frac{\partial u_2}{\partial u_1} < 0 \;;\; \frac{\partial^2 u_2}{\partial u^2{}_1} > 0.$$

So that the slope is given by:

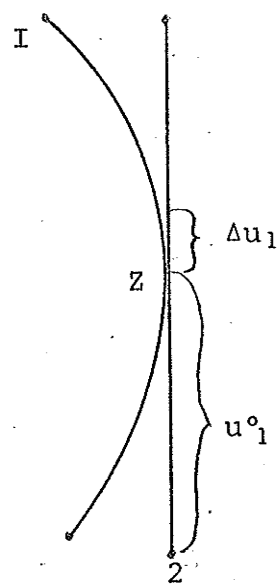
$$(13) \quad \frac{\partial AC(u_1)}{\partial u_1} = a_{1_j}m_1 + a_{2_j}\left(\frac{\partial u_2}{\partial u_1}\right)m_2$$

Figure IX (a)



Unit Cost Curves in the Distance Plane
for Alternative Specifications

Figure IX (b)



The Moses-Weber and Sakashita Spatial Environments and the second derivative is:

$$(14) \quad \frac{d^2AC(u_1)}{d\,u^2{}_1} = a_{2_j} m_2 \left(\frac{\partial^2 u_2}{\partial u^2{}_1}\right) > 0.$$

Thus, the $AC(u_1)$ curve is strictly convex from below in the distance plane. This is easily explained by the fact that $\frac{\partial u_2}{\partial u_1}$ is more strongly negative for small values of u_1 and larger for large values so that the algebraic magnitude of (11) is smaller for lower values of u_1. AC^2 illustrates this in Figure IX. Intermediate locations are observed to be a legitimate result.

MODEL #3 Sakashita Linear Space -- Variable Coefficients in Production

In this case the a_{ij}'s become monotonic functions of the factor price ratio which is in turn influenced by the locational index:

$$(15) \quad AC(u_1) = a_{1_j}(u_1)(\bar{r}_1 + u_1 m_1) + a_{2_j}(u_1)(\bar{r}_2 + u_2(u_2)m_2).$$

At any given u_1 the a_{ij}'s will be optimally adjusted so that the slope of $AC^*(u_1)$ will be:[1]

$$(16) \quad \frac{dAC^*(u_1)}{du_1} = \frac{\partial AC^*}{\partial u_1} = a^*{}_{1_j}\, m_1 + a^*{}_{2_j} m_2 \left(\frac{\partial u_2}{\partial u_1}\right)$$

where: $\frac{\partial u_2}{\partial u_1} = -1$

therefore: $\frac{\partial AC^*(u_1)}{\partial u_1} = a^*{}_{1_j}(u_1) m_1 - a^*{}_{2_j}(u_1) m_2$

---

[1]This is again due to the Wong-Viner theorem since at any u_1:

$$AC(a_{ij};r_i) \text{ and } \frac{\partial AC(a_{ij};r_i)}{\partial a_{ij}} = 0 \quad (i = 1,2)$$

$$\frac{dAC^*(u_1)}{du_1} = \sum_{i=1}^{2} \left(\frac{\partial AC^*}{\partial r_i} + \frac{\partial AC^*}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial r_i}\right) \frac{dr_i}{du_1} = \frac{\partial AC^*}{\partial u_1} + 0$$

and it can be seen that the slope is not independent of u_1 because the a^*_{ij}'s are not.

The second derivative is:

$$(17) \quad \frac{d^2AC^*(u_1)}{du^2_1} = m_1\frac{\partial a^*_{1j}}{\partial u_1} - m_2\frac{\partial a^*_{2j}}{\partial u_1} < 0$$

and the Sakashita result easily obtains.[1] AC^3 in Figure IX depicts this result.

MODEL #4 Moses Location Space -- Variable Coefficients

Here $AC(u_1)$ will again be given by (15) except now the restrictions:

$$\frac{\partial u_2}{\partial u_1} < 0; \quad \frac{\partial^2 u_2}{\partial u^2_1} > 0 \quad \text{apply.}$$

As a change of pace this case will be demonstrated with logic and geometry as the results are ambiguous and have already been shown to be so.

At Z in Figure IX it is clear that (if the optimized factor proportions are equal to the fixed factor proportions at this point) AC^3, AC^2 and AC^1 will be coincident at this point. The effect on AC in Model 4 due to a movement from Z can then be broken into three parts:

---

[1]This is also an illustration of what Samuelson calls the 'generalized le Chatelier principle' since holding factor proportions constant is to introduce a constraint into the system. The le Chatelier principle tells us that if our objective is the minimization of costs that a parametric change once the system is in equilibrium (i.e., once factor proportions are optimally adjusted at a given location) will result in a more desirable level of costs if factor proportions are permitted to adjust than if they are not. See *Foundations*, pp. 36-39.

(i) the change in unit costs as one moves from u°_1 to u'_1 along the Sakashita line while holding production coefficients fixed. AC^1 would give us this change.

(ii) the additional change in costs below ΔAC^1 resulting from permitting factor coefficients to vary as we move a distance Δu_1 along 1-2. AC^3 gives this change.

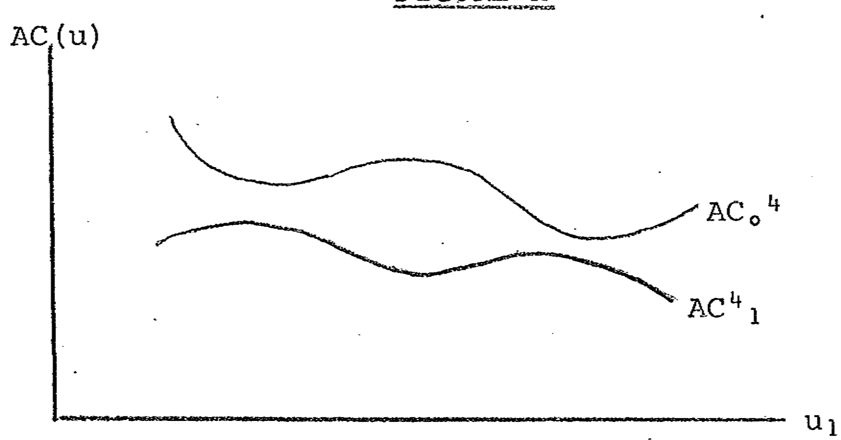
(iii) the change in costs resulting from the fact that for any (u_1, u_2) vector in the Moses space $u^s_1 = u^m_2$ implies $u^m_2 > u^s_2$ and this, in turn, implies $r^m_2 > r^s_2$ (c.i.f.). Thus, for any $u_1 \neq u^\circ_1$

$$AC^m_{a_{ij}\text{ fixed}} > AC^s_{a_{ij}\text{ fixed}}$$

AC^2 gives this change.[1]

Whether or not AC^4 lies above AC^1 at u'_1 will depend on the relative strengths of (i) and (iii). If (i) and (iii) were exactly offsetting then $AC^1 = AC^4$ at the new point. If (iii) always outweighs (i) then AC^4 always lies above AC^1. It may even be true that AC^4 takes on shapes as in Figure X.

FIGURE X



[1]There is actually another small adjustment of factor proportions in (iii) but it would affect cost very little and for expositional purposes is treated as zero.

(ii) *The Underlying Principle*

As one might intuitively surmise from the foregoing there is an underlying principle by which one can exclude intermediate locations or accept them as a possibility. In general unit costs at a given location would be given by:

$$(18) \quad AC(\bar{u}) = \sum_{i=1}^{2} a_{ij}(u) r_i(u) .$$

At any location the optimized a_{ij}'s would be determined by:

$$(19) \quad \frac{\partial AC(\bar{u})}{\partial a^*_{ij}} = 0 \quad i = 1, 2.$$

The effect of a perturbation in location is, by the Wong-Viner theorem:

$$(20) \quad \frac{dAC^*(u)}{du} = \sum_{i=1}^{2} \frac{\partial AC}{\partial a^*_{ij}} \frac{da_{ij}}{du} + \sum_{i=1}^{2} a^*_{ij} \frac{dr_i}{du}$$

$$= \sum_{i=1}^{2} a^*_{ij}(u) \frac{dr_i}{du} = \frac{\partial AC^*(u)}{\partial u}$$

The concavity, convexity or ambiguity of the unit cost function in the distance plane will be determined by the sign of:[1]

$$(21) \quad \frac{d^2AC^*(u)}{du^2} = \underbrace{\sum_{i=1}^{2} \frac{\partial a^*_{ij}}{\partial u} \frac{dr_i}{du}}_{A} + \underbrace{\sum_{i=1}^{2} a^*_{ij}(u) \frac{d^2r_i}{du^2}}_{B}$$

(A) will always be less than or equal to zero (no input substitution) and its magnitude will depend on the responsiveness of factor prices to spatial movement and the responsiveness of the a_{ij}'s to price changes. (B) will reflect the curvature of c.i.f. factor prices in the distance plane. The Sakashita result is merely a case

[1] It should be noted that (21) is also relevant to the case where j must be transported to a market. In this case one would have a_{ij}=1 and the r_j would merely be average revenue received at the plant.

of the first term in (21) being strictly negative (nonzero input substitution) and the second term being zero. The Moses-type space with variable coefficients means that (A) is negative but (B) is positive, ergo an ambiguous result but intermediate locations cannot be excluded.

(iii) *Transport Rates that Decline with Distance*

It has often been heuristically demonstrated that transport rates that decline with distance will enhance the probability of end point solutions. This can be confirmed by using the very simple model #1. For simplicity it will be assumed that transport rates decline linearly with distance over the relevant range. In place of (10) we can write:

$$(22) \quad AC(u_1) = a_{1j}[r_1 + u_1 m_1(u_1)] + a_{2j}[r_2 + u_2(u_1)m_2(u_1)]$$

(11) becomes (23):

$$(23) \quad \frac{dAC(u_1)}{du_1} = a_{1j}m_1(u_1) - a_{2j}m_2 + a_{1j}u_1\frac{dm_1}{du_1} + a_{2j}u_2(u_1)\frac{dm_2}{du_1}$$

Taking a special case of this we know that when (11) equals zero the firm is indifferent between all locations on the line. The same does not apply to (22) because it is not linear but is convex from below:

$$(24) \quad \frac{d^2AC(u_1)}{du^2_1} = a_{1j}\frac{dm_1}{du_1} - a_{2j}\frac{dm_2}{du_1} + a_{1j}\frac{dm_1}{du_1} + a_{2j}\frac{dm_2}{du_1}(-1)$$

$$= 2\left[a_{1j}\underset{-}{\frac{dm_1}{du_1}} - a_{2j}\underset{+}{\frac{dm_2}{du_1}}\right] < 0.$$

Hence end points will now be preferred at all times.

(iv) *Fixed vs. Variable Coefficients Comparisons in Location Theory*

The literature on the theory of firm location contains numerous references to the differences between location optimization when input-coefficients are fixed as opposed to when they are permitted to vary. This section puts forth the view that there is only one legitimate way to compare the two cases and when this approach is used the distinction drawn evaporates.

It is generally said (Moses 1958, Sakashita 1967) that the location problem when production coefficients are given reduces to the problem of minimizing transport costs and that variable coefficients erases this identity. The problem, however, is what constitutes a legitimate comparison between the two cases?

In Figure XI, for example, we may wish to compare fixed coefficient production function I_o with a variable proportions counterpart. Obviously there is no unique variable proportions counterpart to I_o but an infinite number some of which might lead to the same optimum location and some which would not. Similarly there is no unique fixed coefficient production function to correspond to a given variable coefficients function.

The only legitimate comparison that one can make is to select the variable proportions production function that is of interest, and to select the optimized factor coefficients to use in a comparison with the fixed coefficients case. When this is done there is no difference whatever between the fixed and variable coefficients location problem since they both yield the same optimum location.

FIGURE XI

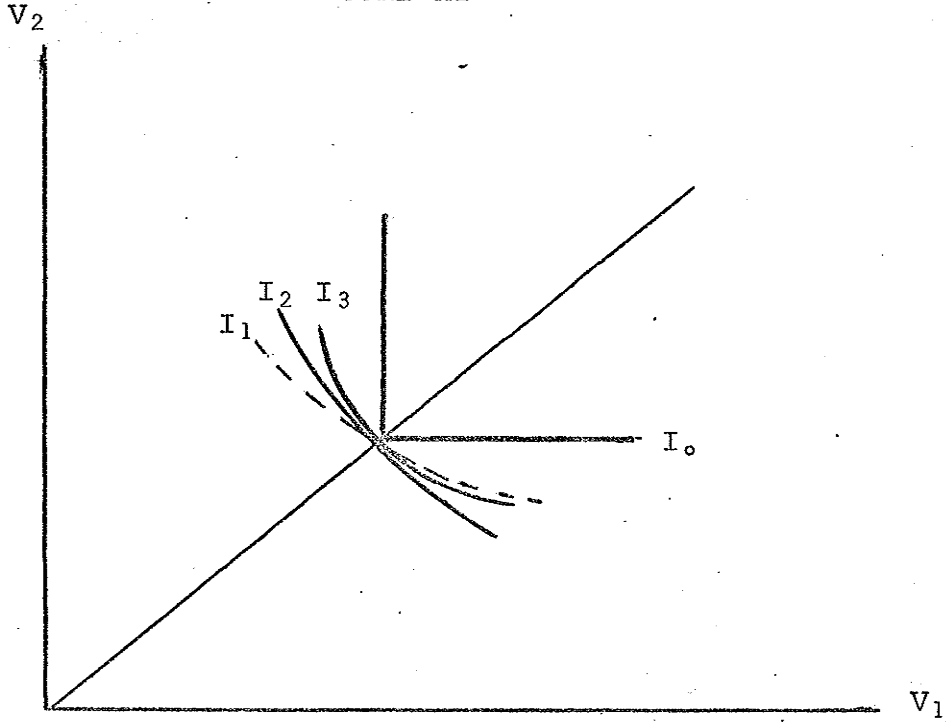
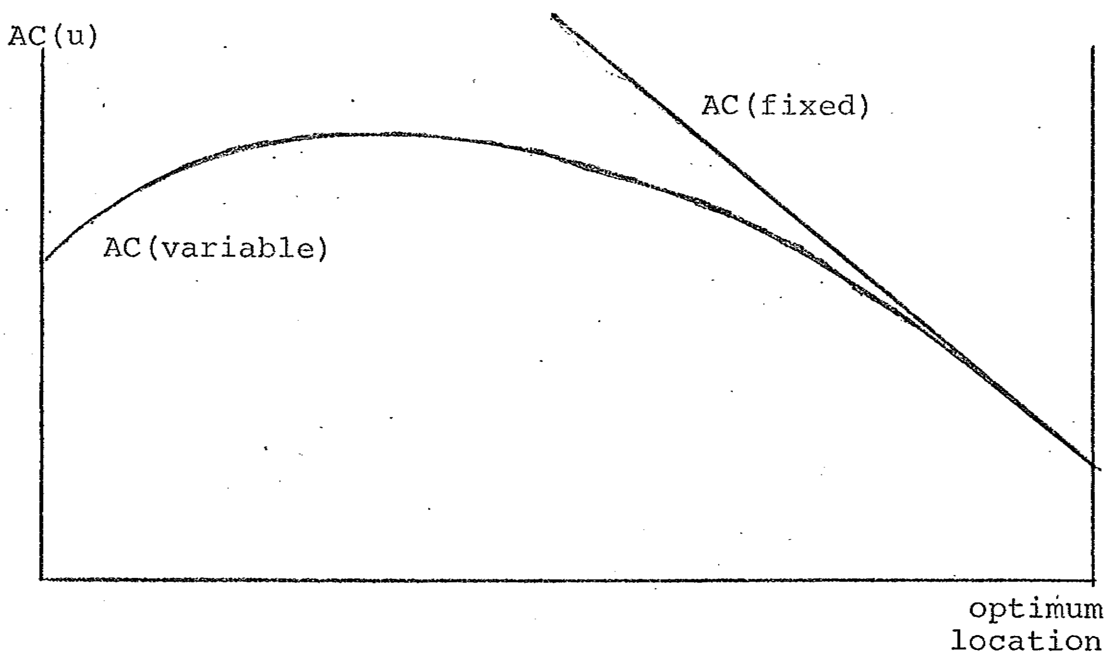


FIGURE XII



In figure XII, the optimum location for AC (variable coefficient) is as indicated and using the coefficients determined at that location we get the corresponding AC (fixed coefficient) and observe the optimum location (the transport cost minimizing location) to be the same. The conclusion that one reaches is that, although there is no global search for the transport cost minimizing location in a variable coefficients model (and the transport cost minimizing coefficients), the optimum location nevertheless corresponds to the transport cost minimizing location *at the optimized production coefficients*.

In the context of location theory literature this implies that the Moses and Weber problems converge when the proper comparison is made. This is particularly important as an empirical matter since it should always be observed that firms locate at the transport cost minimizing location for prevailing production coefficients.

(v) *In What Sense Least-Cost Location?*

In the development of the unit cost model the firm's optimization problem has always been construed as a minimization of costs rather than a maximization of profit. In this section a proposition is developed which justifies this methodology.

PROPOSITION 1 Any spatial movements which reduces costs at a given factor mix and level of output will also increase profits of the firm. The location at which profits are globally maximized, therefore, will also be the location at which a given optimum level of output is produced with the given optimal input mix at least cost.

In general the firm's profit function can be written:

$$(25) \quad \Pi = \Pi[P_j(Q), Q, a_{ij}, u] = P(Q)Q - AC(Q, u)Q$$

where $P'_j(Q) < 0$

and the a_{ij} is defined as the reciprocal of the average physical product of factor i in the production of some given level of output of good j.[1]

At a given location the first-order conditions yield:

$$(26) \quad \frac{\partial \Pi}{\partial Q} = 0$$

$$\frac{\partial \Pi}{\partial a_{ij}} = 0 \quad i = 1, 2.$$

Now the effect on the optimized profit function of a small perturbation in the locational index is, by the Wong-Viner theorem:

$$(27) \quad \frac{d\Pi^*}{du} = \frac{\partial \Pi^*}{\partial Q^*} \cdot \frac{dQ^*}{du} + \frac{\partial \Pi^*}{\partial a^*_{ij}} \frac{da^*_{ij}}{du} + \frac{\partial \Pi}{\partial u} = - \left[\frac{\partial AC(Q^*,u)}{\partial u}\right] Q^*.$$

Which simply says that gross profits will always move in the opposite direction to average costs so that any spatial movement which reduces costs will increase profits and we can conclude that the profit-maximizing location can also be associated with the location at which the profit-maximizing output is produced with the optimized input mix at least-cost.

In intuitive terms what is being said? Take any firm, arbitrarily locate it in space and let it determine the profit maximizing output and factor mix for that location. Because of the first order conditions we know that adjustments in Q^* and a^*_{ij} resulting from a small spatial perturbation will have an effect on profits which are a second-order small hence the direction of change in profits can be predicted by the ceterus paribus effects of the change in costs.

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[1]Output is no longer restricted to the unit level and we do not wish to impose first degree homogeneity on the production function so that $AC(Q, u) = \frac{V_1}{Q_j} [r_1(u)] + \frac{V_2}{Q_j} [r_2(u)]$.

While it might at first appear that the above results are in some sense 'forced' due to the ceterus paribus nature of the formulation such fears have no basis. In fact the Le Chatelier principle assures us that if costs are reduced by a spatial movement in which some variables are constrained to constancy they will be reduced even more if the constraints are removed.

Proposition 1 should not in any way be taken to infer that costs are being globally minimized by the firm. There is nothing to suggest that a firm at a *given location* will select the *level of output* which minimizes costs rather than profits.

Proposition 1 leads quite naturally to the following corollary:

Corollary 1: For a selected factor mix and level of output the firm will minimize costs by minimizing the transport bill associated with bringing inputs to the plant and taking outputs to the market.

From proposition 1 it follows that a complete spatial equilibrium will be attained only if:

$$(28) \quad \frac{\partial \Pi^*}{\partial u} = - \left[\frac{\partial AC(Q^*_j, u)}{\partial u} \right] Q^*_j = 0.$$

This means that the following condition would hold:

$$(29) \quad \frac{\partial AC}{\partial u} (Q^*_j, u) = \sum_{i=1}^{2} a_{ij} m_i - m_j = 0.$$

This, however, is none other than the condition for total transport costs divided by total output to be minimized given the a_{ij}'s and

Q^*_j:

(30) $$T = \sum_{i=1}^{2} a_{ij} m_i u_i + u_j m_j$$

(31) $$\frac{\partial T}{\partial u} = \sum_{i=1}^{2} a_{ij} m_i - m_j = 0.$$

The conclusion one reaches is directly analogous to the earlier notion to the effect that for the solution to every Moses-type problem there is a corresponding Weber problem. Thus, if firms are observed to be in equilibrium with respect to input mix and output level they will also be observed to be ceterus paribus transport-cost-minimizers.

Sakashita made the point that first degree homogeneity of the production function would negate the influence of demand conditions on the choice of plant location. It follows from the above proposition that the only effect demand conditions can have is through Q^*_j which is demand-cost determined. Sakashita was in effect saying that since the level of output does not affect the a_{ij}'s when the production function is linearly homogeneous that the best location for one level of output would be the best for all levels of output. That is, the equation determining the optimum output level for a given location does not feed into the equation for factor mix which in turn feeds into the locational equilibrium condition.

The first remark that should be made is that even first degree homogeneity of the production function combined with transport costs that are linear in the distance plane is not sufficient to negate the role of demand conditons. The earlier point made with regard to

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[1]Sakashita, op. cit., pp. 118-120.

the Bradfield result is equally relevant here since quantity discounts on the transportation of goods and/or factors will lead to the level of output feeding into the locational equilibrium conditon. For demand conditons to be irrelevant it is sufficient that a_{ij}'s *and* the transport rates be independent of the level of output. It is not even necessary that transport rates be constant with respect to the distance variable.

The second remark is that the Bradfield result per se does not hold when the final product must be transported to market. This is so because nonlinear homogeneity will lead to alterations in the a_{ij}'s when the level of output changes and this will alter the relative weighting that the firm places on proximity to product as opposed to factor markets.[1]

PART III AN APPLICATION TO LOCATION IN AN URBAN FIELD

(i) *Setting up the Model and Derivation of Some Propositions*

This section will employ the general method developed in Part II to the problem of location in an 'urban field'. The model will be cast in the context of a fairly large geographic area bounded by a circle of radius R centred on a large urban centre. Initially it will be assumed that there are two factors of production 1 and 2 (land and some arbitrary factor located at the urban centre). The price of land will be given by the inverse exponential form:[2]

$$(32) \quad r_1 = Au^{-\rho}$$

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[1]This point is treated more formally in a specific model in Part III. Section (v).

[2]There is no overriding need for this particular functional form since a more general convex from below specification would generate the same results.

where:

r_1 = the price of land at distance u from the core of the urban centre

A, ρ are parameters; A, $\rho > 0$.

Since input 2 is located at the core it will have a c.i.f. price that is linear in the distance plane.

Production is initially assumed to be described by a linearly homogeneous production function with no input substitutability. The spatial unit cost curve will then be:

$$(33) \quad AC_j(u) = a_{1_j}Au^{-\rho} + a_{2_j}(r_1 + m_2u).$$

The unit cost gradient in the distance plane is then:

$$(34) \quad \frac{dAC_j(u)}{du} = -\rho a_{1_j}Au^{-(\rho + 1)} + a_{2_j}m_2.$$

Obviously it is not independent of the distance variable and hence AC(u) is nonlinear. Application of the principle in 21 reveals AC(u) to be strictly convex from below:

$$(35) \quad \frac{d^2AC_j(u)}{du^2} = \rho(\rho + 1)a_{1_j}Au^{-(\rho + 2)} > 0.$$

Hence, least-cost location need not lead to end points. Indeed, if we removed the boundary restriction on u the only possible corner solution would involve location at the core.

To illustrate the power of the technique developed one must carry the analysis a little further by examining the solution and its immediate neighborhood more closely. The first order condition would be:

$$(36) \quad \frac{dAC_j(u)}{du} = -a_{1_j}\rho Au^{-(\rho + 1)} + a_{2_j}m_2 = 0.$$

Solving for u^*_j:

$$(37) \quad u^*_j = \left[v_j\left(\frac{\rho A}{m_2}\right)\right]^{\frac{1}{\rho+1}}; \quad v_j = \frac{a_{1j}}{a_{2j}}.$$

Now in principle the parameters ρ, A and m_2 are the same for all industries but the v_j will normally differ. The effect of a small perturbation in v thus has the following effect on u*:

$$(38) \quad \frac{du^*}{dv_j} = \left(\frac{1}{\rho+1}\right)\left(\frac{\rho A}{m_2}\right)^{\frac{1}{\rho+1}} v_j^{\frac{1}{\rho}} > 0.$$

The following proposition then emerges:

PROPOSITION 1 The least-cost location for a firm with fixed coefficients in production locating with respect to an inverse exponential rent-distance function and transport costs linearly increasing with distance from the city centre will be further from the city centre the higher is its ratio of land input to factor 2 input.

Proposition 1 also holds when the problem is complicated slightly by the introduction of the costs of transporting the final product to market or transhipment point at the core and by the introduction of a set of 'other costs' that are invariant over space:

$$(39) \quad AC_j(u) = a_{1j}Au^{-\rho} + a_{2j}(\bar{r}_2 + um_2) + m_ju + k_j$$

where: m_j = transport rate on the final good j

k_j = other costs which do not vary over space.

The first-order condition for a minimum is:

$$(40) \quad \frac{dAC_j(u)}{du} = -a_{1j}\rho Au^{-(\rho + 1)} + a_{2j}m_2 + m_j = 0.$$

Solving for u^*_j one obtains:[1]

$$(41) \quad u^*_j = \left[\frac{v\rho A a_{2j}}{a_{2j}m_2+m_j}\right]^{\frac{1}{\rho+1}}$$

Again perturbing v:

$$(42) \quad \frac{du^*_j}{dv} = \left(\frac{1}{\rho+1}\right)\left(\frac{v\rho A a_{2j}}{a_{2j}m_2+m_j}\right)^{\frac{1}{\rho}} \frac{d\left[\frac{v\rho A\ a_{2j}}{a_{2j}m_2+m_j}\right]}{dv}$$

$$\frac{d\left[\frac{v\rho A a_{2j}}{a_{2j}m_2+m_j}\right]}{dv} = \frac{(\rho A a_{2j} + v\rho A a_{1j})(a_{2j}m_2+ m_j) - v\rho a_{2j}a_{1j}m_2}{(a_{2j}m_2 + m_j)^2}$$

Substituting back into (41) gives:

$$(43) \quad \frac{du^*_j}{dv} = \left(\frac{1}{1+\rho}\right)\left(\frac{v\rho A a_{2j}}{a_{2j}m_2+m_j}\right)^{\frac{1}{\rho}} \left(\frac{A\rho a_{2j}^2 m_2 + \rho A a_{2j}m_j + v\rho A a_{1j}m_j}{(a_{2j}m_2 + m_j)^2}\right) > 0.$$

Thus, proposition 1 continues to hold when the final product must be transported back to the city centre and there are other costs unrelated to space.

Another simple but interesting propositon that falls out of the model is the following:

PROPOSITION 2 Upward neutral shifts in the rent-distance function will increase the least-cost distance from the city for all existing firms (and potential entrants) not characterized by corner solutions.

1

$$u^{*\,-(\rho+1)}_j = \frac{a_{2j}m_2}{a_{1j}\rho A} + \frac{m_j}{a_{1j}\rho A}$$

$$u^{*\,(\rho+1)}_j = \frac{a_{1j}\rho A}{a_{2j}m_2 + m_j}$$

$$u^*_j = \left(\frac{v\rho A a_{2j}}{a_{2j}m_2 + m_j}\right)^{\frac{1}{1+\rho}}$$

Differentiating (41) with respect to A:

$$(44) \quad \frac{du^*_j}{dA} = \left(\frac{1}{1+\rho}\right)\left(\frac{v\rho A\ a_{2_j}}{a_{2_j}m_2+m_j}\right)^{\frac{1}{\rho}}\left(\frac{v\rho a_{2_j}}{a_{2_j}m_2+m_j}\right) > 0.$$

Hence, Proposition 2 follows.

Similarly one can deduce a fairly standard proposition regarding the effect on optimum location of an increase in freight rates:

$$(45) \quad \frac{du^*_j}{dm_2} = \left(\frac{1}{1+\rho}\right)\left(\frac{v\rho A\ a_{2_j}}{a_{2_j}m_2+m_j}\right)^{\frac{1}{\rho}}\left(\frac{-a^2{}_{2_j}v\rho A}{(a_{2_j}m_2+m_j)^2}\right) < 0.$$

PROPOSITION 3 Increases in freight rates will lead to a decline in the optimum distance to the source of the input (product market) whose rate has increased.

(ii) *A Note on the Propositions*

Proposition 1 is essentially a static proposition in that it sheds light on equilibrium locational configurations at any point in time. Ceterus paribus, one should observe that firms locating nearer to the core of the principle city tend to have higher v ratios than those in outlying areas.[1]

Propositions 2 and 3 are dynamic in the sense that the rent gradient and freight rates tend to shift over time. If the tendency is for rents to rise, other things remaining constant, then it follows that cost-minimizing firms will be pressured to relocate further from the city's core. This phenomenon has been observed by a number of

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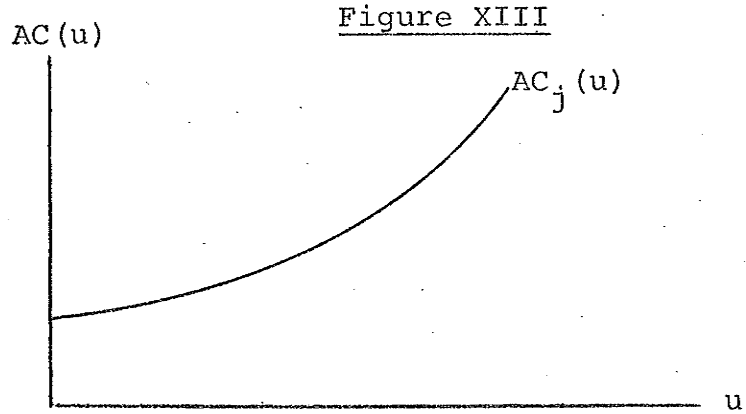
[1]Differing freight rates on the final products could alter this conclusion.

economists[1] and it is quite possible that the simple model outlined above may have considerable relevance in explaining it.

(iii) *Corner Solutions*

With u being unconstrained it was noted that AC(u) would always 'turn up' for some value of u. At u=0, however, it is possible that one could get a corner solution in the sense that $AC_j(u)$ is a minimum minimorum at this point. Of course this type of solution would be particularly relevant for industries with relatively high intensiveness in factor 2. The significance of these corner solutions would be that shifts in the rent gradient would not necessarily affect

Figure XIII



A Corner Solution

the optimum location of these firms. Hence, while much of manufacturing activity, for example, would be subject to pressures to move outward many of the firms located right at the core (such as firms providing a range of professional services) may not be immediately subject to this pressure.

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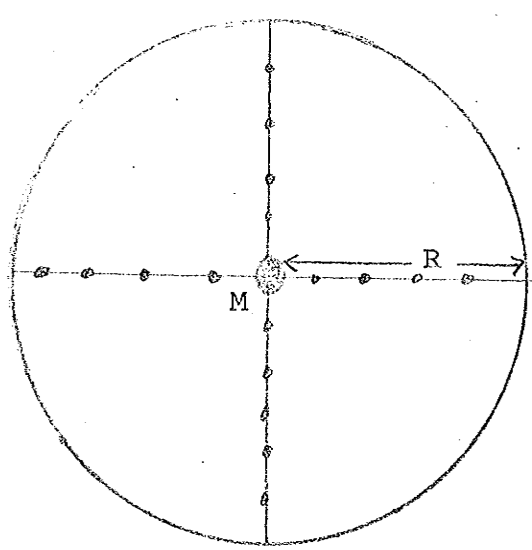
[1]W. R. Thompson, "Internal and External Factors in the Development of Urban Economies", pp. 43-62 of Perloff and Wingo eds., *Issues in Urban Economics* (1968) makes reference to the tendency for certain types of industry to be 'spun-off' as city growth proceeds. Whereas the approach taken here draws on more or less standard economic analysis to explain this process Thompson relied rather more strongly on such elusive concepts as lack of technological dynamism.

(iv) *The Price of Industrial Land Around a City*

In the previous sections dealing with firm location in an urban field the formation of land prices was ignored and a traditional rent-distance function was merely assumed to face the individual producing units. In this section a justification for this assumption will be developed within the locational framework of earlier sections.[1]

Consider some major urban centre M and the entire geographic area withnn an R-mile radius. Consider in addition a number of highways radiating like spokes out of the major centre along each of which is a sequence of smaller order centres. The significance of the smaller centres is that they represent the availability of services industrial land the price of which we want to determine in terms of distance from M.

Figure IV

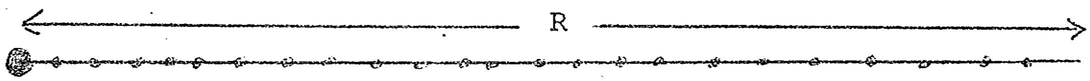


A Hypothetical City and its 'Field'

[1]With the work of Mills, *Studies in the Structure of the Urban Economy* (1972) or *Urban Economics*, Scott, Foresman and Co., (1972) there is doubtful need for such justification.

The information in Figure XIV can be condensed into a one-dimensional diagram as in Figure XV. The dots on the line represent projections of the lesser centres onto the line in terms of their distance (u) from M.

Figure XV



A Uni-dimensional Condensation of Figure XIV

On the production side n goods will be considered each of which is produced by means of a linearly homogeneous-fixed coefficient production function utilizing factor 1 (land) and factor 2 (some arbitrary input located at M). Any good j will have to be transported to M for market or transhipment and will have its price taken as fixed at that point. Factor 2 and good j will be characterized by transport costs that vary linearly over distance and competition in each industry will be assumed to wipe out positive profits.

Competition thus ensures that:

$$(46) \quad P_j = AC_j = a_{1j} r + a_{2j}(r_2 + m_2 u) + m_j u$$

Solving for r_1 one obtains the maximum rent that a firm in industry j can pay at any distance u:

$$(47) \quad \hat{r}_{1j} = \frac{P_j}{a_{1j}} - \frac{a_{2j}(r_2 + m_2 u) + m_j u}{a_{1j}}$$

which is a linear function with slope:

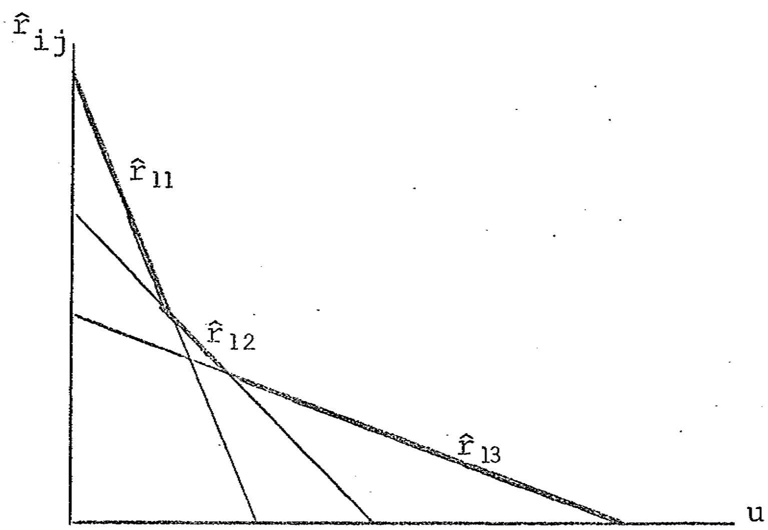
$$(48) \quad \frac{d\hat{r}_{1_j}}{du} = \frac{-a_{2_j} m_2 - m_j}{a_{1_j}} < 0.$$

or to emphasize factor intensity:

$$(48') \quad \frac{d\hat{r}_{1_j}}{du} = \frac{-m_2}{v_j} - \frac{m_j}{a_{1_j}} \; ; \; v_j = \frac{a_{1_j}}{a_{2_j}}$$

Different industries will have characteristic v's, a_{ij}'s and m_j's while m_2 will be identical for all. When all industries are considered, a series of $\hat{r}_{1_j}$ functions as in Figure XVI are obtained.

Figure XVI



Three Hypothetical $\hat{r}_{ij}$ Functions

The rent gradient will be the envelope of all the $\hat{r}_{1_j}$ functions since the industry willing to pay the most for land at distance u will locate there. The rent-distance function will obviously be convex from below and as the number of firms gets large the curve will tend to become smooth and convex from below.

What type of locational configuration is implied by this approach to the rent gradient? If the simplifying assumption that transport costs on the final product are negligible we get the *price relationship* corresponding to our earlier conclusion whereby firms with lower v's would locate nearer to the core. Similarly, industries with lower v's will be able to pay higher land rents. That is:[1]

$$(48)\quad \frac{d\hat{r}_{1_j}}{dv} = -\frac{P_j a_{2_j}}{a^2_{1_j}} + \frac{r_2 + m_2 u}{v^2}$$

$$(49)\quad = -\frac{\hat{r}_{1_j}}{v} < 0.$$

When m_j is of significant size then firms with larger m_j's have more to lose from being further from the core and they will consequently be prepared to pay higher land rents to be there. This being the case there need not be any one-to-one relationship between factor intensity and distance from the core. Some firm j may be prepared to balance a high v_j against a high m_j so as to locate nearer to the core than firm j+1 having the same factor intensity but a lower m_{j+1}. In general, however, one would expect the dramatic variation in land rents around a city to ascribe a dominant role to factor intensity so that there will be some variation in v's at a given u the general tendency will be for high v's to characterize industries further from the core.

(v) *The Effects of Increasing Returns to Scale*

Much of the paper suffers by the assumed linear homogeneity of the production function so that average and/or unit costs were constant for all levels of output. In this section the urban field model is

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$$\frac{d\hat{r}}{dv} = \frac{1}{v}\left(-\frac{P_j}{a_{1_j}} + \frac{r_2 + m_2 u}{v}\right)$$

$$= \frac{1}{v}\left(-\frac{a_{1_j} r_1 + a_{2_j}(r_2 + m_2 u)}{a_{1_j}} + \frac{r_2 + m_2 u}{v}\right)$$

$$= \frac{1}{v^2}\left(-vr_1 - r_2 - m_2 u + r_2 + m_2 u\right) = -\frac{\hat{r}_{1_j}}{v} < 0.$$

utilized to relax this assumption and replace it by two variants of increasing returns to scale.

It will again be necessary to define our a_{ij}'s as the reciprocals of the average physical products for the factors at specified output levels.

The first type of scale economy will be manifested in the model by the restrictions:

$$a_{ij} = a_{ij}(r_i, Q_j) \quad i = 1, 2.$$

$$\frac{da_{ij}}{dQ_j} < 0 \quad i = 1, 2.$$

$$\frac{dv}{dQ_j} = 0.$$

In a variable coefficients world this would correspond to the case where a doubling of the level of output (factor prices constant) would require less than twice the original quantities of inputs and would leave the relative input mix unaltered. Thus, it is a neutral type of scale economy of the sort one would expect in a production function that is homogeneous of degree greater than one.

First, the effect on optimum distance u^*_j of a small change in the level of output. Bearing in mind the above restrictions differentiate (37) with respect to Q_j:

$$(50) \quad \frac{du^*_j}{dQ_j} = \frac{d\left[v_j\left(\frac{\rho A}{m_2}\right)\right]^{\frac{1}{\rho+1}}}{dQ_j} = 0.$$

Since the a_{ij}'s only enter this expression as v_j and v_j is restricted to constancy the change in output level will have no effect on u^*_j. This result is actually non other than the previously mentioned Bradfield result whereby homogeneity to any degree is sufficient to ensure continuous sight preference for all levels of output. Intuitively, the fact that the input coefficients (which combine with transport rates to form locational weighting factors) do not change in relative size means there will not be an increased attraction to either input source.

The more interesting case is where the final product must be transported to market because it is this case where demand conditions become relevant. Take equation (41) and differentiate with respect to quantity of output:

$$(51) \quad \frac{du^*_j}{dQ} = \frac{du^*_j}{da_{2_j}} \frac{da_{2_j}}{dQ} = \left(\frac{1}{1+\rho}\right)\left(\frac{v\rho A a_{2_j}}{a_{2_j}m_2+m_j}\right)^{\frac{1}{\rho}}\left(\frac{m_j v\rho A}{(a_{2_j}m_2+m_j)^2}\right)\frac{da_{2_j}}{dQ} < 0.$$

Thus, economies of scale in this model results in a *pull* toward the market as the level of output expands. This is due to the fact that economies of scale tend to reduce the average quantity of inputs needed to produce each unit of product. Consequentially the transport cost on the final product (which is nevertheless part of the average cost curve) becomes relatively more important than the transport costs on the average quantity of factors required to produce it.

Demand now becomes a factor. Obviously where the firm now locates is going to be determined simultaneously with the decision on the level of output to produce. The level of output is, of course, goint to depend on the nature of demand conditions so that it follows that demand factors will now become significant in the location decision.[1]

[1]This, of course, shows that the Bradfield result is negated when the final product must be transported to market.

This treatment is also interesting in that it provides a theoretical basis for assuming economies of scale in the core as is done often by urban economists. From (51) it follows that firms subject to scale economies may naturally be attracted to the core as markets expand. Further, if scale economies are inconsistent with perfect competition it follows that firms in and around the core may tend to be more monopolistic (greater industrial concentration) than those in peripheral areas. This is a natural extension of the inconsistency of scale economies with atomistic competition.

To this point it has been assumed that scale economies were such as to keep v intact. In reality this is not a very appealing assumption since land is the type of input that would likely expand less than in proportion to the level of output. That is, a doubling of output with factor prices constant would not usually require a doubling of land requirements although it may necessitate a doubling of labor, materials, capital and so forth. This corresponds to an expansion path at a given location (whether input proportions are variable or not) such as depicted in Figure XVII.

Figure XVII