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State-Dependent Risk Aversion

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Abstract

The traditional representative agent, consumption-based asset pricing model with iso-elastic utility has not performed well empirically. Alternative specifications have focused on the rigidities implied by the Von Neuman-Morgenstern axioms of choice under uncertainty, in particular the independence hypothesis, and proposed some degree of generalisation. This has been achieved while retaining constant risk aversion for the within-period utility function. The purpose of this paper is to present further flexibility through the risk aversion specification, within the context of non-expected utility, through relaxation of the iso-elasticity assumption. By allowing attitudes toward risk to reflect the information set used for the decision process, risk aversion is no longer fixed, but responds to the evolution in the state of the world as well as the distributional assumptions governing the state variables. The advantage is that the same individual may be a risk-lover over certain states and distributions, while being risk averse over others. The model is developed within a continuous-time setting for consumption and leverage choices. The closed form solution for the risk aversion function gives results that are appealing on intuitive grounds. In particular, risk aversion increases in the variance of the risky return, and falls in wealth and equity premium. Estimation results are presented.

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1- INTRODUCTION

The consumption-based, representative agent asset pricing model has not performed well in empirical estimations[1]. The standard specification, assuming Von-Neuman Morgenstern (VNM) preferences and iso-elastic utility has been unable to explain certain stylised facts, such as the persistent large differential between equity and riskless returns, or the dichotomy between consumption risk and differentials in expected returns on various risky assets.

In response, a number of generalisations have been proposed. In an effort to reconcile the axioms of choice under uncertainty with observed experimental behavior, the rigidities implied by the linearity in probabilities axiom (independence) has been put forward as possible explanation for the disappointing results of the standard model. Research has concentrated on the development of a non-expected utility theory for choice over uncertain prospects. One the main characteristics of this literature is that it retains the iso-elastic specification for within-period utility, while allowing a more flexible representation of the maximand. Hence, for instance, convex and concave transforms of the VNM index will be allowed, such that probabilities may not enter linearly in the objective function.

The purpose of this paper is to take this generalisation a step further by relaxing the iso-elasticity assumption. To be more precise, we let risk aversion depend on the elements of the information set reflecting state variables and their corresponding distributional processes. This allows us additional flexibility to the extent that risk aversion becomes bi-dimensional. In a static setting, the individual can be a risk-lover for certain problems, depending on his level of wealth and the distribution of the risk (e.g. for lotteries) and risk-averse for other problems, such as with respect to fire insurance. In a dynamic context, his level of risk aversion will change over time as the state variables evolve as well.

The rationale for relaxing constant preferences toward risk is simple. It appears

[1] Grossman and Shiller (1981), Mehra and Prescott (1985), Hansen and Singleton (1983), Grossman, Melino and Shiller (1987).

counter-intuitive to consider that attitudes toward uncertainty remain fixed independently of such factors as the wealth level, and the distribution of risk (such as expected return (loss) and variance). In particular, we might expect a risk-averse individual to be more tolerant toward uncertainty, the wealthier he is, or we could anticipate risk aversion to increase as the distribution of returns to savings becomes more risky, in terms of a mean-preserving spread.

This generalisation cannot take place within the VNM framework. The reason for this is that by permitting the elasticity of utility to depend on the distribution of uncertainty, we effectively recover a maximand which is non-linear in probabilities. It follows that the independence axiom cannot be retained and that other axioms must be invoked. Formal proof of the representation theorem necessary to link the axioms with an operative cardinal index is not provided in the paper, and left for future research.

More general functional forms, such as Hyperbolic Absolute Risk Aversion (HARA) relax fixed risk aversion, with Constant Absolute, and Constant Relative Risk Aversion, treated as special cases. However, this added flexibility takes place at the expense of parsimony, with two additional parameters to estimate. Furthermore, severe restrictions, relating other preference parameters have to be imposed for HARA to yield sensible results considering risk aversion.

The application of State-dependent Risk Aversion (SRA) is derived in the setting of a continuous-time problem. An infinitely-lived agent must choose consumption and leverage between a risky and a riskless asset so as to maximise the discounted present value of future utility. The Arrow-Pratt absolute risk aversion index is specified as depending on the state variable, i.e. wealth, as well as on the parameters of the distribution governing the evolution of wealth.

The parametrisation that is used is derived from the first-order conditions, from the perspective of an external observer who would not know the exact functional form. It imposes risk aversion to be declining in equity premium, while increasing in the variance of the risky return. In addition, it is consistent with declining (increasing) absolute (relative) risk aversion in wealth, as postulated by Arrow (1984:153), Yaari (1969:320), Pratt (1964:122-123), and Friedman and Savage (1948).

We demonstrate that given this particular functional form, the closed-form solutions for optimal consumption and leverage choice exist. In addition, they satisfy regularity conditions such as sufficiency, tranversality and dynamic consistency, while separation results still hold.

The main advantage of specifying the risk aversion function is that it yields sensible results concerning attitudes toward uncertainty, while allowing us to widen the admissible range for risk aversion in Mehra and Prescott's puzzle, by redefining relative risk aversion over wealth. In other words, "unrealistically large" Arrow-Pratt relative indices would be consistent with high aggregate wealth levels, and low discount factors, relative to the risk-free rate of return. From another perspective, a high ratio of discounting over the risk-free rate could reconcile large equity premiums with low levels of relative risk aversion.

The paper is organised as follows. §2 presents the general specification of preferences in a static setting, relating State-Dependent Risk Aversion to other non-expected theories of choice under uncertainty. We indicate in a heuristic manner, which alternative axioms could be used in the theorem necessary for cardinal representation. §3 discusses SRA in an intertemporal setting, with an application in continuous-time leverage and consumption choices. §4 reviews aggregation issues and presents estimation results, while the conclusion offers an overview of the main findings along with possible extensions.

2- SRA PREFERENCES AND NON-EXPECTED UTILITY.

In this section, we attempt to relate SRA to other theories of choice under uncertainty. It is not intended as a formal axiomisation of State-Dependent Risk Aversion, which is left for future research, but rather as a heuristic discussion of where SRA should be located in the array of alternative representations of decision under risk. We begin by defining the environment in which the agent is asked to express his preferences over lotteries, and then define what is intended by SRA. Following this, we discuss which axioms could be consistent with our specification, without referring to formal proofs.

Consider the compact set C, defined as the set of all prizes (outcomes), with P being the set of all probability measures on C. Preferences over P are given by the binary relation $\succeq$ which expresses the agent's preferences on probability distributions. As usual, $\succ$ denotes strict preference and $\sim$ denotes indifference.

The ordering $\succeq$ is assumed to satisfy completeness and transitivity[2]. In addition, following Gul (1991) and Chew, Epstein and Segal (1991), we postulate continuity, i.e.: $\forall F \in P$, the sets $\{G \in P \mid G \succeq F\}$ and $\{G \in P \mid F \succeq G\}$ are closed.

In the simplest case where risk aversion is independent of the prizes themselves, what is implied by SRA is the following mappings:

$$\begin{aligned} \gamma &: P \to \mathcal{R} \\ U &: C \times \mathcal{R} \to \mathcal{R} \\ V &: P \to \mathcal{R}. \end{aligned} \tag{1}$$

Here, we are assuming that $\succeq$ can be represented by $V(F)$, such that:

$$F \succeq G \iff V(F) = \int U[c, \gamma(F)]dF(c) \geq V(G) = \int U[c, \gamma(G)]dG(c), \tag{2}$$

for $c \in C$ and $F, G \in P$. In this notation, γ refers to the risk aversion index, i.e. the relative concavity of $U[c, \cdot]$. Given (1), the problem is to define the additional axioms such that (2) will be verified.

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[2] For lotteries $F,G,H \in P$, completeness implies that $F \succeq G$, or $G \succeq F$ while transitivity can be expressed as: $F \succeq G$, and $G \succeq H \Rightarrow F \succeq H$.

It can be seen from (1) and (2) that the dependence of the concavity factor of the utility index, γ, on the distributional parameters implies that the expected value of $U[\cdot,\cdot]$, will be nonlinear in probabilities. As such, the independence axiom cannot be retained with the consequence that the maximand $V(\cdot)$ corresponds to a non-expected utility setting. This translates to the fact that the iso-expected-utility curves in the three-case probability simplex will be nonlinear and could display fanning-out or fanning-in properties.

As an example, consider the quasi-concave case, with simple probability measures, where an expected-utility maximiser would have his relative risk aversion increase in mean-preserving spreads:

$$\begin{aligned} \gamma(F) &= \gamma_0 + \gamma_1 P_1 + \gamma_3 P_3 \\ U[c,\gamma(F)] &= c^{1-\gamma(F)} \\ V(F) &= P_1 U[c_1,\gamma(F)] + (1-P_1-P_3)U[c_2,\gamma(F)] + P_3 U[c_3,\gamma(F)]. \end{aligned} \tag{3}$$

where, $c_1 < c_2 < c_3$. Then, for $\gamma_1, \gamma_3 > 0$, increased risk aversion as we move to the north-east in the probability simplex, (i.e. reduce P_2) would reflect itself in convex indifference curves, i.e. as the probability of the central prize is reduced in favor of the tail outcomes, the tolerance to risk falls. This simple example nests the standard linear case ($\gamma_1, \gamma_3 = 0$), fanning-out ($\gamma_1 < 0, \gamma_3 > 0$) and fanning-in ($\gamma_1 > 0, \gamma_3 < 0$).

The nonlinearity of the indifference curves also implies that certain generalisations of the independence axiom such as betweenness cannot be invoked. As such, our functional cannot be associated to weighted utility [Chew (1983)] and disappointment-aversion [Gul (1990)], which allow fanning-out and fanning-in but impose linear iso-expected-utility curves. Given this, the preference ordering we are postulating bears closer resemblance to Chew, Epstein and Segal's (1991) mixture symmetry which yields non-linear indifference curves in the probability simplex[3].

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[3] More precisely, for lotteries $F,G,H \in P$, and for $\alpha \in [0,1]$, we have independence if:

$$F \sim G \Rightarrow \alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H,$$

However, mixture symmetry may not be imposed since, given a distribution F, the resulting V must be quadratic in probabilities[4]. In particular, convex transformations of the usual VNM index such as:

$$V(F) = \left(\int U(c)dF(c)\right)^2 + \int W(c)dF(c)$$

for some concave $W(\cdot)$ will be admissible, whereas conditioning the concavity of $U(\cdot)$ on F implies that probabilities do not enter quadratically in the maximand $V(\cdot)$. For this reason, an alternative axiom allowing nonlinear indifference curves without imposing ex-ante quasi-concavity or quasi-convexity will be necessary.

An important additional property of (1) and (2) is that the risk premium will in general not be linear in the variance of the gamble. As is well known, in an expected utility setting, with constant risk aversion, the risk premium is proportional to the variance of the risk[5]. The development of non-expected utility, such as rank-dependent utility, has permitted alternative specifications in which the premium is proportional to the standard error rather than the variance (First-Order Risk Aversion)[6]. The implications are considered important in an asset pricing model, in which the relative smoothness of consumption implies a low degree of volatility, because the standard error will be larger than the variance[7]. In our case, the risk premium is not restricted to be linear in volatility. Hence, in §3, risk *aversion* is proportional to the variance of the risky return, such that the risk premium would be quadratic in volatility. This translates itself into added sensitivity of risk aversion to the

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whereas betweenness implies:

$$F \sim G \Rightarrow \alpha F + (1-\alpha)G \sim F,$$

and mixture symmetry means that $\forall \alpha \in (0, \frac{1}{2})$, $\exists \beta \in (\frac{1}{2}, 1)$, such that:

$$F \sim G \Rightarrow \alpha F + (1-\alpha)G \sim \beta F + (1-\beta)G,$$

Chew, Epstein and Segal (1991:142).

4 Chew, Epstein and Segal, Theorem 5, (1991:149).

5 Pratt (1964:126).

6 Epstein and Zin (1990).

7 Epstein and Zin (1990:389)

underlying generating process for uncertainty. As such, the individual will be much more responsive to increases in risk, such as mean-preserving spreads, and will respond in the form of a lower certainty equivalence. In other words, for a given expected outcome, increasing risk will augment the concavity of the utility function, such that the agent will be willing to pay a higher premium to avoid having to face the uncertainty.

3- INTERTEMPORAL CHOICE UNDER UNCERTAINTY

3.1 The Risk Aversion Function

We now specialise the structure of the risk aversion function for a continuous-time problem of consumption and leverage choices under uncertainty. The model we use closely follows Merton's (1990) work, modified to incorporate state-dependent risk aversion. For tractability, we assume that absolute risk aversion is a function of the state variables and the distribution associated to the diffusion process, but where the functional form is unknown to the external observer.

Consider the following infinitely-lived consumer who allocates wealth, $A(t)$, by choosing consumption, $C(t)$, and leverage $\omega(t)$, so as to maximise the present value of his expected future utility:

$$\max_{C(t),\omega(t)} E(0) \int_0^\infty e^{-\rho t} U(t) dt \tag{4}$$

subject to his budget constraint and:

$$\gamma(t) \equiv \frac{U_{CC}(t)}{U_C(t)} = \gamma[A(t), F], \qquad \gamma(t) \geq 0, \tag{5}$$

where the expectation is taken over the support of F, which represents the CDF governing the evolution of wealth.

Following Merton (1990:97-101), the consumer's budget constraint is given by a two-asset geometric Brownian diffusion process. The cum-dividends price of the risky asset, $P^e(t)$ is assumed to be governed by the law of motion:

$$\frac{dP^e(t)}{P^e(t)} = \alpha dt + \sigma dZ(t),$$

such that the diffusion for wealth is obtained as:

$$dA(t) = \{[\omega(t)\alpha + (1 - \omega(t))r]\, A(t) - C(t)\}\, dt + \omega(t)\sigma A(t) dZ(t) \tag{6}$$

with $A(0)$ given, and where $\omega(t)$ is the share of total savings allocated to the risky asset; α is the constant expected rate of return on the risky asset; r is the riskless rate of return, with $\alpha > r$, and $\rho \neq r$; and σ^2 is the variance of the risky asset's return. Finally, the uncertainty comes from a Wiener process $Z(t)$, where $dZ(t) \sim N.I.D.\ (0, dt)$.

From Itô's lemma, the problem can be expressed as a dynamic programming schedule where the Bellman equation is given by:

$$\begin{aligned} 0 = \max_{C(t),\omega(t)} & \frac{\exp(-\gamma(t)C(t))}{-\gamma(t)} - \rho J(t) + J_A(t)\left\{\left[\omega(t)(\alpha - r) + r\right]A(t) - C(t)\right\} \\ & + \frac{J_{AA}(t)}{2}\omega^2(t)\sigma^2 A^2(t), \end{aligned} \tag{7}$$

where $J(t) \equiv J(A(t))$ is the current value function, with $J_A(t)$ and $J_{AA}(t)$ respectively first and second derivatives with respect to current wealth.

In order to identify the risk aversion function, consider the following postulates:

i) Declining absolute risk aversion (Arrow (1984:153)).

ii) Increasing relative risk aversion (Arrow (1984:153)).

iii) For a risk averter [lover], non-decreasing [increasing] risk aversion for mean-preserving spreads.

iv) Non-increasing risk aversion in equity premium.

The first two hypotheses were considered to be realistic representations of observed behavior by Arrow. The third and fourth constitute a reinterpretation of the well-known result that a risk averter will be willing to pay a non-negative premium to avoid taking a risk. In our case, the size of the premium will not only depend on the distributional parameters, given a fixed level of risk tolerance, but also on the interaction between the distribution of the risk and the degree of concavity of the utility function. It seems reasonable to suppose that a risk averter will not be made more tolerant for a fixed expected loss if the variance of the incurred risk is augmented. Conversely, a reduction in expected loss for a given variability should not make the risk more repulsive to him, other things being equal. At the very least, his level of risk aversion should remain unaffected in both cases.

Returning to the intertemporal problem, the corresponding first-order equations yield the optimal consumption and leverage rules:

$$C^*(t) = \frac{\log(J_A(t))}{-\gamma(t)} \tag{8}$$

and

$$\omega^*(t) = \frac{-J_A(t)(\alpha - r)}{J_{AA}(t)\sigma^2 A(t)} \tag{9}$$

which are substituted back into the objective function (7), such that we obtain a nonlinear second-order differential equation in $J(t)$:

$$0 = -\frac{1}{\gamma(t)}[1 - \log(J_A(t))] - \rho\frac{J(t)}{J_A(t)} + rA(t) - \frac{J_A(t)}{J_{AA}(t)}\frac{(\alpha - r)^2}{2\sigma^2}. \tag{10}$$

Consider a trial solution $J(t)$ such that:

$$\rho\frac{J(t)}{J_A(t)} = rA(t).$$

Upon integrating both sides, it is readily apparent that the objective function will be of the form:

$$J(t) = [A(t)]^{\rho/r} \tag{11}$$

with constants of integration omitted. Substituting back into the Bellman equation gives the maximised value function:

$$0 = -\frac{1}{\gamma(t)}[1 - \log(J_A)] - \rho\frac{J}{J_A} + rA - \frac{J_A}{J_{AA}}\frac{(\alpha - r)^2}{2\sigma^2}. \tag{12}$$

Therefore, for trial solution (10), the closed-form solution for risk aversion that is consistent with this candidate is obtained as:

$$\gamma(t) = \frac{2\sigma^2}{(\alpha - r)^2}\frac{r - \rho}{rA}\left[1 - \log\left(\frac{\rho}{r}\right) + \left(\frac{r - \rho}{r}\right)\log(A)\right] \tag{13}$$

for $\rho \neq r$.

We therefore recover, conditional on (11), an absolute risk aversion function which is consistent with postulates i), iii), and iv). In particular, absolute risk aversion is explicitely declining in wealth and squared equity premium. Furthermore, we show in §3.3 that relative risk aversion increases in wealth. In the next section, we solve for the closed-form solutions for optimal consumption and leverage, and demonstrate that (11) constitutes a valid candidate satisfying the regularity conditions of sufficiency and transversality.

3.2 The consumer's problem

In the preceding section, we obtained the risk aversion function form the perspective of an external observer. We now turn our attention to the consumer's solution of the intertemporal problem, given the functional form for preferences that was recovered.

For the infinitely-lived consumer maximising (4), subject to (6) and (13), the independence of the risk aversion function from the control variables allow us to obtain the same first-order conditions (8) and (9), and maximised value function (10). Given this, the closed-form solutions for consumption and leverage are easily obtained by substituting for trial (11) in (8) and (9) as:

$$C^*(t) = \frac{(\alpha - r)^2}{2\sigma^2}\frac{rA(t)}{r-\rho}\left\{\frac{-\log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right)\log(A(t))}{1-\log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right)\log(A(t))}\right\} \tag{14}$$

and,

$$\omega^*(t) = \left(\frac{r}{r-\rho}\right)\left(\frac{\alpha - r}{\sigma^2}\right), \qquad \rho \neq r. \tag{15}$$

The next step is to show that (14) and (15), which are conditional on candidate (11), satisfy regularity conditions for interior solutions, in terms of sufficiency and transversality.

Proposition. *The solutions* (14) *and* (15) *satisfy sufficiency of the first-order conditions, as well as transversality requirements.*

Proof. By assumption, $\rho \neq r$. Yet, for feasibility, i. e. for nonnegative consumption, we must impose that $\rho < r$. This condition is sufficient to guarantee that the trial function (11) is concave in wealth. Merton (1972:382) uses Arrow's (1968) generalisation of the Mangasarian (1966) sufficiency condition for optimality, to show that concavity of the maximised value function, with respect to the state variables, is sufficient to guarantee that the first-order conditions are optimal[8].

We now show that for our candidate solution (11) of the second-order differential equation, transversality is obtained unconditionally. From the optimal consumption rule

[8] For a proof of Arrow' (1968) and Mangasarian's (1966) result, see Seierstad and Sydsaeter (1977).

(14), we can see that consumption is asymptotically linear in wealth, the reason being that the term inside the brackets asymptotically goes to 1, at rate depending on $\rho - r$, such that the diffusion rule for wealth (6), when evaluated at the optimal controls, can be written in the limit as:

$$dA(t) = aA(t)dt + bA(t)dZ(t),$$

where,

$$a \equiv \left\{ \left(\frac{r}{r-\rho} \right) \frac{(\alpha - r)^2}{2\sigma^2} + r \right\}$$

and

$$b \equiv \left(\frac{r}{r-\rho} \right) \frac{(\alpha - r)}{\sigma}.$$

From the geometric Brownian motion process, a solution for wealth is given by[9]:

$$A(t) = A(0) \exp \left[(a - b^2/2)t + bZ(t) \right].$$

Substituting into our candidate, we obtain:

$$J(t) = A(0)^{\rho/r} \exp \left\{ \left[-\left(\frac{\rho}{r-\rho} \right)^2 \frac{(\alpha - r)^2}{2\sigma^2} + \rho \right] t + \left(\frac{\rho}{r-\rho} \right) \frac{(\alpha - r)}{\sigma} Z(t) \right\}.$$

In the absence of a bequest function, the transversality condition can be written as:

$$\lim_{t \to \infty} E(0) \exp(-\rho t) J(t) = 0,$$

which becomes:

$$\lim_{t \to \infty} \exp \left\{ \left[-\left(\frac{\rho}{r-\rho} \right)^2 \frac{(\alpha - r)^2}{2\sigma^2} \right] t \right\} E(0) \exp \left\{ \left(\frac{\rho}{r-\rho} \right) \frac{(\alpha - r)}{\sigma} Z(t) \right\}.$$

Because $Z(t)$ is stationary, the limit goes to 0 unconditionally[10]. ■

Noticing that the risk function (14) effectively means that preferences will not be stationary in an intertemporal framework naturally raises the issue of time consistency. It

[9] Kamien and Schwartz (1981:245)

[10] It should be noted that other solutions for (10), exist, but that these candidates will not simultaneously satisfy sufficiency and transversality requirements.

is well known that constant tastes are sufficient to guarantee that optimal plans will be time consistent in the sense of Johnsen and Donaldson (1985)[11]. As such, additively time and state separable utilities, with stationary risk preferences used in computing the VNM index are sufficient to guarantee that optimal choices will be time consistent, i.e. an agent will not renege on past decisions. However, as Johnsen and Donaldsen (1985:1454) have shown, expected utility is not necessary for time consistency.

In our context the mapping of the risk aversion function from the state space to the risk tolerance factor means that risk aversion will evolve as state variables (notably wealth) change. Yet, time consistency will be maintained to the extent that this evolution is predictable and fully taken into account by the utility maximiser. It is in this sense that preferences remain constant even though risk attitudes are non-stationary. The consumer will not renege on past decisions as his level of risk aversion changes to the extent that this evolution was anticipated at the time of his decision. In other words, the agent evaluates the desirability of elements of the choice set knowing that each alternative yields a different path for his attitudes toward risk. The optimal set of controls will be consistent with utility maximisation over a path for risk tolerance and thus utility index. This means that in discrete time, we can break an optimal sequence $\{C_0^*, C_1^*, \ldots, C_T^*, \ldots,\}$ and an alternative $\{C_0', C_1', \ldots, C_T', \ldots,\}$ that both agree with state vector $\mathbf{x}_T$ and F at any $T > 0$ and find that $\{C_T^*, C_{T+1}^*, \ldots\}$ will still be preferred to $\{C_T', C_{T+1}', \ldots\}$[12]. The reason for this is that our preferences satisfy weak independence across states. What this implies is that we maintain state and time separability, such that $V(F)$ can be written as a monotone function where each state-contingent utility enters separately.

In §3.3, we compare optimal rules for consumption and leverage as well as risk aversion functions evaluated at the optimum for various utility functionals. We demonstrate that only Hyperbolic Absolute Risk Aversion and State-Dependent Risk Aversion classes satisfy

[11] Epstein (1990:25).

[12] Epstein (1990:24).

postulates i) to iv). However, in the case of HARA, this is achieved at the expense of two additional parameters that have to be estimated, and stringent restrictions that have to be imposed.

3.3 Comparative static results.

The following tables give the optimal decision rules as well as the risk aversion indices evaluated at the optimum for Constant Absolute Risk, Constant Relative and Hyperbolic Absolute Risk Aversion classes. The optimal values have been derived for the two-asset case, using the same diffusion process and distributional assumptions [13].

Table 1: Optimal Decision Rules

	$U(C(t),\gamma(t))$	$C^*(t)$	$\omega^*(t)$
CARA	$\frac{\exp(-\gamma C(t))}{-\gamma}$	$\frac{\rho-r}{\gamma r}+\frac{(\alpha-r)^2}{2\sigma^2\gamma r}+rA(t)$	$\frac{\alpha-r}{\sigma^2\gamma rA(t)}$
SRA	$\frac{\exp(-\gamma(t)C(t))}{-\gamma(t)}$	$\frac{(\alpha-r)^2}{2\sigma^2}\frac{rA}{r-\rho}\left\{\frac{-\log(\frac{\rho}{r})+(\frac{r-\rho}{r})\log(A)}{1-\log(\frac{\rho}{r})+(\frac{r-\rho}{r})\log(A)}\right\}$	$\left(\frac{r}{r-\rho}\right)\left(\frac{\alpha-r}{\sigma^2}\right)$
CRRA	$\frac{C(t)^{1-\gamma}}{1-\gamma}$	$\left\{\frac{\rho}{\gamma}-(1-\gamma)\left[\frac{(\alpha-r)^2}{2\sigma^2\gamma^2}+\frac{r}{\gamma}\right]\right\}A(t)$ $=bA(t),\quad b>0$	$\frac{\alpha-r}{\sigma^2\gamma}$
HARA	$\frac{\gamma}{1-\gamma}\left(\frac{\beta C(t)}{\gamma}+\eta\right)^{1-\gamma}$	$\frac{\eta}{\beta}\left\{\frac{\rho}{r}-\frac{(1-\gamma)}{r}\left[\frac{(\alpha-r)^2}{2\sigma^2\gamma}+r\right]-\gamma\right\}$ $+\left\{\frac{\rho}{\gamma}-(1-\gamma)\left[\frac{(\alpha-r)^2}{2\sigma^2\gamma^2}+\frac{r}{\gamma}\right]\right\}A(t)$ $=a+bA(t),\quad >0$	$\frac{\alpha-r}{\sigma^2\gamma}+\frac{\eta(\alpha-r)}{\beta r\sigma^2A(t)}$

With respect to the decision rules, we observe first that Samuelson's (1969) separation theorem remains valid for SRA utility, as is the case for CARA, CRRA and HARA classes. In fact, separation is verified bilaterally for consumption as well as for leverage choice, i.e. consumption is independent of the portfolio decision and vice versa.

The second important result is that SRA utility yields a non-linear consumption schedule with respect to the wealth parameter, but that this nonlinearity disappears rapidly as wealth increases. In fact, consumption will be convex at low levels of wealth and linear

[13] The CARA, CRRA and HARA decision rules are from Merton (1990: ch. 5, 6).

at higher net worth. As is seen next, this feature does raise some problems for exact aggregation.

This characteristic is not observed in the leverage rule which is expressed as a constant, just as in the CRRA case. As such, it no longer displays the counter-intuitive results that the share of savings invested in a risky asset should fall in wealth as for the CARA and HARA case, while retaining the familiar declining in variance and increasing in premium predictions.

The next step is to discuss the risk functions evaluated at the optimal rules. For this purpose, both Arrow-Pratt indices were evaluated at the maximising control values for the various utility functionals. Table 2 presents the resulting risk functions.

Table 2: Optimal Risk Aversion Functions

	$U(C(t),\gamma(t))$	$R_A^*(t)$	$R_R^*(t)$
CARA	$\frac{\exp(-\gamma C(t))}{-\gamma}$	γ	$\frac{\rho-r}{r}+\frac{(\alpha-r)^2}{2\sigma^2 r}+\gamma r A(t)$
SRA	$\frac{\exp(-\gamma(t)C(t))}{-\gamma(t)}$	$\frac{2\sigma^2}{(\alpha-r)^2}\frac{r-\rho}{rA}\left[1-\log\left(\frac{\rho}{r}\right)+\left(\frac{r-\rho}{r}\right)\log(A)\right]$	$-\log\left(\frac{\rho}{r}\right)+\left(\frac{r-\rho}{r}\right)\log(A)$
CRRA	$\frac{C(t)^{1-\gamma}}{1-\gamma}$	$\frac{\gamma^2}{\left\{\rho-(1-\gamma)\left[\frac{(\alpha-r)^2}{2\sigma^2\gamma}+r\right]\right\}A(t)}$	γ
HARA	$\frac{\gamma}{1-\gamma}\left(\frac{\beta C(t)}{\gamma}+\eta\right)^{1-\gamma}$	$\frac{\gamma\beta}{\beta(a+bA(t))+\gamma\eta}$	$\frac{\gamma\beta(a+bA(t))}{\beta(a+bA(t))+\gamma\eta}$

In §3.2, we suggested four postulates concerning risk aversion functions: declining absolute risk aversion and increasing relative risk aversion in wealth, increasing in variance and falling in equity premium. Bearing these considerations in mind, the first important result to be noted is that for the State-Dependent Risk Aversion case, risk tolerance is no longer expressed as a constant preference parameter, as in the CARA and CRRA cases, but is allowed instead to vary with the state variables of the model. As such, both the absolute and relative indices depend on the state and distributional variables, just as is the case under the general HARA specification.

Risk preferences can be considered as local to the maximisation problem, to the extent that they are functions of the structural and distributional assumptions of the model. The direct implications of this are twofold:

i) the current state of the world directly affects tolerance to risk at the time of decision, and,

ii) the *evolution* of the state variables will have an impact on future preferences.

Under rational expectations, this non-stationarity of preferences is fully taken into account by the individual in assessing the optimality of alternative consumption - savings paths. In other words, under Bellman's principle of optimality, the individual selects optimal consumption and leverage rules given a state-dependent level of risk tolerance, computing its anticipated evolution which will in turn depend on the choices he will make.

Considering first the relation with wealth, we find that, under the maintained hypothesis of risk aversion only SRA and HARA utility functions agree with the first two conjectures. CARA utility will admit increasing relative risk aversion but imposes a fixed preference parameter to the absolute index. On the other hand, the CRRA form forces the elasticity of absolute risk aversion, with respect to wealth to be equal to -1 over the marginal propensity to consume.

Both SRA and HARA are less restrictive on this aspect. For the SRA specification, with respect to the absolute risk aversion parameter, notice that it will be nonnegative, decreasing and convex in current wealth. Therefore, risk aversion is maintained in absolute terms throughout the consumption and wealth ranges, even though a higher level of wealth will be translated into a greater tolerance toward risk. Furthermore, we do not impose the CRRA restriction relating absolute risk aversion to marginal propensity to consume.

Considering the relations with variance and equity premium, we find that for the CARA utility, the relative index displays the counter-intuitive results of increasing in premium and decreasing in variance. For the CRRA case, the relative risk aversion parameter γ has to be sufficiently high, i.e. greater than 1 for hypotheses iii. and iv. to hold with respect to the absolute index. This is not the case for the SRA which maintains the two conjectures for the absolute index, whereas the relative measure is unaffected by the changes in the distribution. This suggest a different interpretation of the Tobin (1985b) and Markowitz (1959) mean-variance analysis. More precisely, it can be seen that the iso-risk preference curves will be linear in variance, squared-spread space, suggesting a homogenous trade-off relation between risk and squared net return of the risky asset with respect to risk tolerance.

The results for the HARA specification are less clear as they depend on the bliss point η which can be either positive or negative, subject to the restriction that marginal utility remain nonnegative.

Summarising these findings, we find that at the optimum, CARA utility should be rejected when considering relations to wealth, variance and equity premium, whereas CRRA functions yield counter-intuitive results, considering the influence of variance and equity premium at low levels of risk aversion. HARA utility gives ambigous results that depend crucially on the bliss point. As such, we could conclude that both CARA and CRRA are too restrictive in imposing constant preferences with respect to risk, which leads to counter-intuitive results at the optimum, whereas HARA utility is too general to allow for realistic predictions unless restricted further. By comparison, SRA has the advantage of being more parsimonious, with 3 parameters less to estimate than HARA, and 1 less than CRRA and CARA, while giving clear predictions which make economic sense, with the only restrictions imposed, being that r be greater than ρ, for transversality and sufficiency, and that α be greater than r, for feasibility of the leverage schedule.

4- AGGREGATION AND ESTIMATION

4.1 Aggregation

Despite their shortcomings, constant risk preferences specifications remain widely used in the literature. An important reason explaining this popularity relates to their quasi-homotheticity. This translates itself into the fact that the resulting consumption and leverage rules will be linear in wealth, as was shown by Merton (1971;391). It follows that linear aggregation will be made possible, such that the maximisation program may be applied to a single representative consumer. This feature is particularly important to the extent that aggregate time-series data can then be used at the estimation stage.

Generalising the risk aversion parameter immediately raises the question of aggregation. This can be seen from the standard CARA optimal consumption and portfolio choices. If, under the alternative hypothesis, the risk tolerance parameter γ depends on current wealth, then a test for state-dependency becomes a joint test for state- and distributional dependency *as well as* aggregation. Indeed, the constant term will vary with wealth, such that a redistribution of total wealth will affect aggregate demand.

The model derived thus far has been applied to a single consumer, without addressing the issue of his representativeness. In our derivation, it can be seen that, for optimal consumption, exact Gorman aggregation is no longer holding. In fact, by allowing risk preferences to depend on current wealth, we are effectively moving away from the Hyperbolic Absolute Risk Aversion class utility. In our case, we recover a consumption demand function which is not linear in wealth (although the leverage decision still is) such that exact linear aggregation which would yield the representative agent result cannot be invoked.

The reason for this is that the marginal propensity to consume becomes a function of wealth to the extent that the curvature of the utility function depends on current wealth. Therefore, the indirect utility function's curvature as well will be affected by the wealth level and will not be of the general linear Gorman form. From Roy's identity, the derived

Marshallian demand will therefore not be affine in wealth[14].

Yet, the problem is not as acute is it may seem in our case, because the nonlinearity of the Engel curve dissipates very rapidly, being manifest only in the lower ranges of wealth[15]. Bearing these considerations in mind, several alternatives are possible. First, we could assume that individual wealth levels are sufficient to guarantee that aggregation over all agents will take place over the linear range. Considering that wealth will be defined as the sum of tangible assets, including durable goods, it might not seem unreasonable to suppose that net worth for the poorest individuals is still sufficiently high that consumption is homogeneous in wealth. A second option would be to restrict aggregation to the highest wealth levels. Hence, for instance, we could postulate that individual agents invest in risky assets through institutional organisations such as mutual funds, whose combined assets is sufficiently high. A third possibility could be to restrict individual preferences to be identical across agents, and to use an *average* agent, given per capita wealth as the consumer solving the intertemporal problem. Finally, because leverage choice is linear in wealth, estimation could take place for optimal portfolio choice only. This approach would however have the disadvantage of losing the information contained in the form of cross-equations restrictions.

In the next section, we present estimation results for three options, i.e., we start by estimating an "average" agent's consumption and leverage rule, where per-capita consumption is regressed on per-capita wealth. We then restrict estimation over the linear range, by regressing a linearised version of the consumption rule. Finally, we estimate the model for the leverage rule, which does satisfy conditions for exact linear aggregation/representative agent.

Using the linear approximation is unlikely to result in serious bias in estimated pa-

14 This objection does not hold for the optimal leverage choice which is expressed as a constant. Furthermore, using cross-sectional data for estimation obviously eliminates the need to address the aggregation issue.

15 The reason for this is that the marginal contribution of the term term in bracket goes to 0, such that the dominating effect of wealth on consumption is linear.

rameters, as can be seen by rewriting optimal consumption (14) for individual i, where $i = 1 \ldots n$ is ranked according to increasing levels of wealth:

$$C_i^* = \begin{cases} \phi_0 A_i \phi(A_i), & \text{for } i = 1 \ldots l; \\ \phi_0 A_i, & \text{for } i = l+1 \ldots n, \end{cases} \tag{16}$$

where,

$$\phi_0 \equiv \frac{(\alpha - r)^2}{2\sigma^2} \frac{r}{r - \rho}$$

and,

$$\phi(A_i) \equiv \left\{ \frac{-\log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right) \log(A_i)}{1 - \log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right) \log(A_i)} \right\}.$$

In this representation, we assume that for some criteria, the Engel curve can be considered as linear after the first l poorest individuals.

Summing over the n individuals, we can define aggregate consumption, $\bar{C}$ in function of aggregate wealth $\bar{A}$ as:

$$\bar{C} = \phi_0 \sum_{i=1}^{l} (\phi(A_i) - 1) A_i + \phi_0 \bar{A}. \tag{17}$$

This means that at the estimation stage, the consumption equation becomes:

$$\bar{C} = \phi_0 A(\phi - \iota) + \phi_0 \bar{A} + \epsilon, \tag{18}$$

with ϕ being $l \times 1$, ι being $l \times 1$ unit vector and A, $T \times l$. If individual data on consumption and wealth are not available, then the estimated equation proceeds by omitting the relevant first regressor A, such that $\tilde{\phi_0}$, the estimator of ϕ_0 will in general be biased. In fact, the linearisation bias is easily obtained as:

$$E(\tilde{\phi_0} - \phi_0) = \phi_0 E[(\bar{A}'\bar{A})^{-1} \bar{A}' A(\phi - \iota)]. \tag{19}$$

It follows that, since the individual $\phi(A_i)$ are less than 1, and because, by definition, $\bar{A}$ is positively correlated with A, $\tilde{\phi_0}$ will underestimate the true ϕ_0. Yet, the estimator will still be consistent because, as time evolves, individual wealth is increasing. If the first l

individuals' wealth is growing at the same rate as the rest of the population's, this means that l must be falling at rate equal to the growth rate of wealth, for a given non-linearity criterion. In the case of a worsening distribution of wealth, the correlation between $\bar{A}$ and A is falling as well. It follows that estimated bias will fall asymptotically.

We present estimation results in §4.2, discussing first the data sources.

4.2 Estimation

The estimation proceeds with discrete-time approximations of the diffusion processes for equity and the risk-free asset, as well as the two optimal rules, which are estimated as a bivariate system. As such, we have four equations to estimate:

$$\frac{P_t^e - P_{t-1}^e}{P_{t-1}^e} = \alpha + \epsilon_{e,t}, \qquad \epsilon_{e,t} \sim N.I.D.(0, \sigma^2), \tag{20}$$

for the risky asset,

$$\frac{P_t^f - P_{t-1}^f}{P_{t-1}^f} = r + \epsilon_{f,t}, \qquad \epsilon_{f,t} \sim I.I.D.(0, \sigma_f^2), \tag{21}$$

for the risk-less asset, and

$$C_t^* = \frac{(\alpha - r)^2}{2\sigma^2} \frac{rA_t}{r - \rho} \left\{ \frac{-\log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right)\log(A_t)}{1 - \log\left(\frac{\rho}{r}\right) + \left(\frac{r-\rho}{r}\right)\log(A_t)} \right\} + \epsilon_{C,t}, \qquad \epsilon_{C,t} \sim I.I.D.(0, \sigma_C^2), \tag{22}$$

for consumption, and,

$$\omega_t^* = \left(\frac{r}{r - \rho}\right)\left(\frac{\alpha - r}{\sigma^2}\right) + \epsilon_{\omega,t}, \qquad \epsilon_{\omega,t} \sim I.I.D.(0, \sigma_\omega^2), \tag{23}$$

for leverage. Due to the presence of cross-equations restrictions, the most efficient estimator will to be estimate (20)-(23) jointly.

Having recourse to discrete-time estimation will imply that we are using an approximation to the continuous-time process. This follows for the diffusion processes for wealth (6), as well as for the riskless and risky assets, for which integrals appear on the right-hand sides, when changes in the state variables are computed over finite time intervals[16].

[16] This problem is known as the temporal aggregation issue, Grossman, Melino and Shiller (1987).

However, the importance of this problem is likely to be minimal, assuming continuity of the diffusion processes[17]. Furthermore, discrete-time approximation can be justified by asuming that the agent's decisions are taken at the same periodicity as the available data[18].

4.2.1 Data

The series used in the estimation process were obtained from the City Bank, and the International Financial Statistics databases. The series are US quarterly observations for the period 1957:4 to 1992:4.

Wealth We use IFS-I1110172, Fixed assets / real estate as a proxy for total wealth, A_t. The original series consists of monthly observations form 1957:1 to 1993:3, which were reorganised into quarterly data to replicate the periodicity of the other series. We also divide by total population (CITI92-GPOP) and the implicit price index for consumer nondurables (CITI92-GDCN) to obtain real per-capita assets.

This proxy could however underestimate the true wealth, to the extent that no provisions have been made for other components, such as consumer durables, claims to pension and insurance plans as well as social security wealth. [19]

Consumption The consumption series, C_t consists of real per-capita expenditures on nondurables, with aggregate consumption given by CITI92-GCN.

Leverage Following Chou, Engle and Kane (1992:214), we proxy the share of total wealth allocated to equity, ω_t, by dividing corporate profits (CITI92-GPBT) by GNP (CITI92-GNP). The argument put forward by Chou *et al* is that the income flow from stock ownership can be approximated from corporate profits, whereas income from total wealth is

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[17] Chan et al (1992:1213).

[18] Melino (1991:9).

[19] A more comprehensive wealth series has been computed by Wolff (1989), which includes estimates of total assets and liabilities. Unfortunately, the series, which is annual, is incomplete, with 18 estimates over the 1900-1983 period. Rather than interpolating the missing data points, we prefer to use the complete IFS series.

National Income. Assuming both series are integrated of order 1, and for identical discounting, ω_t can be represented by their ratio.

Returns We use Standard and Poor's Common Stock price indices (CITI92-FSPCOM) and dividend yield (CITI92-FSDXP) to calculate the rate of return for the risky asset. The real return, α, is obtained by subtracting the rate of inflation calculated from the nondurables deflator. The real rate of return on the riskless asset is proxied from the 90-days T-Bills series (CITI92-FYGM3), minus the inflation rate.

Table 3 presents summary statistics for the series used in estimation. In addition, we present the correlation matrix.

Table 3: Preliminary Statistics, Correlation Matrix.

SERIES	MEAN	VAR.	ω_t	C_t	A_t	$\Delta P_t^f/P_{t-1}^f$
ω_t	0.086	0.35E-5				
C_t	3713	30E+4	-0.60			
A_t	9511	81E+5	-0.77	0.77		
$\Delta P_t^f/P_{t-1}^f$	0.021	28E-4	-0.10	0.006	0.10	
$\Delta P_t^e/P_{t-1}^e$	0.071	0.079	-0.12	-0.07	0.07	0.14

Notice that the equity premium is evaluated at 5%, and that we obtain a strong correlation between the consumption, wealth and leverage data, which is mainly due to the non-stationarity present in these series. We now present the estimation results, followed by the imputed risk aversion. We close the section by model specification inference, in which SRA is tested against CARA and CRRA specifications.

4.2.2 Estimation results

Tables 4.1 and 4.2 present estimation results, for the four estimated parameters, α, σ^2, r, and ρ, with White's Heteroscedasticity-Consistent standard errors in parentheses. The first reported results in 4.1 concern the non-linear consumption. 4.2 reports estimated parameters for the linearised consumption, i.e. using the asymptotic approximation, where the term in brackets is set equal to 1. The estimation procedure is by Multivariate Maximum Likelihood.

Table 4.1: Estimation Results, Non-linear Consumption.

EQUATIONS.	α	σ^2	r	ρ	LLF
(20),(21)	.0703 (.031)	.0784 (.010)	.0207 (.45E-02)		297
(22),(23)				.0199 (.75E-02)	-1630
(20),(21), (22),(23)	.0858 (.75E-02)	.0735 (.50E-02)	.0200 (.98E-02)	.0186 (.47E-02)	
(20),(21),(22)	.1459 (.031)	.0607 (.99E-02)	.0222 (.45E-02)	.0167 (.53E-02)	-840
(20),(21),(23)	.0703 (.031)	.0784 (.010)	.0207 (.45E-02)	.0192 (.42E-02)	-206

Table 4.2: Estimation Results, Linear Consumption.

EQUATIONS.	α	σ^2	r	ρ	LLF
(22),(23)				.0199 (.19E-04)	-1727
(20),(21),(22)	.0781 (.031)	.0796 (.010)	.0225 (.45E-02)	.0214 (.44E-02)	-832
(20),(21), (22),(23)	.4245 (.029)	-.0734 (.010)	.0185 (.45E-02)	.7478 (.216)	-661

We first obtain starting values by separating the full system, (20) to (23), into two sub-systems: prices [(20), (21)], and optimal rules [(22), (23)]. From the prices system, we obtain α, σ^2, and r, which are then substituted back into the optimal rules to obtain an estimate of ρ. The four estimated parameters are then used as starting values for the Gaussian optimisation procedure[20]. The starting values thus obtained appear consistent with standard results, with an estimated discount factor of .0199, which corresponds to a discrete-time analog of $\beta = e^{-\rho}$ of .98.

The initial parameters, however, failed to yield convergence for the full system (20)-(23). The presence of logs, in the non-linear part of consumption, impaired the search procedure, by affecting the calculation of residuals. For this reason, we used instead a Minimum-Distance estimator, which fixes the contemporaneous covariance matrix at its initial estimate, obtained from the starting values. Evidently, the reported standard errors will not be valid. In this case, convergence was achieved very rapidly (3 iterations) and the result retained their consistency, even though α appears slightly biased upward, with respect to the unconditional mean of the risky rate of return.

We then segmented the optimal rules sub-system and estimated successively prices and consumption, followed by prices and leverage. In the case of consumption the upward bias on α becomes more apparent, whereas for leverage, the result are similar to the starting values.

As suggested in §4.1, a potential answer to the aggregation bias could be to restrict optimal consumption to its asymptotically linear form. For this purpose, the term insides the brackets is restricted to 1, such that consumption becomes affine in wealth. We present the results for the linear model in Table 4.2, first for starting values and then, for the full system, to be followed by the 3-equations system of asset prices and consumption.

The Maximum-Likelihood estimators for the initial ρ from the joint linear consumption/leverage system is very similar to its nonlinear counterpart. However, when the full system is estimated, the results are clearly biased, with an excessive α and *negative* σ^2 [21].

[20] The reported error for the discount factor is not valid, as it fails to take into account the fact that the rates of return and variance have been estimated ex-ante.

[21] It should be noted that the variance was estimated from a separate scedastic function, subject to the estimated α

Yet, for the consumption sub-system, the estimated parameters appear more acceptable.

We conclude that the cross equations restrictions implied by SRA appear too stringent to jointly explain consumption and leverage. The fault seems to be stemming from consumption, with linear approximation doing little to improve performance. On the other hand, leverage results are much more acceptable, and have the advantage of allowing for exact aggregation. We therefore continue our analysis of model specification based on the portfolio sub-system [(20), (21), (23)]. Our first task is to test the independence assumption concerning the error term, and then to test sufficiency and transversality, i.e. $\alpha > r > \rho$. Following this, we compare SRA results for the leverage sub-system, with its CRRA and CARA counterparts[22].

The testing procedure used here is based on artificial regression, or Gauss-Newton regression (GNR) for multivariate models[23]. The GNR merely expresses orthogonality conditions from the maximisation procedure between the residuals and the matrix of derivatives with respect to the parameters in regression form. It can be used for various specification tests, including serial correlation, and non-nested hypotheses testing. More specifically, for the general m-equations nonlinear multivariate case,

$$\mathbf{y}_t = \mathbf{x}_t(\theta) + \mathbf{U}_t, \qquad E\mathbf{U}_t'\mathbf{U}_t = \Sigma,$$

with upper-tringular weighting matrix ψ, where

$$\psi\psi' = \Sigma^{-1},$$

with i'th column denoted ψ_i the GNR can be run in stacked form as:

$$\begin{pmatrix} \mathbf{y} - \mathbf{x}(\hat{\theta})\psi_1 \\ \vdots \\ \mathbf{y} - \mathbf{x}(\hat{\theta})\psi_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} \mathbf{X}_j(\hat{\theta})\psi_{j,1} \\ \vdots \\ \sum_{j=1}^{m} \mathbf{X}_j(\hat{\theta})\psi_{j,m} \end{pmatrix} \mathbf{b} + residuals, \tag{24}$$

22 Unfortunately, the HARA specification cannot be tested for the portfolio system, due to the inclusion of two additional parameters which can be identified only through the full system.

23 Davidson and MacKinnon (1993:313-15).

where $\mathbf{X}_j(\hat{\theta})$ is the $n \times k$ matrix of derivatives evaluated with respect to the k parameters, at the estimated values, in the j'th equation.

The specification tests results are summarised in Tables 5.1 and 5.2. The first table presents tests results for the leverage sub-system, with State-Dependent Risk Aversion, whereas 5.2 gives results for specification tests against CARA, and CRRA models.

Table 5.1: Specification tests, SRA, Leverage sub-system.

TEST	RESULTS								
GNR	T	$\mathbf{X}_\alpha$	$\mathbf{X}_{\sigma^2}$	$\mathbf{X}_r$	$\mathbf{X}_\rho$	$\mathbf{U}_{t-1}$	$\mathbf{U}_{t-2}$	$\mathbf{U}_{t-3}$	$\mathbf{U}_{t-4}$
	.26E-03	-.21E-02	.21	-.29E-02	0.0	.22	-.01	.05	.03
	(.11E-02)	(.03)	(.76)	(.01)	(0.0)	(.09)	(.05)	(.05)	(.05)
ARCH(4)	const.	T	$\mathbf{U}^2_{t-1}$	$\mathbf{U}^2_{t-2}$	$\mathbf{U}^2_{t-3}$	$\mathbf{U}^2_{t-4}$			
	1.13	-.57E-03	.22E-02	.55E-02	-.01	-.19E-02			
	(.41)	(.16E-02)	(.04)	(.04)	(.04)	(.04)			
$\alpha > r > \rho$	$\chi^2_{(2)}$	P-value (2-tails)							
	2.73	.25							

We start by testing for serial correlation, in the form of an AR(4) process, common to the 3 equations. Re-organised in stacked form, the residuals from the three equations are regressed on the derivatives with respect to the parameters, using the appropriate weightting structure, and lagged residuals. We also included a simple trend component to test for deterministic nonstationarity of the error terms. Under the assumption that an optimum has been achieved, the derivatives should have no explanatory power, from first-order conditions, and under the null of serially independent errors, the coefficients on the lagged residuals should be 0.

As can be seen from the GNR results, no derivative is significant, indicating that an optimum has been obtained[24]. The addition of a time trend, T, did not improve the fit significantly. Furthermore, the null of no AR(4) process is not rejected, with only the first lagged residual being significant.

[24] Notice the coefficient on the derivative w.r.t. the discount factor which is exactly 0, indicating perfect collinearity in the matrix of derivatives, due to the multiplicative structure.

We also test for Autoregressive Conditional Heteroscedasticity (ARCH) in the form of an ARCH(4) process. The scedastic function is obtained by expressing squared residuals on a constant, time trend, and lagged squared errors, up to lag 4. Again, no violations from the spherical errors assumption were detected.

Finally, we test model adequacy, in the form of sufficiency, transversality, and feasibility. It will be recalled from §3.2 that sufficiency and transversality conditions can be expressed as a test that $\rho < r$, whereas feasibility of leverage plans requires that $\alpha > r$. We therefore tested the two assumptions jointly, through a Wald test which yielded a two tails marginal value of .25, indicating that the null is not rejected.

These results lead us to conclude that the SRA specification performs reasonably well, in terms of explaining leverage choices. The next step is to test SRA against the other utility specifications. In order to be consistent, we restrict estimation to portfolio rules only. This forces us to abandon estimation of HARA specifications, for identifiability reasons. We therefore turn our attention to CRRA and CARA. We start by estimating the leverage sub-systems for both alternative specifications, and then perform straightforward restrictions tests, in the case of CRRA, and non-nested hypothesis testing for CARA. Full results are presented in Table 5.2.

Testing for SRA against CRRA can be implemented easily as a nonlinear restriction that:

$$\gamma = \frac{r - \rho}{\rho}, \tag{25}$$

in which case CRRA reduces to SRA, if estimation is restricted to the leverage sub-system. This test was performed in two steps, first by regressing the SRA model, and by fixing the γ parameter, estimated from the CRRA model, and secondly, by inversing the procedure, i.e., by holding α, σ^2, and ρ fixed, and using the estimate γ, along with its standard error.

Testing for SRA against is somewhat more complicated due to the impossibility of nesting the two hypotheses within a single test equation. In this case, we use artificial nesting, through multivariate stacked GNR. The P test used here is simply devised a t

statistic on the parameter on the differences in predicted values between the two models, added to the stacked GNR introduced earlier[25]. To simplify, consider the univariate case, with two competing hypotheses, H_1, and H_2:

$$\begin{aligned} H_1: &\qquad y = x(\beta) + u_1 \\ H_2: &\qquad y = z(\zeta) + u_2 \end{aligned}.$$

The two hypotheses may be nested within a more general model:

$$H_C: \qquad y = \pi x(\beta) + (1-\pi)z(\zeta) + u,$$

such that when $\pi = 0$, we recover H_2, and when $\pi = 1$, we recover H_1. To test H_1 against H_C, we simply use the t statistic on a in the artificial regression:

$$y - \hat{x} = \hat{X}b + a(\hat{z} - \hat{x}) + residuals,$$

whereas to test H_2 against H_C, we reverse the procedure as:

$$y - \hat{z} = \hat{Z}c + a(\hat{x} - \hat{z}) + residuals.$$

Keeping in mind that these tests may have poor finite sample properties, we present specification tests results for SRA, first against CRRA, and subsequently against CARA in the following table.

[25] Davidson and MacKinnon (1993:381-88).

Table 5.2: Specification tests, CRRA and CARA, Leverage sub-systems.

MODEL/TEST	RESULTS					
CRRA	α	σ^2		r	γ	LLF
	.0704 (.030)	.0748 (.010)		.0207 (.45E-02)	.1354 (.087)	662
CARA	α	σ^2		r	γ	LLF
	.0722 (.030)	.0806 (.010)		.0204 (.45E-02)	.0422 (.087)	706
H_0 : SRA			$\chi^2_{(1)}$ 1.07			P-value .30
H_0 : CRRA			$\chi^2_{(1)}$ .049			P-value .82
GNR	T	$\mathbf{X}_\alpha$	$\mathbf{X}_{\sigma^2}$	$\mathbf{X}_r$	$\mathbf{X}_\rho$	$\hat{\mathbf{y}}^S - \hat{\mathbf{y}}^C$
H_0 : SRA	.35E-03 (.15E-02)	-.42E-02 (.039)	-.08 (.98)	-.39E-02 (.02)	0.0 0.0	.52 (2.92)
GNR	T	$\mathbf{X}_\alpha$	$\mathbf{X}_{\sigma^2}$	$\mathbf{X}_r$	$\mathbf{X}_\gamma$	$\hat{\mathbf{y}}^C - \hat{\mathbf{y}}^S$
H_0 : CARA	-.42E-03 (.16E-02)	.010 (.04)	.23E-02 (.07)	.50E-02 (.02)	0.0 0.0	.15 (.09)

Considering first the relevance of CRRA as an alternative specification, we can see from the Wald tests of the nonlinear restriction (25) that we do not reject the null that the SRA restriction is valid. Fixing γ, we obtain a P-value of .30, for estimated α, σ^2, r and ρ, whereas using the estimated relative risk aversion index, for fixed rates of return, variance, and discount factor, yielded a P-value of .82. We conclude that SRA appears to explain the leverage data better than CRRA.

Testing for SRA against CARA via artificial nesting also yielded results in favor of SRA. Controlling for serial correlation (not reported), we find that both tests lend support to State-Dependent Risk Aversion[26]. In the first case, SRA is tested against the compound alternative. The coefficient on the differences in predicted values, $\hat{\mathbf{y}}^C - \hat{\mathbf{y}}^S$, thus obtained was .52, with a t statistic of .17, indicating that the null of SRA is not rejected. In the second case, CARA is held as the null against the compound. We obtain a coefficient of

[26] We also tested for ARCH(4) processes and could not reject the null of constant variance.

.15, with a t statistic of 1.52, which lends weak evidence against CARA, in favor of the compound.

Summarising these results, we find the following:

1- SRA gives mixed results when estimated over the full system. In particular, the non-linearities implied by the consumption schedule impair convergence. The problem can be overcome by fixing the rates of return and variance and by estimating consumption and leverage, separated from the diffusion processes for asset prices, or by using a minimum-distance estimator, with fixed contemporaneous covariance matrix.

2- Estimating the leverage sub-system yields sensible results which favor SRA over CARA and CRRA alternatives. In particular, we find that the regularity conditions underlying the model are verified, and that the leverage data used in the estimation is better explained by SRA than other utility specifications.

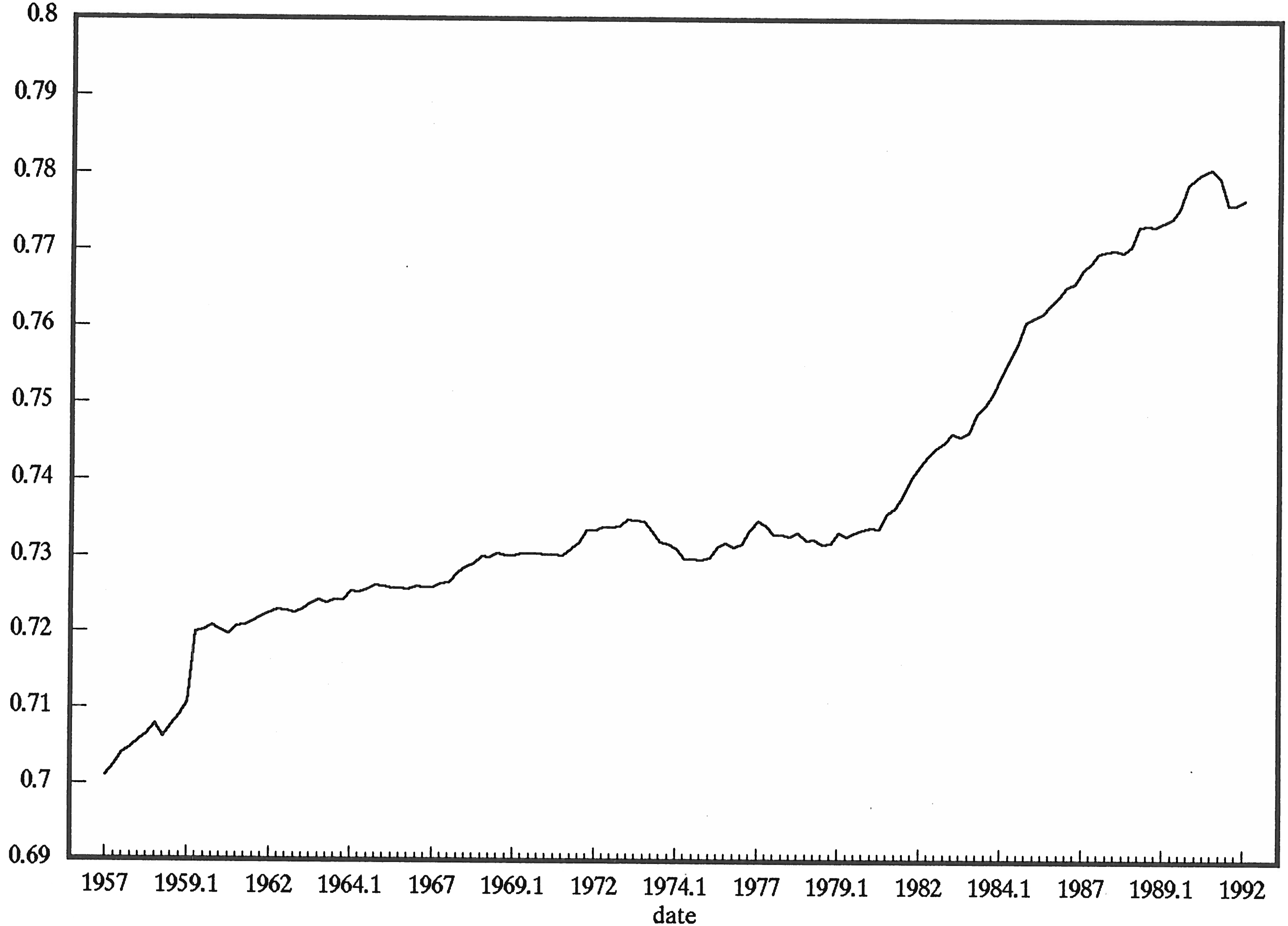
4.2.3 Imputed Risk Aversion

The results presented in the preceding section have important implications for risk aversion. In particular, they suggest that both relative and absolute risk aversion may not be constant over the sample. This can be seen from the following figure which plots relative risk aversion against time. By substituting $\hat{\alpha}$, $\hat{\sigma}^2$, $\hat{r}$, and $\hat{\rho}$, into the SRA relative introduced in Table 2, we recover a relative measure of risk aversion which increases as per-capita wealth trends upward. This suggests that imposing constant risk preferences might be inadequate in explaining consumption and leverage choices.

A natural extension to the model will be to consider the dual of the consumer's problem, i.e. to express SRA implications in prices space, rather than goods space, as was done here. Only then will we be able to verify whether SRA improves the empirical performance of the asset-pricing model in terms of observed paradoxes of the traditional specifications. Yet, initial results suggest that allowing time-variant risk preferences might be a step in the right direction.

IMPUTED RELATIVE RISK AVERSION

SRA UTILITY



5- CONCLUSION

SRA appears to be a promising alternative to constant risk preferences. The idea of more general utility functionals is not new, having been developed by Uzawa (1968) and more recently by Bergman (1985) in the context of the discount factor, as well as by Nason (1991) with respect to the bliss point. Yet, the crucial implications of time-dependent risk aversion have not been explored. The objective of this paper was to show that alternative risk preferences can yield sensible and tractable expressions which retain many of the desirable features of the more standard models.

In our view, the main advantages of SRA are twofold. First, the added flexibility provided by liberalising the risk preference parameter allows for an alternative set of relations between the states, control and preference variables. This provides a new method to look at certain problems associated with standard approaches such as the excessively large equity premium.

Secondly, SRA suits well with the rational expectations hypothesis. In our application, the individual not only uses all available information to determine optimal consumption and diversification paths, given fixed utility parameters, but also uses it to assess his own preferences, given a candidate solution to his maximising problem. In other words, with a known distribution and wealth level, the level of risk tolerance is determined, knowing that it will evolve across time as the opportunity set is modified. This non-stationarity is taken into account when deciding upon alternative consumption and leverage schedules. This translates itself into intertemporally consistent decision-making which uses all available information to a greater extent than is the case with more standard approaches. What this implies is that the agent adjusts his tolerance to risk, fully taking into account the characteristics of the problem that he faces.

Our specification for the risk aversion function yields both intuitively and empirically appealing results. By postulating absolute risk aversion to be declining in wealth and equity premium, and to be increasing in variance, we were able to obtain better estimation results than CARA and CRRA specifications, in terms of explaining leverage choices. Consumption, on the other hand proved more difficult to estimate, due to the presence of nonlinearities in the parameter space, which impaired convergence.

As such, other applications can also be considered. In the demand for insurance case, interesting moral hazard results could be obtained if the distributional parameters are related to the control variables (such as preventive actions) and affect in return the degree of risk aversion. For the real business cycles case, prolonged inertia at the end of a recession could be related to the increased reluctance to take risks when state variables are in an averse situation. Finally, within a static framework, Allais' paradox could be associated to wealth effects on risk tolerance for almost-sure lotteries, in which the secure prizes may effectively be counted as wealth by the individual. In this case, the demand for the risky lottery could exhibit kinks over the distributional range indicating that once a prize becomes sufficiently secure, it is treated as part of total wealth, with other prizes ranked accordingly. Furthermore, SRA could be applied in the labor studies to analyse the impact of changing state variables on the desire for union employment protection.

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