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UNSUSPECTED PERVERSITIES IN THE THEORY OF LOCATION

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DISCUSSION PAPER NO. 88

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SUMMARY

On an infinitely-extensible plane (with uniform customer-density) the socially-optimal configuration of firms is a regular hexagonal lattice. Will the free market necessarily produce this configuration? We argue that the currently-accepted, affirmative answer has been erroneously derived from models in which equilibrium is undefined, and in which equilibrium conditions are asserted rather than being derived from behavioural postulates. We answer the question negatively, showing that in a standard location model: (a) many configurations, including the hexagonal, satisfy the equilibrium conditions (and in no case is zero profits a necessary condition for equilibrium); and (b) if a hexagonal configuration is initially imposed, it is much less likely to persist through successive rounds of entry than is a square or a rectangular configuration.

The great emphasis placed on regular hexagonal market networks in location theory indicates the existence of (1) a strong presumption that this network represents the unique equilibrium configuration when the number of firms in the market is given and/or (2) a strong presumption that free entry will cause this network to prevail. A related presumption in free entry models of location is that (3) free entry will drive profits to zero.

Each of these presumptions is wrong. There is an infinite number of possible equilibrium configurations. Free entry does not necessarily produce regular hexagons, nor does free entry drive profits to zero. The erroneous presumptions have arisen because of methodological errors which seem to be pervasive in the Löschian location literature. Methodology is considered in section I and is followed by a free entry model in section II.

I. Methodology

Condition 1: "The location for an individual [firm] must be as advantageous as possible."

Condition 2: "The locations must be so numerous that the entire space is occupied."

Condition 3: "... in all activities that are open to everyone <u>abnormal profits must disappear</u>." (Lösch's own emphasis)

Condition 4: "... the areas of supply, production, and sales must be as small as possible."

Condition 5: "At the boundaries of economic areas it must be a matter of indifference to which of two neighbouring locations they [consumers] belong."

These are August Lösch's "general conditions of equilibrium that are valid for independent producers and consumers, for agriculture as well as for industry..." [7 , pp. 94-97]. Lösch's five "conditions" are a strange mix of behavioural postulates and conditions which he asserts follow from them. The two behavioural postulates in this list are condition 1, that producers maximize profits, and condition 5, that yields the market boundary between two firms. Lösch reasons that, under "competitive" or free-entry conditions, the equilibrium configuration of firms

> "is determined by two fundamental tendencies: the tendency as seen from the standpoint of the individual firm and hitherto alone considered, to the maximization of advantages; and as seen from the standpoint of the economy as a whole, the tendency to maximization of the number of independent economic units. The latter is affected by competition from without, the former by industrial struggle within" [7 , p. 94]. (Emphasis added).

He then asserts that the "number of independent economic units" is maximized if conditions 2, 3 and 4 are met. Of the three configurations of firms which Lösch considers, the regular hexagonal network alone fulfills these three conditions.

It is well known that (subject to the condition that every point in the space be served) the hexagonal network of market boundaries would be the planner's solution, since it minimizes transport costs for any given size of each firm's market.(1) This result is not at issue here. The question that we consider is whether or not free competition, with each firm seeking to maximize its own private profits will bring about the socially-optimal configuration of firms. If this question is to be answered satisfactorily it is important not to assume the answer at the outset but rather to make behavioural assumptions suitable to the decisions of independent firms and households and then to demonstrate whether or not the configuration of firms that results from the assumed behaviour is socially-optimal.

With the above question in mind, at least two major objections may be raised against Lösch's theory. First, the behavioural assumptions that he does make are not sufficient to permit the definition of equilibrium in his model. Second, the equilibrium conditions that he does give are merely asserted; indeed they cannot be deduced from his behavioural assumptions (even when these assumptions are completed in the manner that seems most in line with the rest of his model). We take these objections in turn.

(a) With respect to a firm's locational decision Lösch's behavioural postulate is that "the location ... must be as advantageous as possible". How is this postulate to be interpreted? A new entrant, or an existing firm considering relocation, must fit into an existing network of firms and hence it will have neighbours. The firm's maximizing behaviour will be dependent upon the assumption it makes about its neighbours' reactions to its own entry or relocation. Lösch does not, however, specify a conjectural variation. The absence of an explicit assumption with respect to conjectural variation is critical,for the concept of equilibrium has no meaning in its absence. We cannot know what profits a firm expects to earn in alternative locations unless we know what reactions it expects from its neighbours.

Similar omissions occur in almost all of the free entry models of spatial competition. Mills and Lav [8], for example, fail to introduce a conjectural variation. They note that

> "we consider only static industry equilibriums in this paper. We do not consider adjustment processes, and we make no attempt to ascertain whether any adjustment process will converge to industry equilibrium from any particular arbitrary initial arrangement of firms and market areas. One way to envisage the adjustment is to assume a tâtonnement process in which no plants are actually built until equilibrium is reached".

Because they fail to specify a conjectural variation, equilibrium is undefined and thus their caveat that they "consider only static industry equilibriums" is without force. Although Beckmann [1] explicitly introduces a conjectural variation with respect to price in his one-dimensional model, he does not do so with respect to location in either his one or two-dimensional model. Telser's one-dimensional model [9] is essentially similar to Beckmann's. He assumes zero conjectural variation with respect to price but does not introduce a conjectural variation with respect to location.

The assumption concerning conjectural variation that seems implicit in the models referred to above is zero conjectural variation (ZCV): each firm in selecting the location that seems best to it takes the location of all other firms as fixed. This is the assumption that we shall adopt.(2)

(b) Lösch's conditions (2), (3) and (4) are not basic assumptions about the profit and utility-maximizing behaviour of firms and customers. Neither, however, are they equilibrium conditions deducible from his behavioural assumptions. It is well known that the space-filling condition (2) is not deducible (see Mills and Lav [8] and Beckmann [1]).(3) Although Beckmann [2 , p.44] has demonstrated that the zero-profit condition (3) is not a necessary condition in Lösch's model, this condition continues to be employed by many writers, including Beckmann [1]. Finally, the densest packing condition (4) is not, as we show below, a necessary equilibrium condition in the model. It seems to us, therefore, that all three conditions are nothing more than arbitrary assertions.(4)

In confining his attention to equilateral triangles, squares and regular hexagons, Lösch implicitly assumes that the network of market areas must be made up of identical, regular, space-filling polygons. We shall show below that this assumption is also incorrect.

Lösch's critical conditions (3) and (4) have been built into virtually all the free entry models of spatial competition. Mills and Lav [8], for example, impose densest packing and zero profits as equilibrium conditions, and the only space filling configurations that they consider are the three regular polygons.(5) The only space-filling configuration considered by Beckmann in [1] is the hexagonal configuration, a procedure which implicitly introduces densest packing. Furthermore, he determines the distance between two firms in this configuration under free entry "by the condition that profits are wiped out." [1, p. 16]. In his one-dimensional model, Telser [9] imposes the condition that free entry drives profits to zero.

II. The Model

We now lay out a locational model, similar in general form to Lösch's model, and investigate its behaviour. We use the simplest model that is consistent with revealing the relations in which we are interested.

(a) The infinitely extensible plane is uniformly populated with customers such that the density is 1 per unit area.

(b) Each firm is faced with the same cost function

$$C = K + cQ$$

where K is fixed costs and c is the constant marginal cost of production. This cost function is commonly used in location theory and we can take c as zero without loss of generality. One way of rationalizing the form of the function is to assume that the only fixed costs are those associated with capital, and that there is an indivisibility in plant size such that the smallest possible plant is large enough to serve any of the markets that we consider. Thus $K \equiv rI$ where r is the rate of return on capital elsewhere in the economy and I is the investment associated with the minimum possible size of plant.

(c) All firms charge the same mill price and consumers bear the cost of transport. The common mill price must exceed c and it is taken to be unity.

(d) Transport costs per unit distance are t, a constant.

(e) All customers buy one unit of the product per period of time. This assumption greatly simplifies the numerical methods employed in this paper. The assumption is not, however, necessary for the results that we obtain and the consequences of dropping it are discussed later.

(f) Consumers buy from the firm whose delivered price is lowest; given the common price, this means that consumers buy from the nearest firm.

(f) In selecting its location, each firm seeks to maximize its profits and takes the location of all other firms as given, i.e., it adopts the assumption of zero conjectural variation (ZCV). Given assumptions (e) and (f), the maximization of the firm's profits is the same thing as the maximization of its market area.

First, consider the case in which firms are allowed neither to enter nor to leave the market. The necessary and sufficient condition for all existing firms to be in an equilibrium configuration (assuming ZCV) is:

(i) No firm can find a new location that offers it a larger market area than that obtained in its present location.

Second, consider the case in which firms are permitted to enter and to leave the market. Condition (i) remains an equilibrium condition, but there are now two further conditions.

(ii) All possible locations for a new entrant within the network of existing firms offer gross profits, π, of less than K. (Gross profits are $\pi = PQ - cQ = Q$, since $c = 0$ and $P = 1$. Because there is one customer per unit of market area and each customer buys one unit of the product per period of time, $\pi = Q \equiv$ market area.)

(iii) No existing firm should earn gross profits of less than K.

Condition (iii) is needed because it is possible for existing firms to have their gross profits reduced to less than K by the entry of new firms. It is also possible for new firms to enter expecting $\pi \geq K$ but to find after entry that $\pi < K$ because of the simultaneous entry of other new firms. Taken together (i), (ii) and (iii) are necessary and sufficient conditions for equilibrium in our free entry model: (i) ensures that no existing firm wishes to relocate elsewhere in the market; (ii) ensures that no new firm wishes to enter; and (iii) ensures that no existing firm wishes to exit.

Equilibrium when the number of firms per unit of area is arbitrarily fixed. We first consider condition (i) and begin by asking: which, if any, of the three configurations of regular, space-filling polygons satisfy the condition? It is interesting that (to our knowledge at least) no answer to this question exists in the literature. The reason is probably that it is extremely difficult, if not impossible, to answer it using conventional analytical methods.(6). We employ computer simulation techniques. The core of the simulation model is an algorithm, described in the appendix, which answers the following question: Given that there are n-1 firms located at the points (X_i, Y_i), i = 1, ..., n - 1, what market can the nth firm expect to control if it locates at an arbitrary point (X_0, Y_0)? Using this approach it is an easy matter to produce a map which describes the market area that the nth firm could expect to have if it located at any one of a large number of alternative points, given the fixed locations of the other n-1 firms. We refer to such maps as market area maps.(7)

The question may now be dealt with as follows. Let all the firms be arranged so that the network of market boundaries is composed of equilateral triangles. Select one firm, and make its location the origin. Allow the firm to consider a large number of alternative locations and calculate the market area that the firm would obtain in each location. The triangular configuration is an equilibrium one if and only if the best location for the firm is at the origin; if the firm wishes to relocate the configuration is not an equilibrium one. Repeat the experiment with the firms arranged so that there is first a square network of market boundaries and second a regular hexagonal network of boundaries.

Figure 1, 2 and 3 show the results. Intuition suggests and calculation confirms that the firm is always better off remaining within the area defined by its present neighbours rather than moving outside of that area to relocate in an already complete portion of the lattice of firms. Our maps are thus confined to this area. The firm's neighbours are shown by circled crosses. The unbracketed numbers indicate the market area that would be obtained by the firm in various alternative locations. (The bracketed numbers give the scales on the X and Y axes.) The broken lines indicate the firm's market boundaries when it locates at the origin, thus completing the regular lattice of firms.

Figure 1 shows that the triangular configuration does not satisfy condition 1: the firm at the origin and hence, any existing firm, wishes to relocate. Figures 2 and 3 show that both the square and hexagonal configurations do satisfy condition (i): the firm at the origin, and hence any existing firm, does not wish to relocate. Thus two of the possible configurations of regular, space-filling polygons satisfy condition (i), and one does not. It is not possible, however, to deduce from our model the necessity that the markets of all firms should be identical, regular, space-filling polygons. We now demonstrate that other types of configurations will fulfill equilibrium condition (i).

An infinite number of configurations of identical rectangles, for example, will fullfill the condition. We know from maps that we have produced that the condition is fulfilled by any rectangular lattice in which the ratio of the long to the short side of the rectangle is 26:10 or less. Figure 4 provides an example in which the ratio of the sides is 9:5. The figure shows the market areas for alternative locations for a firm that is surrounded by a rectangular lattice of other firms. The firm maximizes its market area by locating at the origin thus completing the regular rectangular lattice.

Arrangements of firms that give irregular hexagons as the network of market boundaries also fulfill condition (i). In Figure 5 the firm that is free to move chooses to locate at the origin thus completing the lattice of irregular hexagons. When it does this each firm is separated from two of its neighbours by .40 and from the remaining four neighbours by .45 .

The above results show that equilibrium condition (i) can be fulfilled by an infinite number of configurations that give the firms identical but non-regular hexagonal market boundaries.

Our results suggest the further questions: "can condition (i) be fulfilled (a) if firms have markets that are not identical in shape but which are equal in area? and (b) can the condition be fulfilled if firms do not even have equal market areas?" Although we have not been able to show that identical markets, or even equal market areas, are required by our assumptions, neither have we yet succeeded in finding a configuration of non-identical markets that satisfies condition (i).

It should be noted at this point that our assumption of a uniform parametric mill price in no way affects our results with respect to the multiplicity of equilibrium configurations that we have so far established. As long as firms have identical market areas they must be charging the same mill price. Thus even if price is taken as a variable, the equilibrium situations would be indistinguishable from the ones we have established in all cases in which firms serve identical markets.

Equilibrium configurations under freedom of entry and exit. We have seen that the equilibrium configuration is not unique when the number of firms per unit of market area is arbitrarily determined. We now allow free entry of firms into the market and ask which of the configurations that satisfy condition (i) will also satisfy (ii) and (iii). We restrict our attention to configurations that give all firms identical market areas (since we have not yet established the existence of any other configuration that satisfies condition (i)).

For any given configuration of firms (e.g., squares, rectangles, hexagons) the market area, and hence the gross profits of each existing firm (π^x), is a monotonically decreasing function of the number of firms per unit of market area (n). We write this

$$\pi^x = \pi^x(n)$$

with the restrictions

$$\frac{d\pi^x}{dn} < 0, \quad \lim_{n \to 0} \pi^x(n) = \infty \text{ , and } \lim_{n \to \infty} \pi^x(n) = 0 \text{ .}$$

Condition (ii) is fulfilled whenever the anticipated gross profits of new entrants (π^e) are less than K:

$$\pi^e(n) < K \text{ .}$$

Since the profits that a new firm can expect to earn can be expressed as a fraction of the profits earned by existing firms which are in turn a function of n, we express π^e also as a function of n. Condition (iii) is fulfilled whenever existing firms are earning gross profits of at least K:

$$\pi^x \geq K \text{ .}$$

Any new entrant must fit into an already completed lattice of firms and must expect to earn lower profits than those earned by existing firms before entry occurs. Thus $\pi^e < \pi^x$ and it follows immediately that we can find an n that will allow conditions (ii) and (iii) to be fulfilled simultaneously.

Thus the answer to our question is simply that any of the configurations that give identical market areas and that satisfy condition (i) can be made to satisfy (ii) and (iii) by packing the firms closely enough together so that the expected gross profits of any new entrant is less than K and far enough apart so that the gross profit of any existing firm is K or more. Thus our free entry model has the same multiplicity of equilibrium configurations as does our model where n is fixed.

It follows from the above that not only is there a range of configurations that satisfy all the equilibrium conditions of our model but also for each such configuration there is a range of density of packing (n), and hence of profits of existing firms (π^x), that is consistent with equilibrium. Any n that satisfies the following inequalities is consistent with equilibrium

$$\pi^x(n) \geq K > \pi^e(n) \ .$$

To illustrate we calculate the range of n and of $\pi^x(n)$ compatible with equilibrium when the firms are arranged in a square configuration. We first determine the best point of entry for a new firm. In Figure 6 the existing firms are shown by circled crosses and the figures show the market areas (π^e) for a new entrant in various alternative locations. The market boundaries of the existing firms coincide with the X and Y axes over the segment of the market that is mapped. The four best points of entry for a new firm are circled and they are the points that bisect each of the sides of the existing firms' market boundaries. (Note that the best entry point is not the point equi-distant between the four existing firms - i.e., the origin in Figure 6.)

A firm entering a square network at one of these "best-entry points" expects to get the market boundaries illustrated by dot-dashed lines in Figure 10 (if no other firms are expected to enter the market). The market area it expects is equal to 9/16 of the markets of the existing firms before the new firm enters. Now let X represent the side of the existing firm's square market area. Then, if

$$(9/16)\ X^2 < K, \quad \text{or} \quad X < (4/3)\ K^{1/2}$$

condition (ii) is satisfied. The existing firms need a side of a least $K^{1/2}$ to satisfy condition (iii). Thus the range of possible X which is consistent with free entry equilibrium is

$$K^{1/2} \leq X < (4/3)\ K^{1/2}\ .$$

This implies that the maximum number of firms per unit of area that is consistent with entry equilibrium in the square network is 16/9 (= 1.79) the minimum number. If we take r (the normal rate of return on capital) as 10%, this implies that the maximum rate of profit consistent with a free entry equilibrium is just less than 7/90 (= 6.17%) and the minimum is zero.(8)

The same type of reasoning can be applied to any of the configurations which satisfy condition (i) to determine the range of profits within which (ii) and (iii) are satisfied.(9) We have now seen that equilibrium in a free entry model is consistent with a wide range of configurations of firms (e.g., squares, rectangles, regular and irregular hexagons) and with a wide range of density of packing of firms (and hence of profits for each firm) in each of these configurations. The reason for this latter result is simple: entry stops when potential new entrants can only expect negative profits; but since a new entrant must always expect lower profits than existing firms the absence of entry is consistent with a range of positive profits earned by the existing firms.

The effect of the process of entry on the configuration of firms: We now ask if the dynamic process of entry is more likely to produce one of the possible equilibrium configurations (in particular regular hexagons) rather than any of the others. Many entry dynamics are possible. We shall investigate only one. Following a dynamic suggested for one-dimensional space by Steven Grace [6], we discover the set of points of entry offering the highest possible profits. We then assume that one firm enters simultaneously at each of these points.

We first consider the hexagonal network. Beckmann [2] conjectured that a new entrant would wish to locate at the centroid of the equilateral triangle defined by three contiguous firms. If we let a firm enter at each such point the number of firms is tripled and all firms again have identical, regular, hexagonal market areas. This is illustrated in Figure 7. The original firms are shown by circled crosses and, as a visual aid, lines are drawn between the firms that form hexagons around other firms. The new entrants are shown by dots. In part (B) a small segment of the market is enlarged and the new firms are shown forming a regular hexagonal lattice around each of the original firms.

When we applied our simulation model to the entry problem in a hexagonal network we discovered that Beckmann's conjecture was wrong. Figure 8 shows the market areas for a new entrant in a number of alternative locations when existing firms are located at the circled crosses. Beckmann's conjectured entry points are shown by the six points marked with triangles. The market-maximizing entry points turn out to be much closer to the existing firms than Beckmann conjectured. The best entry points close to the firms located at the origin are marked by squares,(10) and there are six such points around each of the existing firms. If we let a new firm enter at each of these points, we obtain the configuration shown by the circled crosses and the dots in Figure 9. This is not an equilibrium configuration since condition (i) is no longer satisfied for the original firms who now find themselves surrounded by six very near neighbours and who could substantially increase their market areas by relocating outside of their present neighbours. (The best places to relocate are shown by the triangles in Figure 9.) Furthermore, since the new entrants obtain a market smaller than they expected (due to the simultaneous entry of many firms) they will not satisfy condition (iii) unless they entered in the expectation of gross profits sufficiently in excess of K. Since the market of existing firms is smaller than that for the new entrants, it is even more likely that condition (iii) will not be satisfied for them.

In order to carry the analysis a stage further, however, let us assume that the original hexagonal network was so loosely packed that condition (ii) still does not obtain after one round of entry. We now consider the effects of further rounds of entry that occur before existing firms are allowed to relocate or exit. The second and third rounds of entry are shown in Figure 9 by the triangles and the squares respectively. The second round still leaves a very irregular network of firms but a third round, if it occurs, re-establishes a hexagonal network. The three rounds, however, require a 12 fold increase in the number of firms and, furthermore, the new configuration is not even one of regular hexagons. There are no less than four types of market areas. The original firms have regular hexagonal markets, the firms that entered in the three rounds of entry all have irregular hexagonal markets.[11] The firms that entered in the first and second rounds of entry are not even at the centroids of their irregular market areas so that condition (i) is not satisfied for them.

Finally consider allowing those existing firms for which condition (i) is not fulfilled to relocate. We allow the firms that entered in the first round to relocate to their market maximizing position and then do the same for all the firms that entered in the second round of entry. The first-round-entry firms are shown with a dot and the desired relocation is shown for six of them by the arrows in Figure 9. Calculation of the market-maximizing locations for these firms shows that this one set of movements establishes a regular hexagonal network of firms and no further movement is desired by any of the firms in the market. The re-establishment of a regular hexagonal lattice thus requires two sets of adjustments. (1) There must be three rounds of entry coming one after the other before existing firms have a chance to relocate. This means that the firms in the original hexagonal lattice would need to have been earning profits of at least 12K before entry occurred. If they had been earning less entry would stop after one or two rounds and the network would break up as firms for whom condition (i) was not satisfied relocated. (2) Once the irregular hexagonal network is established by three rounds of entry there must be one set of relocations of all those firms that entered the market on the first round of entry.

Thus regular hexagons beget irregular hexagons after three rounds of entry and a twelve-fold increase in the number of firms and regular hexagons only after a further round of relocations of the existing firms. The transitional configuration after one round of entry does not satisfy condition (i) for the original firms who, given a chance, would relocate outside of the area defined by their present neighbours. Thus, if relocation of existing firms is allowed, the whole configuration will break up as existing firms radically shift their locations.

We next consider entry into a square network. Figure 6 shows that the most profitable position for a new entrant is at the midpoint of each of the market boundary segments. Letting a firm enter at each of these points produces the configuration shown by the circled crosses and the dots in Figure 10A. Condition (i) is no longer satisfied for the original firms. (The diamond in the top left hand corner of Figure 10A shows one of the best possible points for relocation of the original firms, such points recur throughout the market in similar locations.) Furthermore, for exactly the same reasons as with the hexagonal network, condition (iii) may not be satisfied for both the new entrants and the original firms. The new entrants are all making profits of $3/8\pi^x$ while the original firms are making profits of $1/4\pi^X$ (where π^X stands for each firm's profits before entry). If profits after entry were high enough to encourage a second round of entry and if this entry occurred before any relocation or exit, the square network would then be re-established. The second round of entry is shown by the triangles in Figure 10B. Condition (i) would now be satisfied for all firms and they would have gross profits of 1/4 of those earned when the firms were in the original square network. Conditions (ii) and (iii) may or may not be satisfied in this configuration. Thus, squares beget squares but only after two generations of entry.

The same experiment can also be conducted in a network of rectangular markets (in which the lengths of the long and short sides of the market boundaries are X and Y). A new entrant's most profitable location is at the midpoint of the short side of an existing firm's market boundary. When firms enter at each such point the number of firms is doubled, and a new configuration of identical rectangular market areas (with sides of lengths $X/2$ and Y) is established. Thus the ratio of the long to the short side is changed by entry(12) but a new configuration of identical rectangular market areas is established. If the configuration satisfied condition (i) before entry it must satisfy it after entry.(13) As in previous cases, the new firms obtain a smaller market than expected so that condition (iii) will not always be satisfied after entry. This process is illustrated in Figure 11 for a rectangular configuration in which the ratio of market sides is 9:5. Each firm enters on the first round expecting the hexagonal market area shown by the dot-dashed lines in the Figure, but, due to the simultaneous entry of other new firms, all firms new and old end up with identical rectangular markets whose sides are in the ratio 10:9 (one such market is shown by the broken lines in the Figure). Thus rectangles beget rectangles (which may or may not satisfy conditions (ii) or (iii)) after each round of entry.

There are of course many assumptions that can be made about the behaviour of new firms entering the market. Our analysis so far, however, does not seem to establish any strong presumption that the process of entry, motivated by the maximization of private profits, will tend to produce hexagons rather than any of the other possible equilibrium configurations.

Entry into a square or a rectangular lattice does not tend to turn it into a hexagonal lattice. Entry into a hexagonal (or a square) lattice produces disequilibrium which will destroy the lattice unless further rounds of entry occur before existing firms relocate. To re-establish hexagons (albeit irregular ones) the number of firms must be increased by twelve fold. Of the three lattices, squares, rectangles and hexagons, the rectangular lattice seems to be the most robust in the face of the type of entry that we have considered since a rectangular configuration that satisfied condition (i) is re-established after each round of entry (and the number of firms is only doubled).

If one wishes to argue a general presumption that the free market would produce a hexagonal configuration the only other obvious remaining possibility would be to hold that if the firms were initially placed haphazardly in the market and allowed to go through a dynamic adjustment process they would end up in a hexagonal configuration rather than in any of the other possible configurations. We see no reason for such a presumption, and to study it would be a very difficult task - if for no other reason than that it is impossible to simulate numerically a dynamic adjustment process where there is an infinite number of firms.[14]

III. Conclusions

The simulation model we have used in this paper was developed for analysis of location problems in a bounded space. We have adapted that model to analyze location in infinite two-dimensional space. The model has the apparently unfortunate property that, because demand is perfectly inelastic and the customer bears the costs of transport, the firm cares only about the size and not the shape of its market. Thus a firm would be indifferent between a hexagonal market and, for example, a square market of equal area. If, however, individual demand is anything but completely inelastic, the hexagonal market would be preferred. It is, therefore, critical that we show that our results hold for at least some non-zero elasticities. With respect to the static equilibrium results we must show that for some non-zero elasticities, condition (i) is satisfied when all firms are placed in a square, rectangular, or irregular hexagonal network. With respect to our dynamic results we must show that for some non-zero elasticities, a new entrant would choose the locations that we have determined for zero elasticities. Both of these results can be established by the following general argument.

Assume that firms continue to sell at a parametric mill price and that the customers bear the transport costs. Now, however, drop the assumption of a zero elasticity of demand and assume that all customers have the same downward-sloping, linear demand curve in which quantity is a function of delivered price.

The importance of making this change is that the firm now does care how far away its customers are. Faced with a choice between a hexagonal and, say, a square market of equal area, the firm would choose the hexagonal market. Firms are not usually presented with such choices, however, and we wish to know whether or not their individual behaviour will give rise to a hexagonal configuration.

Let the individual demand function be linear

$$q = a - b\ (1 + td)$$

where 1 + td is delivered price - the parametric price of 1 plus transport costs of t times the distance d. For any shape of the market area we can write the firms' total demand curve as

$$Q = N[a - b\,(1 + tD)] = Q(D)$$

where N is the number of customers served and D is the average distance from firm to customer. Let P_1 denote the most profitable location under zero elasticity of demand and let P_2 be any other point. We wish to demonstrate that in some circumstances

$$N_1 Q(D_1) \gtrless N_2\, Q(D_2),$$

where subscripts refer to points P_1 and P_2. Since the number of customers is maximized at P_1 we have $N_1 > N_2$. D_2 may or may not be larger than D_1, but for the purpose at hand let $D_2 < D_1$. Then $Q(D_2) > Q(D_1)$. Rewrite the inequality as

$$N_2\, Q(D_1) + (N_1 - N_2)\, Q(D_1) > N_2\, Q(D_1) + N_2[Q\,(D_2) - Q\,(D_1)]$$

or

$$(N_1 - N_2)\, Q(D_1) > N_2[Q\,(D_2) - Q(D_1)]$$

or
$$\frac{N_1 - N_2}{N_2} > \frac{Q(D_2) - Q(D_1)}{Q(D_1)},$$

and finally,

$$\frac{N_1 - N_2}{N_2} > \frac{bt(D_2 - D_1)}{Q(D_1)}.$$

In our model b = 0 and the inequality necessarily holds. For any value of t > 0, there exists a range of positive b for which the inequality holds. Similarly, for any b > 0, there exists a range of positive t for which the inequality holds.

Thus for any t, there is a range of b over which our results hold exactly. In particular, condition (i) can be satisfied by a wide range of configurations which includes squares, regular and irregular hexagons, and rectangles. Further, all of these configurations can be made to satisfy conditions (ii) and (iii) for a range of spacing between the firms and hence a range of non-zero profits. If condition (ii) is not satisfied entry will occur. For b in the relevant ranges: a rectangular configuration prevails after each round of entry; a square configuration is re-established on every other round of entry providing intervening relocation or exit is not allowed; a hexagonal pattern requires three rounds and a twelve fold increase in the number of firms (with no intervening relocations of existing firms) before it is re-established and then only in a non-regular form. Condition (i) does not hold in the configuration established after one round of entry when the original configuration had been square or hexagonal, but condition (i) does hold after one round of entry when the original configuration had been rectangular. If entry occurs in rounds, as it does in our model, the potential oversupply of firms due to excess entry is greatest with hexagons, less with squares and least with rectangles.

These results have obvious but nonetheless interesting applications to markets in which the demand is increasing over time (because, e.g., customer density is everywhere increasing). One round of entry into a hexagonal or a square market configuration produces a situation in which condition (i) is <u>not</u> fulfilled for the firms originally in the market. This is not the case in a rectangular configuration. Thus it seems that even if it were imposed initially, a hexagonal pattern would not long persist in a gradually growing market where existing firms were able to relocate. Furthermore, simultaneous entry of firms at the points offering highest expected profit always results in some disappointment of expectations. Thus, if firms enter a market that is growing as soon as they expect gross profits of K, condition (iii) will necessarily be unfulfilled for all firms after entry whatever the initial configuration. Actual exit may or may not occur depending on how fast the market is growing and upon the durability of the fixed capital.[15]

APPENDIX

The Algorithm is described in Appendix A to our paper "The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spacial Competition", Queen's Discussion Paper No. 87 and so is not repeated here.

(1) See [3] for a careful and rigorous demonstration of this proposition.

(2) ZCV is probably an appropriate assumption either when a stable equilibrium is approached very rapidly so that firms do not have time to learn the opponent's reactions or when relocation occurs only with a very long time lag as in many locational problems. Other conjectural variations can be analysed but in each case the conditions of equilibrium will have to be suitably amended.

(3) Mills and Lav [8] demonstrate the existence of non-space filling configurations that are consistent with the maximization of the number of firms. They unfortunately commit a serious error with respect to the types of market areas which are consistent with the maximization of the number of firms. They assert that any regular polygon with number of sides, s, which is an integer multiple of 6 is a possible market area which is non-space filling and, under certain conditions, yields the largest possible number of firms. Simple inspection of their Figure 2, however, reveals that any firm in the configuration they are considering can have only 6 linear boundary segments since it has only 6 neighbours [8, p. 285]. Beckmann has resolved the "space filling controversy" by demonstrating that "the general shape of the optimal sales region is thus a hexagon <u>with its corners possibly</u> rounded off" ... [1, p. 13].

(4) Michael Webber [10] interprets Lösch's condition 2, 3, and 4 to mean that "society maximizes the number of firms" [10, p. 24]. Our reading of Lösch is that Lösch intended them as equilibrium conditions.

(5) It is interesting that Mills and Lav [8] note that the space-filling condition does not follow from Lösch's behavioural postulates but they seemingly do not apply the same methodological standards to the zero- profit and densest-packing conditions.

(6) Proper analysis of location problems in even one-dimension is very difficult owing to the large number of simultaneous equations involved. "Simplifying" assumptions are usually used to reduce the problem to manageable proportions. These "simplifying assumptions can lead to wrong conclusions. (See Eaton [4].) In two dimensions, the analytical problems become quite intractable with conventional techniques.

(7) The maps can be produced in as much detail as is desired. Those drawn here give only a fraction of the actual points for which market areas were calculated. When a first set of calculations leaves too much uncertainty about the exact location of the market-maximizing point, a second set of calculations can be made for a very large number of points placed in the vicinity of the suspected location.

(8) Rate of net profit $= \frac{\text{gross profits - fixed costs}}{I}$.

But $K = rI$, and r is assumed to be 0.10. Thus the maximum rate of net profit is

$$\frac{16/9\ K - K}{\frac{K}{.10}} = 7/90 \ .$$

If the firm knows that other firms will enter at the same time then it expects to obtain the market area shown by the broken lines in Figure 10, and to earn profits of only 6/16 of those earned by the existing firms before entry. The maximum rate of profit that can be earned without encouraging entry is then 16.66%.

(9) Beckmann [2 , p. 44] has produced a demonstration showing the range of indeterminacy in the hexagonal lattice. Unfortunately, he conjectured the wrong point of entry for a new firm within the lattice, (See our discussion of Figures 7, 8 and 9 below.) and the range that he calculated thus overstates the true range of indeterminacy.

(10) The exact location of the six points required the two rounds of calculations referred to in footnote 7. The location of one of the points was checked

analytically for variations parallel to the X and Y axes and the location of the other five followed from considerations of symmetry.

(11) Beckmann's conjectured entry points located the new firms one third of the distance from the base to the apex of the equilateral triangle joining three neighbouring firms. If the entry point were two thirds of this distance regular hexagons could be re-established by three rounds of entry. As it turns out the firms go less than this distance and a regular hexagonal network can never be established unless some existing firms are relocated.

(12) The original ratio of the long to the short side of the market is X:Y. There are several cases. (a) $Y < X < 2Y$. After entry, Y becomes the long side and the new ratio becomes 2Y:X. After a second round of entry the ratio becomes 2X:2Y which is, of course, the original ratio. In the special case in which $X:Y = \sqrt{2}$ the ratio is unchanged after each round of entry since if $X/Y = \sqrt{2}$ then $2Y/X = \sqrt{2}$.
(b) $X = 2Y$. One round of entry creates a square configuration and after this the analysis of entry into square markets is appropriate. This is the special case in which rectangles do not beget rectangles. (c) $2Y < X$. In this case the short side remains the short side after entry, and the ratio of the long to the short side becomes X:2Y. Entry proceeds through r rounds until $X < 2^r Y$ and after this the analysis of case (a) applies.

(13) (a) If $X < 2Y$ the ratio of the long to the short sides alternates between two values X:Y and 2Y:X which must both lie between the bounds of 1 and 2. (b) if $X > 2Y$ the ratio declines through successive rounds of entry until it reaches a value of less than 2 after which it alternates as in (a) above. Thus if the ratio of the sides is small enough so that condition (i) is satisfied in the initial configuration, entry can never cause it to increase to the point at which condition (i) is not subsequently satisfied.

(14) We have argued elsewhere [5] that there may be no equilibrium configuration when the model is transferred to a disc. The problem of disequilibrium in infinitely extensible space cannot be studied by putting a boundary around one part of the space because the imposition of such a boundary appears to remove the possibility of the existence of an equilibrium configuration.

(15) We have demonstrated some conditions under which our results hold exactly. If these conditions are not met we see no reason to believe that there would cease to be multiple equilibria, or that entry into an imposed hexagonal lattice would recreate such a network.