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Implementation in Generic Environments

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Abstract

In this paper, we study implementation in "economic environments". It is shown that there is a dense subset of the set of preference profiles such that given an arbitrary social choice function, f, and $\varepsilon > 0$, $\exists$ another social choice function f_ε, f_ε within ε of f uniformly, and f_ε implementable in Nash equilibrium on the dense subset.

1 Introduction

In the classical implementation problem, a social planner's objective is represented by a social choice function (or correspondence) which associates outcomes to the characteristics of the agents as represented by their preferences. Since preferences are unobservable, the problem arises of designing a mechanism whereby agents with any given collection of preferences are led via the mechanism to the outcome specified by the social choice function at those preferences. In the case where such a mechanism exists, the social choice function is said to be "implementable". The work of Maskin (1977) considers the question of characterizing those social choice functions that can be implemented in normal form games using Nash equilibrium as the solution concept. This is described in further detail below.

Recent literature has addressed the problem of implementation in two ways. In one approach the solution concept is varied while in the second, the problem is reformulated. The first approach is adopted by Moore and Repullo (1988) and Palfrey and Srivastava (1991). Moore and Repullo study the question of implementation using extensive form games and subgame perfection as the solution concept while Palfrey and Srivastava consider the impact of using undominated Nash equilibrium in the normal form. An important conclusion of this line of research is that the set of implementable social choice functions is greatly enlarged (relative to those implementable in Nash equilibrium), when solution concepts other that Normal Form Nash equilibrium are used. A second approach is given in Abreu and Sen (1991) and Matsushima (1988) where preferences on lotteries over states is considered and the criterion of virtual implementation used (implementation of a social choice function which is arbitrarily close) with Nash equilibrium as the solution concept. (See also Abreu and Matsushima (1990).) Moore (1991) gives a useful survey of the literature and an extensive list of references.

In this paper we provide quite a different perspective on the implementation problem. Rather than vary the solution concept, we consider the impact of varying the underlying set of preference profiles. This is done in the context of economic environments - where agent's preferences are continuous monotone preferences on some subset of euclidean space. The main result is that the Maskin requirements (that a social choice function satisfy weak no veto power and monotonicity to be implementable in Nash equilibrium), are not robust to perturbations of the preferences and social choice function.

2 Preliminaries

The environment consists of a set of outcomes A, a set of preferences over outcomes, $\mathcal{U}$, and a collection of agents, $\{1, 2, \ldots, n\}$. In this environment, a preference profile is a vector: $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{U}^n$. In addition, there is a planner. The planner's preferences are represented by a social choice function which relates outcomes to preference profiles. Formally:

Definition 1 *A social choice function f is a mapping from $\mathcal{U}^n$ to* $\mathbf{A}$.

Thus, when the preference profile is $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, the desired outcome of the planner is $a = f(\mathbf{u})$. However, the planner faced with n individuals does not know the preferences of each individual. Typically, agents if asked have an incentive to misrepresent preferences. Suppose that f satisfies $a = f(u_1, u_2, \ldots, u_n)$ and $a' = f(u_1', u_2, \ldots, u_n)$, and that the "true" preference profile is $(u_1, u_2, \ldots, u_n)$. Then if $u_1(a') > u_1(a)$, agent 1 has the incentive to claim the preference u_1'. In the implementation literature, the task is the design of an environment (game form) whereby the interaction of agents leads to the desired outcome: when the preference profile is $\mathbf{u}$, the unique equilibrium outcome in the environment is $a = f(\mathbf{u})$. This issue is discussed in Maskin (1977) where two key requirements of a social choice function are identified: "weak no veto power" and "monotonicity". These properties are sufficient to characterize "implementable" social choice functions.

Definition 2 *A social choice function f satisfies* **weak no veto power** *on $\mathcal{U}^n$ if $\forall \mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{U}^n$, if given i, $\forall j \neq i$, $u_j(a) \geq u_j(b)$, $\forall b \in A$, then $a = f(u)$.*

Weak no veto power is considered a very mild assumption. In economic environments with three or more agents, where the social choice function specifies allocations of goods and where goods are desirable the condition is satisfied trivially since no two agents will agree on the "best outcome".

Definition 3 *A social choice function f is* **monotonic** *on $\mathcal{U}^n$ if $\forall \mathbf{u}, \mathbf{u}' \in \mathcal{U}^n$, $\forall a \in \mathbf{A}$,*

1. *$a = f(\mathbf{u})$ and*

2. *$\forall b \in \mathbf{A}, \forall i = 1, 2, \ldots, n,\ u_i(a) \geq u_i(b) \Rightarrow u'_i(a) \geq u'_i(b)$*

Then $a = f(\mathbf{u}')$.

Whether a social choice function, f, is monotone or not depends critically on the domain $\mathcal{U}$. It may be useful to express monotonicity in a slightly different manner. Let $u \in \mathcal{U}$ and define $\mathbf{L}(u, a) = \{b \in \mathbf{A} \mid u(b) \leq u(a)\}$. $\mathbf{L}(u, a)$ is the "lower contour set of u at a". With this notation, observe that a social choice function is monotonic if $a = f(\mathbf{u})$ and $\forall i, \mathbf{L}(u_i, a) \subseteq \mathbf{L}(u'_i, a)$, then $a = f(\mathbf{u}')$. In words, if at a point $a = f(\mathbf{u})$, the lower contour set of each agent at profile $\mathbf{u}$ is nested in the lower contour set at profile $\mathbf{u}'$, then a must be in the social choice function at the preference profile $\mathbf{u}'$.

To formulate the implementation problem, a "game form" specifies the environment in which agents interact. To each player $i \in \{1, 2, \ldots, n\}$ is attached a strategy space. In addition, a rule is given which associates outcomes to actions chosen by the agents.

Definition 4 *A game form $G = (\{S_i\}_{i=1}^n, g)$, is a collection of action spaces, $S_1, S_2, \ldots, S_n$, and a mapping $g : S \rightarrow \mathbf{A}$, $S = \times_{i=1}^n S_i$.*

A preference profile $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and a game form G define a game where the action vector $s \in S$ leads to outcome $g(s)$ with associated payoff $u_i(g(s))$ for agent i. The outcome a^* is a Nash equilibrium outcome if $\exists s^* \in S$, $g(s^*) = a^*$ and $\forall i, \forall s_i \in S_i$, $u_i(g(s^*)) \geq u_i(g(s_i, s^*_{-i}))$, where (s_i, s^*_{-i}) is obtained from s^* by replacing the i^{th} component, s^*_i with s_i. Given a preference profile, $\mathbf{u}$, and a game form G, the associated set of Nash equilibrium outcomes is denoted $\mathbf{N}_G(\mathbf{u})$.

Definition 5 *The game form G* **implements** *the social choice function f on $\mathcal{U}^n$ if $\mathbf{N}_G(\mathbf{u}) = f(\mathbf{u}), \forall \mathbf{u} \in \mathcal{U}^n$. A social choice function f is* **implementable** *on $\mathcal{U}^n$ if $\exists$ a game form G which implements f on $\mathcal{U}^n$.*

The following theorem (Maskin (1977)) characterizes implementable social choice functions.

Theorem 1 *Let f be an n person social choice function. If f is implementable in Nash equilibrium then it is monotonic. Conversely, when $n \geq 3$ and f satisfies both weak no veto power and monotonicity, then it is implementable.*

Proof: See Maskin (1977).

■

3 Economic Environments

Throughout the paper we focus exclusively on "economic environments", where an economic environment is essentially the classical exchange model. Let $X \subset \mathcal{R}^k$ where $\mathcal{R}$ is the real line, $\mathcal{R}^k = \times_{i=1}^k \mathcal{R}$ and X a compact connected set with nonempty interior (and we assume that $k \geq 2$). Let $\mathcal{F} = \{u : X \to \mathcal{R}\}$. Thus X is the consumption set of each agent and $\mathcal{F}$ the set of "possible preferences" over points in X. The sets of continuous, continuous differentiable (ie. $\partial u / \partial x_i$ continuous for each i) and continuous twice differentiable (ie. $\partial^2 u / \partial x_j \partial x_i$ continuous for each i, j) functions from X to $\mathcal{R}$ are denoted, respectively, $\mathcal{C}^o(X, \mathcal{R})$, $\mathcal{C}^1(X, \mathcal{R})$ and $\mathcal{C}^2(X, \mathcal{R})$. The set of utility functions, $\mathcal{U}$, is assumed to be continuous (at least), so that $\mathcal{U} = \mathcal{C}^j(X, \mathcal{R})$ for some $j = 0, 1, 2$. In addition elements of $\mathcal{U}$ are assumed to be (strictly) monotonic: if $u \in \mathcal{U}$, $x, y \in X, x \neq y$ and $x_j \geq y_j, j = 1, \ldots, k$, then $u(x) > u(y)$. An n-person preference profile is an element $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{U}^n$. This relates to the previous notation by letting $\mathbf{A} = X^n$. Also, taking $a = (x_1, x_2, \ldots, x_n)$, with some abuse of notation $u_i(a) = u_i(x_i)$. If we impose feasibility or endowment constraints, let $e^k > 0$ be the total endowment of good k. Define $F \subset \mathcal{R}^{nk}$ according to $F = \{(x_i, x_2, \ldots, x_n) \mid x_i \in \mathcal{R}^k, \sum_{i=1}^n x_{ij} \leq e_j, j = i, \ldots, k\}$. In this case, set $A = X^n \cap F$. For much of the discussion, we use the sup norm on $\mathcal{U}$ to determine "closeness" of preferences.

Definition 6 *Let $u \in \mathcal{F} \cap \mathcal{C}^o(X, \mathcal{R})$, then $\|u\| = sup_{x \in X} \mid u(x) \mid$.*

4 Preference Approximation

Recall that a social choice function is monotonic if $a \in f(\mathbf{u})$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, and for each agent i, the lower contour set of u_i through a lies below the lower contour set of some other preference u_i' for each i, $\mathbf{u}' = (u_1', u_2', \ldots, u_n')$, then $a \in f(\mathbf{u}')$. If for each point a in the range of f the lower contour sets of different preferences are nonnested, or equivalently, if the level surfaces (indifference curves) intersect at each a, then the condition of monotonicity imposes no requirements on f.

Cobb-Douglas preferences provide is a simple example illustrating "good intersecting properties". Let $X = [0,1]^2$ and let $\mathcal{U} = \{u : X \to \mathcal{R} \mid u(x,y) = x^\alpha y^{1-\alpha}, 0 < \alpha < 1\}$. In this case, the marginal rate of substitution at (x, y), given α is $(\frac{\alpha}{1-\alpha})(\frac{y}{x})$, so that at every point in the domain (every (x, y) pair), the indifference curves cut and the lower contour sets are never nested. In this case every social choice function satisfies monotonicity: definition 3 is satisfied trivially since 2 of definition 3 is never satisfied. The content of the following discussion is to show that this situation is "generic". Before presenting the details, we first give a brief summary.

If $g : X \to \mathcal{R}$ denote the gradient of g by $\nabla g = (\partial g/\partial x_1, \partial g/\partial x_2, \ldots, \partial g/\partial x_n)$. If two preferences $u, v \in \mathcal{U}$ have the same tangency at a point $x \in X$, then the normals to the indifference curves of u and v are the same and $u(x) = \varphi \cdot v(x)$. Conversely, if the normals differ, then the indifference curves of u and v cut at x. If every preference pair has this property at x, then the requirement of monotonicity can never be violated at x. The denseness result gives a dense subset of $\mathcal{U}$ such that u, v in this dense set implies that u and v have intersecting indifference curves at "most" points in X: There is no open set, $\mathcal{O}$, and function φ defined on $\mathcal{O}$ such that $u(x) = \varphi(x)v(x)$ on $\mathcal{O}$. The main results of this section are:

1. If $\mathcal{U} = \mathcal{C}^0(X, \mathcal{R})$ is the set of preferences, then $\exists$ a dense (in sup norm) set of functions, $\mathcal{U}_d$ in $\mathcal{U}$, such that if $u, v \in \mathcal{U}_d$, then the gradients of u and v agree on at most a set in X with empty interior. Precisely, $\nabla u(x) = \varphi(x)\nabla v(x)$ on at most a closed set with empty interior. See theorems 4 and 5.

2. This result **cannot** be extended from "dense" to "open dense": If $\mathcal{D}$ is an open set in $\mathcal{U}$, then there are functions $u, v \in \mathcal{D}$, and open sets $Y_1, Y_2 \subset X$, $Y_1 \cap Y_2 = \emptyset$ such that $u = v$ on Y_1 and $u \neq v$ on Y_2. See theorem 6.

3. If $\mathcal{U} = \mathcal{C}^1(X, \mathcal{R})$ and $\mathcal{U}^* = \mathcal{U} \times \mathcal{U} = \mathcal{C}^1(X, \mathcal{R}^2)$ with the Whitney topology, then $\exists$ an **open dense** set $\mathcal{U}_d^*$ in $\mathcal{U}^*$, such that for each $w = (u, v) \in \mathcal{U}_d^*$, the associated Jacobian is singular only on a set of lower dimension than X (and hence has empty interior). See theorem 7.

4. Comment 2 remains valid if the Whitney topology is used instead of the sup norm topology. The property of open denseness typically in equilibrium theory does not carry over to the social choice environment.

The remainder of this section proves these results. We use the following notation. If u and v are two functions, $u, v : \mathcal{R}^k \to \mathcal{R}$, define $\mathbf{J}_{uv}^{ij}$ by:

$$\mathbf{J}_{uv}^{ij} = \begin{bmatrix} u_{x_i} & u_{x_j} \\ v_{x_i} & v_{x_j} \end{bmatrix}$$

Given a function $h = (h_1, h_2, \ldots, h_n) : \mathcal{R}^k \to \mathcal{R}^n$, denote the Jacobian matrix $\mathbf{J}_h$ or $\mathbf{J}_h(x)$ if evaluated at x.

$$\mathbf{J}_h(x) = \begin{bmatrix} \frac{\partial h_1(x)}{\partial x_1} & \frac{\partial h_1(x)}{\partial x_2} & \cdots & \frac{\partial h_1(x)}{\partial x_k} \\ \frac{\partial h_2(x)}{\partial x_1} & \frac{\partial h_2(x)}{\partial x_2} & \cdots & \frac{\partial h_2(x)}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n(x)}{\partial x_1} & \frac{\partial h_n(x)}{\partial x_2} & \cdots & \frac{\partial h_n(x)}{\partial x_k} \end{bmatrix}$$

The denseness result is based on approximation by polynomials. A key property of polynomials and (more generally) of analytic functions, is that the parameters of such functions are fully determined by the value of the function on *any* open set.

Let $\mathcal{Z}$ denote the set of nonnegative integers, $\mathcal{Z}^k$ the n-fold product of $\mathcal{Z}$. Given $\gamma = (j_1, j_2, \cdots, j_k) \in \mathcal{Z}^k$ and $x \in X \subset \mathcal{R}^k$, write $x^\gamma = x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$.

Definition 7 *A function f on $\mathcal{R}^k$ is analytic if for any $a = (a_1, a_2, \ldots, a_k) \in \mathcal{R}^k$, $\exists \mathcal{N}_a$, a neighborhood of a and $c_\gamma \in \mathcal{R}, \gamma \in \mathcal{Z}^k$ such that for $x \in \mathcal{N}_a$ the series $\{c_\gamma(x-a)^\gamma\}_{\gamma \in \mathcal{Z}^k}$ is absolutely summable ($\sum_{\gamma \in \mathcal{Z}^k} \|c_\gamma(x-a)^\gamma\|$ converges) and*

$$f(x) = \sum_{\gamma \in \mathcal{Z}^k} c_\gamma(x-a)^\gamma, \forall x \in \mathcal{N}_a$$

An important class of analytic function is the class of polynomials. A polynomial p on $\mathcal{R}^k$ is a function of the form:

$$p(x) = \sum_{j_1, j_2, \cdots, j_k}^{m_1, m_2, \cdots, m_k} b_{j_1 j_2 \cdots j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}.$$

The key theorem on analytic functions is:

Theorem 2 *Let f and g be two real valued analytic functions on $Y \subset \mathcal{R}^k$, Y an open connected subset. If there exists an nonempty open set, $\mathcal{O} \subseteq \mathcal{R}^k$, such that $f(x) = g(x)$ on $\mathcal{O}$, then $f(x) = g(x)$ on Y.*

Proof: See Dieudonné.

■

An immediate implication of theorem 2 is the following corollary 1.

Corollary 1 *If f and g are two real valued analytic functions which do not agree on some open set in $\mathcal{R}^k$, then they do not agree on any open set in $\mathcal{R}^k$.*

The next theorem connects the representation of preferences to the slopes of indifference curves.

Theorem 3 *Let u, v be continuous functions with continuous partial derivatives on $X \subset \mathcal{R}^k$. If $\exists \mathcal{O} \subset X$, $\mathcal{O}$ open, $\mid \mathbf{J}_{uv}^{ij} \mid = 0$ on $\mathcal{O}$ and some $\bar{x} \in \mathcal{O}$ with $v_i(\bar{x}) \neq 0$ for some i, then $\exists$ an open neighborhood of $\bar{x}$, $\mathcal{O}_{\bar{x}}$, with $u(x) = \psi(v(x))$ on $\mathcal{O}_{\bar{x}}$ (and so $\nabla u(x) = \psi'(v(x)) \nabla v(x), \forall x \in \mathcal{O}_{\bar{x}}$). Conversely, if $\exists \mathcal{O} \subset X$, $\mathcal{O}$ open and $\varphi : \mathcal{O} \to \mathcal{R}$ such that $\nabla u(x) = \varphi(x) \nabla v(x), \forall x \in \mathcal{O}$, then $\exists \psi : \mathcal{R} \to \mathcal{R}$ such that $u = \psi \circ v$ on $\mathcal{O}^*$, where $\mathcal{O}^*$ is an open subset of $\mathcal{O}$.*

Proof: Let u, v be as given. By assumption, there is some non zero partial derivative at a point $\bar{x} \in \mathcal{O}$, say $v_1 \neq 0$ (otherwise both u and v are constant on $\mathcal{O}$ and the theorem is trivially true). Continuity of the partial derivatives implies that $v_1 \neq 0$ on a (open) neighborhood of $\bar{x} \in \mathcal{O}$, so we can apply the implicit function to the equation

$$0 = G(v, x_1, x_2, \ldots, x_n) = v - v(x_1, x_2, \ldots, x_n)$$

to obtain $x_1 = x_1(v, x_2, x_3, \ldots, x_n)$. This equation becomes an identity on a neighborhood of $\bar{x}$ if we write $x_1 = x_1(v(x_1, x_2, \ldots, x_n), x_2, x_3, \ldots, x_n)$. Define

$$F(v, x_2, x_3, \ldots, x_n) = u(x_1(v, x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n).$$

Again, we have an identity on a neighborhood of $\bar{x}$,

$$u(x_1, x_2, \ldots, x_n) = F(v(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n),$$

so that on this neighborhood, $u_{x_1} = F_v v_{x_1}$, and $u_{x_j} = F_v v_{x_j} + F_{x_j}, j = 2, 3, \ldots, n$. Therefore, on a neighborhood of $\bar{x}$

$$\mathbf{J}^{1j}_{uv} = \begin{bmatrix} u_{x_1} & u_{x_j} \\ v_{x_1} & v_{x_j} \end{bmatrix} = \begin{bmatrix} F_v v_{x_1} & F_v v_{x_j} + F_{x_j} \\ v_{x_1} & v_{x_j} \end{bmatrix} = \begin{bmatrix} F_v & F_{x_j} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{x_1} & v_{x_j} \\ 0 & 1 \end{bmatrix}$$

Since $\nabla u(x) = \varphi(x) \nabla v(x)$ on a neighborhood of $\bar{x}$, $\mathbf{J}^{1j}_{uv}$ has a zero determinant on a neighborhood of $\bar{x}$: $\mid \mathbf{J}^{1j}_{uv} \mid = 0 = -F_{x_j} v_{x_1}$. Because the choice of $j \in 2, 3, \ldots, k$ is arbitrary, and $v_{x_1} \neq 0$, this implies that on an open neighborhood of $\bar{x}, F_{x_j} = 0$, for $j = 2, 3, \ldots, k$. Thus, on this neighborhood,

$$u(x_1, x_2, \ldots, x_n) = F(v(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n) = \psi(v(x_1, x_2, \ldots, x_n)).$$

Conversely, Let u, v be as given with $\nabla u(x) = \varphi(x) \nabla v(x)$ on $\mathcal{O}$ for some function φ. Then $\mid \mathbf{J}^{ij}_{uv} \mid = 0$ on this neighborhood, and we have $u = \psi(v)$ on some open subset in $\mathcal{O}$.

■

In view of theorem 2 and the observation following it (corollary 1), in the present context we have:

Corollary 2 *Suppose that u, v are polynomials on an open connected set, $Y \subset \mathcal{R}^k$. If the matrix $\mathbf{J}_{uv}^{ij}$ is nonsingular on some open set $\mathcal{O} \subset Y$, then there is no open set $\mathcal{N} \subset Y$ on which this matrix is singular.*

Proof: Suppose that $|\mathbf{J}_{uv}^{ij}| = 0$ on some open set $\mathcal{N} \subset Y$. Observe that $|\mathbf{J}_{uv}^{ij}| = (u_{x_i}v_{x_j} - u_{x_j}v_{x_i})$ is a polynomial, and since polynomials are analytic, $|\mathbf{J}_{uv}^{ij}| = 0$ on $\mathcal{N}$ implies $|\mathbf{J}_{uv}^{ij}| = 0$ on Y, (using theorem 2).

■

Thus, if we can find an open set on which $|\mathbf{J}_{uv}^{ij}| \neq 0$, there is no open set on which this is zero and hence no open set on which $\nabla u(x) = \varphi(x)\nabla v(x)$ for some function v. In this case, we have intersection of indifference curves "almost everywhere" (meaning except on a closed set with empty interior). This is the content of the next theorem.

Theorem 4 *Let $\mathcal{U} = \mathcal{C}^o(X, \mathcal{R})$. There exists a dense subset of $\mathcal{U}$, $\mathcal{U}_d$, such that for any $u, v \in \mathcal{U}_d$, $\mathbf{J}_{(u,v)}$ has full rank on an open dense subset of X.*

Proof: Since $\mathcal{U}$ are continuous functions on X, X a compact subset of $\mathcal{R}^n$, the space of polynomials on X is dense in $\mathcal{U}$ by the Stone Weierstrass theorem. Taking only rational coefficients implies that there is a countable collection of polynomials dense in $\mathcal{U}$. Let $\{p_i\}_{i\in\mathcal{Z}}$ be the collection. For each $u \in \{p_i\}_{i\in\mathcal{Z}}$ we can assume that there is no open set on which $\partial u/\partial x_j = 0, j \in \{1, 2, \ldots, k\}$. (If for some $u \in \{p_i\}_{i\in\mathcal{Z}}$, $\partial u/\partial x_j = 0$ on an open set, then u is independent of x_j on an open set, and hence by theorem 2 is independent of x_j on the entire domain. Let $\mathbf{j}_u$ denote the set of variables with respect to which u has a zero derivative on an open set. Define $u^n = u + \frac{\eta}{n}\sum_{j\in\mathbf{j}_u} x_j$, η a small positive number, and replace u by the collection $\{u^n\}_{n=1}^{\infty}$).

Now, consider p_1 and p_2. Define $P(1) = \{p_1\}$. If there is no open set $\mathcal{O}$ in X such that for all i, j, $(p_{1i}p_{2j} - p_{2i}p_{1j}) = 0$ on $\mathcal{O}$ (where $p_{ij} \equiv \partial p_i/\partial x_j$), then define $P(2) = \{p_2\}$. Otherwise, pick some i and let r be the highest power of any variable appearing in p_1 or p_2, and define $p_2^n = p_2 + \frac{\varepsilon}{n}x_i^{2r+1}$, $0 < \varepsilon$, ε small. Thus on the open set,

$$\begin{aligned}(p_{1i}p_{2j}^n - p_{2i}^n p_{1j}) &= (p_{1i}p_{2j} - [p_{2i} + (2r+1)\frac{\varepsilon}{n}x_i^{2r}]p_{1j}) \\ &= -(2r+1)\frac{\varepsilon}{n}x_i^{2r}p_{1j}\end{aligned}$$

Since $p_{1j} \neq 0$, this is non zero on an open set. By corollary 2, there is no open set on which this determinant is zero. Next, if we consider the sequence $\{p_2^n\}$, then for $m \neq n$,

$$\begin{aligned}(p_{2i}^n p_{2j}^m - p_{2i}^m p_{2j}^n) &= (p_{2i}^n p_{2j} - p_{2i}^m p_{2j}) \\ &= [p_{2i} + (2r+1)\frac{\varepsilon}{n} x_i^{2r}]p_{2j} - [p_{2i} + (2r+1)\frac{\varepsilon}{m} x_i^{2r}]p_{2j} \\ &= (2r+1)[\frac{\varepsilon}{n} - \frac{\varepsilon}{m}]x_i^{2r} p_{2j}\end{aligned}$$

This is nonzero on an open set, and hence nonzero on every open set. Define $P(2) = \cup_n \{p_2^n\}$. Now, consider p_3. If for each $u \in P(1) \cup P(2)$, there is no open set such that for all i, j $p_{3i}u_j - u_i p_{3j} = 0$, then set $P(3) = \{p_3\}$. Otherwise, there is such an i, j pair. Let r be the highest power of i appearing in $P(1) \cup P(2) \cup \{p_3\}$. Define $p_3^n = p_3 + \frac{\varepsilon}{n} x_i^{2r+1}$. Then, given $u \in P(1) \cup P(2)$,

$$(p_{3i}^n u_j - u_i p_{3j}^n) = [(p_{3i} + (2r+1)\frac{\varepsilon}{n} x_i^{2r}]u_j - u_i p_{3j}.$$

Since, $[(2r+1)\frac{\varepsilon}{n} x_i^{2r}]u_j$ is nonzero and the only term in the expression with x_i raised to the power of $(2r+1)$ or greater, the expression is nonzero on an open set. Similarly,

$$\begin{aligned}(p_{3i}^n p_{3j}^m - p_{3i}^m p_{3j}^n) &= (p_{3i}^n p_{3j} - p_{3i}^m p_{3j}) \\ &= (2r+1)[\frac{\varepsilon}{n} - \frac{\varepsilon}{m}]x_i^{2r} p_{2j}\end{aligned}$$

so the gradients of p_3^n and p_3^m disagree on open sets when $n \neq m$.

Proceed inductively to define $\{P(s)\}_{s=1}^{\infty}$, and set $P = \cup_{s=1}^{\infty}\{P(s)\}$. Let $\mathcal{U}_d = P$, $\mathcal{U}_d$ is the required dense set.

■

An alternative representation avoids distinguishing between different utility functions which generate the same indifference curves (ie. one function is a monotone transformation of the other). Write $u \sim v$ if $\exists \psi$ such that $u = \psi \circ v$. The equivalence class of $v \in \mathcal{U}$ is denoted $[v]$ and defined: $[v] = \{u \in \mathcal{U} \mid u \sim v\}$. Let $[\mathcal{U}]$ denote the family of equivalence classes. Say that two elements, $[u], [v]$ of $[\mathcal{U}]$ intersect if there does not exist $u, v \in \mathcal{U}$ with $[u] \neq [v]$ such that $\nabla u(x) = \varphi(x)\nabla v(x)$ on an open set in X.

Theorem 5 *Let $[\mathcal{U}]$ be the set of "$\sim$" equivalence classes, with the quotient topology. Then there exists a dense set in $[\mathcal{U}]$, $[\mathcal{U}]_d$ such that no two elements of $[\mathcal{U}]_d$ intersect.*

Proof: Take the dense set, $\mathcal{U}_d$ given in theorem 4 and let $[\mathcal{U}]_d$ be the equivalence classes in $\mathcal{U}_d$ determined by "$\sim$". By construction, if $p, q \in \mathcal{U}_d$, then $\mathbf{J}^{ij}_{pq}$ has a nonzero determinant on an open subset of X. Thus, there is no open set on which $p = \psi \circ q$ for some function ψ so that $p \not\sim q$, and so no two elements of $\mathcal{U}_d$ are in the same equivalence class. Thus, no two elements of $[\mathcal{U}]_d$ intersect. Now, let $\wp$ be the identification mapping which associates to each $u \in \mathcal{U}$, the equivalence class $[u] \in [\mathcal{U}]$. Thus, $\wp : \mathcal{U} \to [\mathcal{U}]$. With the identification (quotient) topology on $[\mathcal{U}]$, $\wp$ is continuous. Given $[u] \in [\mathcal{U}]$, there is some sequence $v^n \subset \mathcal{U}_d$, with $v^n \to u$. Since $\wp$ is continuous, $[v^n] \to [u]$, so that $[\mathcal{U}_d]$ is dense in $[\mathcal{U}]$.

■

One might expect that theorems 4 and 5 can be strengthened from "dense" to "open dense". The next result shows that this is not possible.

Theorem 6 *Let $\mathcal{D}$ be an open set in the space of continuous differentiable functions (with the sup norm) on X. Then, given $\varepsilon > 0$, $\exists f, g \in \mathcal{D}$, $\|f - g\| < \varepsilon$ and two open sets $Y_1, Y_2 \subset X$ such that $f = g$ on Y_1 (so that $\nabla f = \nabla g$ on Y_1) and $f \neq g$ on Y_2.*

Proof: Let $f \in \mathcal{D}$. Pick $\beta \in \mathcal{O} \subset X \subset \mathcal{R}^k$, $\mathcal{O}$ open, and define $\theta(x) = max\{0, \times_{i=1}^{k}(x_i - \beta_i)^3\}$. Thus, θ is a continuous differentiable function on X, $\theta_{x_i} = max\{0, \times_{i=1}^{k}(x_i - \beta_i)^3\} \cdot [\times_{j \neq i}(x_j - \beta_j)^3] \cdot 3(x_i - \beta_i)^2$, and since X is compact, $max_{x \in X}\theta(x) = \bar{\theta} < \infty$. Note that there is an open set, $Y_1 = \mathcal{O} \cap \{x \in X \mid x_i < \beta_i, \forall i\}$ on which θ is zero. Define $f^n(x) = f(x) + \frac{1}{n}\theta(x)f(x)$ and set $Y_2 = \mathcal{O} \cap \{x \in X \mid x_i > \beta_i, \forall i\}$. Thus, $f^n \to f$, $f^n \neq f$ and $\nabla f = \nabla f^n$ on Y_1, for all n.

■

In other areas of economic theory, it is common to require genericity in *product spaces* of preferences, $\mathcal{U}^n$ (for example, in equilibrium theory). In that literature it is usual to obtain "open denseness" rather than "denseness" results (for example, Smale (1974)). In the present context, there is an analogous open denseness result. For this, assume that the space of utility functions is continuously differentiable: $\mathcal{U} = \mathcal{C}^1(X, \mathcal{R})$ and let the topology on $\mathcal{U}$ be the Whitney topology.

Theorem 7 *There is an open dense set of preferences $\mathcal{U}_d^n \subseteq \mathcal{U}^n$ such that if $\mathbf{u} \in \mathcal{U}_d^n$, then $\mathbf{M_u} = \{x \mid rank[\mathbf{J_u}(x)] < min(n,k)\}$ is a manifold with $dim(\mathbf{M_u}) < dim(X)$.*

Proof: The proof is a straightforward application of the jet transversality theorem (see Hirsch (1976), p80-82). The vector space of $k \times n$ matrices, $LA(k,n)$ is a differentiable manifold of dimension kn. The subspace of $k \times n$ matrices of rank r, $LA(k,n;r)$ is a submanifold of $LA(k,n)$ of codimension $(k-r)(n-r)$, $r \leq min(k,n)$. The submanifold of $LA(k,n)$ of matrices of rank lower than $min(k,n)$ is closed. The "1-jet" of a function $\mathbf{u}$ at x consists of x, the value of the function at x, $\mathbf{u}(x)$, and it's Jacobian at x, $\mathbf{J_u}(x)$. If we look in the range space of the mapping, identify the submanifold in $LA(k,n)$ of matrices of rank less than $min(k,n)$,this gives a closed submanifold. Using the transversality theorem (see 2.8, 2.9 of Hirsch), gives an open dense set of functions intersecting this manifold transversely (openness is due to the fact that the submanifold in the range is closed).

■

Thus, there is an open dense set of functions $\mathcal{U}_d^n$ in $\mathcal{U}^n$, such that $\mathbf{v} \in \mathcal{U}_d^n$ implies that $\mathbf{J_v}(x)$ has full rank except on a set of dimension less that X.

Note however, that this result does not address the issue of monotonicity. To see this suppose that there are two agents. Then it may be that there are $\mathbf{u}, \mathbf{v} \in \mathcal{U}_d^2$ with $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, such that u_1 and v_1 have nested lower contour sets. Then $\mathbf{u}, \mathbf{v} \in \mathcal{U}_d^2$ is consistent with the possibility that monotonicity of a social choice function fails at those preferences. Or, in the approached used above, it may be that for some function φ, $\nabla u_1(x) = \varphi(x)\nabla v_1(x)$ on open sets in X. It is worth making one last point of clarification: this result arises from the fact that open denseness is in a product space, and not from the fact that the Whitney topology is used. To see this, observe that if $u \in \mathcal{C}^1(X, \mathcal{R})$, then a neighborhood basis of u is given by the collection $\{\mathbf{B}_{1/r}(u)\}_{r=1}^{\infty}$, where $\mathbf{B}_{1/r}(u)$ is defined:

$$\mathbf{B}_{1/r}(u) = \{v \in \mathcal{C}^1(X, \mathcal{R}) \mid \|u - v\| < \frac{1}{r}, \|\partial_j u - \partial_j v\| < \frac{1}{r},\ j = 1, 2, \ldots k\}$$

where $\partial_i u$ denote the partial derivative of u with respect to x_i (a function from X to $\mathcal{R}$). With this notation, we can repeat theorem 6 (essentially).

Theorem 8 *Let $\mathcal{D}$ be an open set in the space of continuous differentiable functions (with the Whitney topology) on X. Then, $\exists f, g \in \mathcal{D}$, and two open sets $Y_1, Y_2 \subset X$ such that $f = g$ on Y_1 (so that $\nabla f = \nabla g$ on Y_1) and $f \neq g$ on Y_2. Therefore, $dim(\mathbf{M}_{(f,g)}) = dim(X)$, where $\mathbf{M}_{(f,g)} = \{x \mid rank[\mathbf{J}_{(f,g)}(x)] < 2\}$.*

Proof: Let $f \in \mathcal{D}$. Define f^n as in theorem 6. Observe that since X is compact and $\theta, \theta_{x_i}, \forall i$ continuous, $\exists \theta^*$ such that $max_{\{x \in X, i \in \{1,\ldots,n\}\}}\{\theta(x), \theta_{x_i}(x)\} = \theta^* < \infty$. Thus given $r, \exists \bar{n}$ such that $f^n \in \mathbf{B}_{1/r}(f), n \geq \bar{n}$. Consequently, in any open neighborhood of f, there are functions $g, h \in \mathbf{B}_{1/r}(f)$ with $\nabla h = \nabla g$ on open sets in X.

■

5 Implementation on Dense Domains

The next result shows that there is a dense subset ($\mathcal{U}_d$) of $\mathcal{U}$ such that if f is an arbitrary social choice function, there exists another social choice function, f_ϵ, approximately equal to f and which is implementable on $\mathcal{U}_d^n$.

Recall that the consumption set of agent i is X, a compact subset of $\mathcal{R}^n$ and that preferences are monotonic. Choose X be sufficiently large to contain the set of feasible allocations, $F = \{(x_i, x_2, \ldots, x_n) \mid x_i \in \mathcal{R}^k, \sum_{i=1}^n x_{ij} \leq e_j, j = i, \ldots, k\}$. Thus $e = (e_1, \ldots, e_k) \in X$ and $A = X^n \cap F = F$. Write $X(e) = \times_{j=1}^k [0, e_j]$. For the remainder of the paper, we normalize preferences in $\mathcal{U}$ with a monotone (affine) transformation in the following way. Let β be a small positive number and given $x \in \mathcal{R}^k$, let $B_\beta(x) = \{y \in \mathcal{R}^k \mid d(y, x) < \beta\}$ be the open sphere of radius β centered on x. Given $u \in \mathcal{U}$, let $u_\beta = max\{u(x) \mid x \in X(e) \setminus B_\beta(e)\}$, where $X(e) \setminus B_\beta(e)$ is the set of points in $X(e)$ not in $B_\beta(e)$. Similarly, $u_{2\beta} = max\{u(x) \mid x \in X(e) \setminus B_{2\beta}(e)\}$. With u monotonic, $u_\beta > u_{2\beta}$. Choose a_u and b_u so that $a_u u_\beta + b_u = 1$ and $a_u u_{2\beta} + b_u = 0$. With some abuse of notation we leave notation unchanged and take all functions in $\mathcal{U}$ to satisfy this normalization. Note then that given $u \in \mathcal{U}$, $y \in X(e)$ and $u(y) > 1$ implies that $y \in B_\beta(e)$. Similarly, $u(y) > 0$ implies that $y \in B_{2\beta}(e)$.

The dense set of (approximating) preferences, $\mathcal{U}_d$ given in theorem 4 is dense in this set. Fix some $\gamma > 0$, γ small, and assume that the set of approximating functions in $\mathcal{U}_d$ contain no polynomials p for which $sup_x \mid p(x) - u(x) \mid > \gamma > 0, \forall u \in \mathcal{U}$. If any such polynomials are present, they may be discarded and $\mathcal{U}_d$ remains dense in $\mathcal{U}$ since such polynomials approximate no function in $\mathcal{U}$ "well". Thus, $\mathcal{U}$ is the set of normalized functions as described above, and $\mathcal{U}_d$ a dense set of preferences with any point in $\mathcal{U}_d$ within γ of some point in $\mathcal{U}$.

Observe that in the case of three or more players, the *weak no veto power* condition is always satisfied on $\mathcal{U}_d$. To see this, let $p_i, p_j \in \mathcal{U}_d$, the preferences of agents i and j. Suppose the allocation $x = (x_1, x_2, \ldots, x_n)$ is top-ranked by i. Let $u \in \mathcal{U}$ be uniformly within $\gamma < \frac{1}{2}$ of p_i, so p_i satisfies $\mid p_i(y) - u(y) \mid \leq \gamma, \forall y \in X(e)$. Since $max_{y \in X(e)} u(y) > 1$, it must be that $max_{y \in X(e)} p_i(y) = p_i(x_i) \geq 1 - \gamma$. Furthermore, $\mid p_i(y) - u(y) \mid \leq \gamma, \forall y \in X(e)$, implies $p_i(x_i) \leq u(x_i) + \gamma$. Thus, $u(x_i) + \gamma \geq 1 - \gamma$ so $u(x_i) \geq 1 - 2\gamma > 0$. Recall that $u(x_i) > 0$ implies that $x_i \in B_{2\beta}(e)$. Similarly, since x_j is top-ranked by j, it must be that x_j is within 2β of e. With β sufficiently small, this is impossible since $x_i + x_j \leq e$ (and β is chosen so).

Theorem 9 *$\exists$ a dense set, $\mathcal{U}_d \subset \mathcal{U}$, such that if f is a social choice function on $\mathcal{U}^n, n \geq 3$, then given $\epsilon > 0, \exists f_\epsilon : \mathcal{U}^n \to X^n$, with $\|f(\mathbf{u}) - f_\epsilon(\mathbf{u})\| \leq \epsilon, \forall u \in \mathcal{U}^n$, such that f_ϵ is implementable on $\mathcal{U}_d^n$.*

Proof: Take the dense set $\mathcal{U}_d$ in $\mathcal{U}$ given by theorem 4. This set can be taken to be countable (using rational coefficients in the polynomials). If $u, v \in \mathcal{U}_d$ then $\{x \mid \nabla u(x) = \nabla v(x)\}$ is nowhere dense in X. For each $u, v \in \mathcal{U}_d$ let $\mathcal{N}_{(u,v)} = \{x \mid \nabla u(x) = \nabla v(x)\}$ and let $\mathcal{Q} = \cup_{\{u,v \in \mathcal{U}_d \mid u \neq v\}} \mathcal{N}_{(u,v)}$. $\mathcal{Q}$ is the union of nowhere dense sets in X - a set of "first category" - and since X is a complete metric space, the complement of $\mathcal{Q}$ in X, $\mathcal{Q}^c$, is dense in X. This implies that given $f : \mathcal{U}^n \to X^n$, $\exists f_\epsilon$, $\|f(\mathbf{u}) - f_\epsilon(\mathbf{u})\| \leq \epsilon, \forall u \in \mathcal{U}^n$ such that f_ϵ has range $[\mathcal{Q}^c]^n$. At every point in the range of each agents indifference curves in $\mathcal{U}_d$ intersect, so that monotonicity is satisisfied on $\mathcal{U}_d^n$. In addition, by the earlier discussion, weak no veto power holds. Thus, theorem 1 applies.

■

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