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Markov Puppy Dogs and Other Related Animals

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Abstract

It is shown that steady state Markov perfect equilibria of discrete time, infinite horizon, quadratic, adjustment cost games differ from equilibria of their infinitely repeated counterpart games with zero adjustment costs even though no adjustment costs are paid in the steady state. In contrast to continuous time games, the limit of these equilibria as adjustment costs approach zero is the same as the equilibria of their static counterpart games. A classification scheme is presented and it is shown that the taxonomy is identical to that of analogous two stage games such as those analyzed by Fudenberg and Tirole (1984). This classification is useful in that it implies that steady state equilibria need not be explicitly calculated to analyze qualitatively the effects of adjustment costs in strategic environments. It is also argued that estimated conjectural variations parameters may capture a well defined property of strategic interaction in a dynamic game.

JEL Classification No: C73

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I. Introduction

Two stage competition has become a standard methodology for studying strategic behaviour in industrial organization. The approach has also had extensive applications in other fields such as international trade, public finance, and macroeconomics.[1] Much progress has been made in analyzing the equilibria of two stage games as a function of the payoff structure of the game (Bulow, Geanakopolos and Klemperer (1985), Fudenberg and Tirole (1984)). The Fudenberg and Tirole analysis provides an influential system for classifying perfect Nash equilibria to asymmetric two stage games. In this paper, we provide an analogous taxonomy, based on properties of payoff functions, for a class of discrete time, infinite horizon, dynamic games with Markov strategies.

Generally, in two stage competition, a "strategic" action is committed to in the first period (or stage) and is observed by all rivals. Perfect Nash equilibria are derived and first period choices influence the second period equilibrium. The Fudenberg and Tirole system classifies the perfect Nash equilibria of asymmetric games according to whether the second stage competition is between strategic complements or substitutes and whether investment (the commitment stage of the game) makes the strategic player "tough" or "soft." Symmetric two stage games can be similarly classified, but the interpretation is not as straightforward. The value of these classification systems arises because by studying the primitive properties of the payoff functions, the qualitative features of the equilibrium can be derived without fully computing equilibria.

In infinite horizon games, the physical environment can remain unchanged from period to period, as in the supergame framework, or the past can have a direct influence on current opportunities, through so called "payoff-relevant" strategies. We adopt the latter dynamic game framework (which is analogous to two stage competition) and restrict attention to Markov strategies. In a Markov strategy, the past affects current choices only through its effect on the current physical environment, i.e. on a state variable. Supergame strategies are thus ruled out.

We show that the steady state Markov perfect equilibria[2] of discrete time quadratic

[1] See, for example, Shapiro (1989) for applications.

[2] Maskin and Tirole (1988) includes a critical discussion of the Markov perfect equilibrium concept.

adjustment cost games will differ from the equilibria of their infinitely repeated counterpart game with zero adjustment costs. This occurs even though no adjustment costs are paid in the steady state and arises as the presence of such costs provide a strategic incentive to deviate from the repeated equilibrium. A similar result has been noted in differential games by Driskill and McCafferty (1989), Fershtman and Kamien (1987), Reynolds (1987), and others. These three authors also obtain an interesting limit result that, as the strategic connection between periods goes to zero, the Markov perfect equilibrium approaches a limit *not equal* to the Nash equilibrium of the corresponding static game. In contrast, in our discrete time framework, the limit of steady state Markov perfect equilibria as adjustment costs approach zero *is equal* to the Nash equilibria of their static counterpart games without adjustment costs. Finally, it is shown that the structural classification for steady state Markov perfect equilibria of quadratic adjustment cost games is *identical* to that of the analogous two period games.

Consider a comparison between the infinite horizon dynamic game and the analogous two stage game. In the latter, a strategic choice in the first period is made anticipating the effect on the second period sub- game. In a game with symmetry between periods, but with adjustment costs of changing one's action between periods, the perfect Nash equilibrium will not be characterized by the same choice in both periods. In the infinite horizon game, a current choice is made taking into account the effect on the value of all future sub-games. Although these are infinite in number, their value can be summarized, using a dynamic programming approach, by a single value function. In the steady state, the Markov perfect equilibrium will be characterized by the same actions across periods. Despite these fundamental differences between the two games, Markov perfect equilibria can be classified in exactly the same way as the perfect Nash equilibria to two stage games. In fact, evaluating the derivative of the equilibrium choice variable with respect to the scale of adjustment costs, we find for small deviations around zero adjustment costs, that the derivative is identical to the analogous derivative obtained for the equilibrium to the two stage game.

The steady state equilibria of our infinite horizon games consist of deviations around the one shot (non-strategic) Nash equilibrium. As such, they predict the same outcome as a static conjectural variations model, infinitely repeated, with the appropriate choice of the conjectural variations (cv) parameter. We show that a function can be defined from the underlying payoff functions and economy parameters to a cv parameter, such that the sign of this cv parameter can be classified in a similar manner to that of Markov perfect equilibria of dynamic game. Thus, an empirically estimated cv parameter implies restrictions on the structure of an underlying Markov perfect equilibria.

The remainder of the paper is organized as follows. Section II reviews the model of two stage competition and the "Puppy Dogs" classification system due to Fudenberg and Tirole. Section III presents a class of infinite horizon games, derives equilibria, and constructs an classification system. Section IV applies the general framework to several example games, which are commonly referred to in industrial organization. Section V relates conjectural variations equilibria to the Markov perfect equilibria of the infinite horizon game and Section VI concludes.

II. Benchmark and Review: Two Stage Competition

As a benchmark for comparison, we review a simple two period model of strategic competition, which has had extensive applications in Industrial Organization and other fields. There are two players. Each player chooses an action, u, simultaneously with her rival, in each of two periods. The overall payoff to each player is given by:

$$\text{player 1}: \quad R^1(\tilde{u}_1, \tilde{u}_2) + \beta[R^1(u_1, u_2) + \alpha_1 P^1(u_1, u_2, \tilde{u}_1, \tilde{u}_2)] \tag{1.1}$$

$$\text{player 2}: \quad R^2(\tilde{u}_1, \tilde{u}_2) + \beta[R^2(u_1, u_2) + \alpha_2 P^2(u_1, u_2, \tilde{u}_1, \tilde{u}_2)] \tag{1.2}$$

where $\tilde{u}_j$ is player j's choice in period 1, and u_j is player j's choice in period 2. $R^j(\cdot)$ is a concave one period payoff function, symmetric across players. The concave function $P^j(\cdot)$ captures the dynamic component of the game; the link between period 1 and period 2. We restrict $P^j(\cdot)$ to a generalized adjustment costs form, namely

$$P^j(u_1, u_2, \tilde{u}_1, \tilde{u}_2) = -(g(u_1, u_2) - g(\tilde{u}_1, \tilde{u}_2))^2 \tag{2}$$

where $g(\cdot)$ is affine. The discount factor is β and $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are parameters which determine the strength of the dynamic interaction.

As an example of a model with the above structure, consider a model of Cournot competition with adjustment costs to changing output levels. In this case,

$$R^j(q) = (a - b(q_i + q_j) - w)q_j \tag{3.1}$$

$$P^j(q\ \tilde{q}) = -(q_j - \tilde{q}_j)^2. \tag{3.2}$$

First period choices are known when the second period decisions are made, so the appropriate equilibrium concept is Perfect Nash. When $\alpha_1 = \alpha_2 = 0$, the model reverts to a one period game with payoffs $R^j(\cdot)$, repeated for two periods.

We begin with an asymmetric case, in which $\alpha_2 = 0$ and in which $P^1(\cdot)$ is independent of $\tilde{u}_2$ and u_2. In this game, adjustment costs confer a strategic advantage only to player 1 [3]. The structure of this asymmetric game is consistent with the analysis of Fudenberg and Tirole (1984). The solution to this model is well known, so that we present only the relevant details here. We can write the period 1 problem for player 1 as

$$\max_{\tilde{u}_1}\{R^1(\tilde{u}_1, \tilde{u}_2) + \beta[R^1(u_1(\tilde{u}_1), u_2(\tilde{u}_1)) + \alpha_1 P^1(u_1(\tilde{u}_1), \cdot, \tilde{u}_1, \cdot)]\}$$

where $u_1(\tilde{u}_1)$ and $u_2(\tilde{u}_1)$ are Nash equilibrium functions from the second period of the game. Letting subscript j denote the derivative of a function with respect to its j^{th} argument, and using the envelope theorem, the first order conditions in the first period can be written:

$$\tilde{R}^1_1 + \beta\alpha_1 P^1_3 + \beta R^1_2 \frac{\partial u_2}{\partial \tilde{u}_1} = 0 \tag{4.1}$$

$$\tilde{R}^2_2 = 0. \tag{4.2}$$

where $\tilde{R}^j_k \equiv R^j_k(\tilde{u}_1, \tilde{u}_2)$. Using standard comparative static techniques, we can evaluate $\partial u_2/\partial \tilde{u}_1$ in equation (4.1) and rewrite the first order condition as:

$$\tilde{R}^1_1 + \beta\alpha_1 P^1_3 + \frac{\beta\alpha_1}{\Omega} R^1_2 P^1_{13} R^2_{12} = 0 \tag{5}$$

where in these equations R^j_{ik} and P^j_{ik} denote second derivatives of the return functions. $\Omega = R^2_{22}(R^1_{11} + \alpha_1 P^1_{11}) - R^1_{12}R^2_{12}$ and is positive in a stable second period game.

[3] The analogy here is a model in which only one player can commit; for example the well known entry deterrence model of Dixit (1980).

The conventional analysis of such a strategic game, as pioneered by Fudenberg and Tirole (1984), would examine the sign of the final term in equation (5), the *strategic term*. This term indicates how the equilibrium differs from the equilibrium of the open loop version of the game, in which the final term would not appear. The Fudenberg and Tirole (FT) analysis classifies the sign of the strategic term as demonstrated by Table 1 below. "Investment makes player 1 tough," in the context of our Cournot adjustment cost example, means that an increase in production in period 1 increases the costs of reducing production in period 2, and so acts as a commitment to overproduction in the second period, which in turn makes the rival worse off.

Table 1: FT Classification

	Investment Makes Player 1	
	Tough	**Soft**
	$R_2P_{13} < 0$	$R_2P_{13} > 0$
$R_{12} > 0$	*Underinvestment*	*Overinvestment*
(Strategic Complements)	*(Puppy Dog)*	*(Fat Cat)*
$R_{12} < 0$	*Overinvestment*	*Underinvestment*
(Strategic Substitutes)	*(Top Dog)*	*(Lean & Hungry)*

An alternative method for classifying the outcomes of these games is to examine the derivative of first-period Nash equilibrium strategies with respect to α_1, evaluated at $\alpha_1 = 0$. The sign of this derivative suggests the effects of dynamic elements (small adjustment costs facing player 1) on the equilibrium of the two period game. A positive derivative is associated with overinvestment and a negative derivative with underinvestment.

The solution to the two period model when $\alpha_1 = 0$ is just the repeated Nash equilibrium characterized by the first order conditions

$$R_j^j(\cdot,\cdot) = 0 \qquad j = 1,2.$$

Let the solution for $\alpha_1 = 0$ be denoted (u_1^o, u_2^o). It is straightforward to show that

$$\left.\frac{\partial \tilde{u}_1}{\partial \alpha_1}\right|_{\alpha_1=0} \equiv \frac{\partial \tilde{u}_1^o}{\partial \alpha_1} = \frac{-\beta R_{11}}{\Delta^2} R_{12} R_2 P_{13} \tag{6}$$

where all derivatives in (6) are evaluated at (u_1^o, u_2^o) and $\Delta = (R_{11})^2 - (R_{12})^2$. Given the symmetry of the solution, we have dropped the identifying superscripts.[4] Given concavity of $R(\cdot)$ and stability of the one period game, we have

$$Sign \ \frac{\partial \tilde{u}_1^o}{\partial \alpha_1} = Sign \ R_{12}R_2P_{13}. \tag{7}$$

Therefore, the classification given in Table 1 is again valid for this derivative approach.

Consider a more general asymmetric game in which $\alpha_2 = 0$, but $P^1(\cdot)$ may now depend on $\tilde{u}_2$ and u_2. If $\tilde{u}_2$ enters player 1's payoff function in the second period, then both players' choice in the first period affects the Nash equilibrium in the second period. In this game, *both* players have a strategic incentive to deviate from the equilibrium of the repeated game. We follow our earlier methodology and study the sign of the change in the first period equilibrium choices with the introduction of a small dynamic element.

Evaluating the derivatives at $\alpha_1 = 0$ gives:

$$\frac{\partial \tilde{u}_1^o}{\partial \alpha_1} = \frac{-\beta}{\Delta^2} R_{11}R_{12}R_2[P_{13} + P_{14}] \tag{8.1}$$

and

$$\frac{\partial \tilde{u}_2^o}{\partial \alpha_1} = \frac{\beta}{\Delta^2} R_2[(R_{12})^2 P_{13} + (R_{11})^2 P_{14}] \tag{8.2}$$

where Δ is defined above.[5] Concavity of $R(\cdot)$ implies that

$$Sign \frac{\partial \tilde{u}_1^o}{\partial \alpha_1} = Sign R_{12}R_2[P_{13} + P_{14}] \tag{9.1}$$

and

$$Sign \frac{\partial \tilde{u}_2^o}{\partial \alpha_1} = Sign R_2[(R_{12})^2 P_{13} + (R_{11})^2 P_{14}]. \tag{9.2}$$

Equation (9.1) allows the classification given by Table 2 for the general two period game with asymmetric dynamics. The table indicates how first period perfect Nash equilibrium choices of player 1 in the presence of asymmetric adjustment costs will differ from his equilibrium choices in the two period repeated game (when $\alpha_1 = 0$).

[4] Given the symmetry of the solution and of the payoff functions, we also have $R^j_i = R^i_j = R_2$; $R^1_{11} = R^2_{22}$; $R^1_{12} = R^2_{21}$; $P^1_{13} = P^2_{24}$; $P^1_{14} = P^2_{23}$.

[5] Note that (8.1) reduces to (6) when $P_{14} = 0$.

Table 2: Two Stage Asymmetric Dynamics - Player 1

	$R_2[P_{13}+P_{14}]<0$	$R_2[P_{13}+P_{14}]>0$
$R_{12}>0$	$(\partial\tilde{u}_1^o/\partial\alpha_1)<0$	$(\partial\tilde{u}_1^o/\partial\alpha_1)>0$
$R_{12}<0$	$(\partial\tilde{u}_1^o/\partial\alpha_1)>0$	$(\partial\tilde{u}_1^o/\partial\alpha_1)<0$

Note that when player 2's first period choices have no effect on second period returns ($P_{14}=0$), Tables 1 and 2 are identical.

We also consider the symmetric two period game,[6] in which both players face the same adjustment cost function, given by (2), and $\alpha_1=\alpha_2=\alpha$. In this game, both players have a strategic incentive to deviate from the equilibria of the repeated game. The first order conditions for period 1 choices for the symmetric problem are identical for the two players and can be derived as:

$$\tilde{R}_1+\beta\alpha P_3+\frac{\beta\alpha}{\Phi}(R_2+\alpha P_2)[P_{13}(R_{12}+\alpha P_{12})-P_{14}(R_{11}+\alpha P_{11})]=0 \qquad (10)$$

where $\Phi=(R_{11}+\alpha P_{11})^2-(R_{12}+\alpha P_{12})^2$. From this equation, we can obtain the following derivative which captures the effects of adding a small dynamic component to the payoffs of *both* players:

$$\left.\frac{\partial\tilde{u}_j}{\partial\alpha}\right|_{\alpha=0}\equiv\frac{\partial\tilde{u}_j^o}{\partial\alpha}=\frac{\beta(R_{11}-R_{12})}{\Delta^2}R_2(R_{11}P_{14}-R_{12}P_{13}) \qquad for\ j=1,2 \qquad (11)$$

where Δ is defined as in the asymmetric case. Stability of the one-period game requires $|R_{11}|>|R_{12}|$ and, therefore, $R_{11}-R_{12}<0$. Hence, we have

$$Sign\frac{\partial\tilde{u}_j^o}{\partial\alpha}=Sign R_2[R_{12}P_{13}-R_{11}P_{14}].$$

As in the asymmetric game, equilibria in the symmetric game can be classified according the sign of this derivative. This classification is presented in Table 3.

[6] An early example of symmetric, simultaneous two period competition is Brander and Spencer (1983).

Table 3: Two Stage Symmetric Dynamics: j=1,2

	$R_2 < 0$	$R_2 > 0$
$(R_{12}P_{13} - R_{11}P_{14}) > 0$	$(\partial\tilde{u}_j^o/\partial\alpha) < 0$	$(\partial\tilde{u}_j^o/\partial\alpha_1) > 0$
$(R_{12}P_{13} - R_{11}P_{14}) < 0$	$(\partial\tilde{u}_j^o/\partial\alpha_1) > 0$	$(\partial\tilde{u}_j^o/\partial\alpha_1) < 0$

Tables 2-3 provide classifications of two-period games in which adjustment costs are paid in the second period either by a single firm (Table 2) or by both firms (Table 3). Adopting the classification system introduced by Fudenberg and Tirole, these tables allow us to predict the outcomes of various types of strategic interactions. We now turn to an analysis of infinite horizon games with adjustment costs.

III. The Infinite Horizon Game

Consider a quadratic, infinite horizon version of the two-period game analyzed in the previous section. In this section, we determine the effects of small asymmetric and symmetric adjustment costs on steady state variables. In addition, we suggest a classification method for Markov perfect equilibria analogous to the classification scheme for two-period games. Let the strategy vector at time t be given by

$$u^t = (u_1^t \; u_2^t)' \,\epsilon\, \Re_+^2$$

where u_j^t is the choice variable of player j at time t. For expositional purposes, time superscripts will be suppressed henceforth. Let u_j denote the current choice and let $\tilde{u}_j$ denote the previous period choice for player j. Hence, $(u \; \tilde{u})$ denotes the 4x1 vector of current and previous period choices.

The one-period payoff function for player j=1,2 is written:

$$Q^j(u \; \tilde{u}) = R^j(u) + \alpha_j P^j(u \; \tilde{u}). \tag{12}$$

The function $Q^j(u \; \tilde{u})$ is restricted to be quadratic in its arguments. The function $P^j(u \; \tilde{u})$ reflects dynamic elements and when $\alpha_1 = \alpha_2 = 0$, the game is a repeated game. In this environment, $\tilde{u}$ is the state vector and player j faces the following dynamic programming problem:

$$V^j(\tilde{u}) = \max_{u_j} \left\{ Q^j(u \; \tilde{u}) + \beta V^j(u) \right\} \tag{13}$$

where $V^j(.)$ is the value function for player j and β is the discount factor. A first-order necessary condition for a solution to this problem is

$$\frac{\partial Q^j}{\partial u_j} + \beta \frac{\partial V^j}{\partial u_j} = 0. \tag{14}$$

Kydland (1975) has shown existence of a linear solution for linear-quadratic discrete time dynamic games with finite horizons. Existence results for infinite horizon games are less general. We focus on equilibria with linear solutions and show, in the appendix, that such equilibria exist in a neighborhood of $\alpha_1 = \alpha_2 = 0$, where it is trivial to show that the equilibrium coincides in each period with the equilibrium of the static game. Furthermore, equilibria in this neighborhood will be continuously differentiable functions of α_1 and α_2. Since linear Markov perfect equilibria are *continuous* functions of these dynamic parameters, the limit of these equilibria when these parameters approach zero will be the same as the equilibrium of the static game. This is in contrast to discontinuities found in differential games such as those analyzed by Driskill and McCafferty (1989), Fershtman and Kamien (1987), Reynolds (1987), and others.

In the case of linear strategies, equilibrium Markov strategies are affine, and value functions are quadratic. The derivatives of the value functions can be written as:

$$\frac{\partial V^j}{\partial u_j} = a_{1j} + b_{1j}u_j + c_{1j}u_i \qquad for\ i \neq j \tag{15.1}$$

$$\frac{\partial V^j}{\partial u_i} = a_{2j} + b_{2j}u_i + c_{1j}u_j \qquad for\ i \neq j \tag{15.2}$$

where a_{1j}, a_{2j}, b_{1j}, b_{2j}, and c_{1j} for j=1,2 are to be determined by methods described below. Given (14), (15.1), (15.2) and the quadratic form of the $R^j(\cdot)$ and $P^j(\cdot)$ functions, the first-order condition for player j=1,2 is a linear equation in $(u_1, u_2, \tilde{u}_1, \tilde{u}_2)$. Assume further that the second order conditions for j=1,2 are satisfied. The system of equations given by (14) can be solved for the current choice variables, u, as a function of the previous period (state) variables, $\tilde{u}$. The solution yields Nash equilibrium current period choices, conditioned on the state vector $\tilde{u}$:

$$\hat{u} = F\tilde{u} + G \tag{16}$$

where F is a 2x2 matrix of coefficients, and G is a 2x1 matrix of constants. Note that these matrices depend on the coefficients in the derivatives of the value functions given by (14). The matrices are derived in the appendix.

In a steady state, $\hat{u} = \tilde{u} = \bar{u}$. Solving the system of equations given by (16) under this restriction yields steady state values as functions of economy parameters:

$$\bar{u} = (I - F)^{-1} G = MG \tag{17}$$

where $M = (I - F)^{-1}$. The value of the program in steady state equilibrium (discounted value of $R^j(\bar{u}\ \bar{u})$) is given by

$$\bar{V}^j = \left(\frac{1}{1-\beta}\right) R^j(\bar{u}\ \bar{u}). \tag{18}$$

Our purpose is to determine the effect on these steady state choices and returns of the presence of dynamic elements as captured by $\alpha_j > 0$.

Substituting the Nash equilibrium functions into equation (13) yields

$$V^j(\tilde{u}) = Q^j(\hat{u}(\tilde{u})\ \tilde{u}) + \beta V^j(\hat{u}(\tilde{u})).$$

Differentiating this with respect to $\tilde{u}_j$ and using the envelope theorem gives

$$\frac{\partial V^j}{\partial \tilde{u}_j} = \left[\frac{\partial Q^j}{\partial u_i} + \beta \frac{\partial V^j}{\partial u_i}\right] \frac{\partial \hat{u}_i}{\partial \tilde{u}_j} + \frac{\partial Q^j}{\partial \tilde{u}_j} \qquad for\ i \neq j. \tag{19.1}$$

Symmetrically, we have for $i \neq j$

$$\frac{\partial V^j}{\partial \tilde{u}_i} = \left[\frac{\partial Q^j}{\partial u_i} + \beta \frac{\partial V^j}{\partial u_i}\right] \frac{\partial \hat{u}_i}{\partial \tilde{u}_i} + \frac{\partial Q^j}{\partial \tilde{u}_i}. \tag{19.2}$$

Using the method of undetermined coefficients, we compare the two derivatives (15.1) and (15.2) with (19.1) and (19.2) and obtain ten non-linear equations in the ten unknown parameters a_{1j}, a_{2j}, b_{1j}, b_{2j}, c_{1j} for $j = 1, 2$. When a solution to this system of equations exists, steady state Markov perfect equilibria (MPE) given by equation (17) exist. The details are presented in the appendix.

We consider two cases. The first case is one in which the dynamic elements are asymmetric across players ($\alpha_1 > 0, \alpha_2 = 0$). As in the two-period model, if the adjustment costs function facing player 1 is independent of u_2 and $\tilde{u}_2$, then only player 1 has strategic power, and player 2's behaviour is passive. If, however, player 2's previous choice enters player 1's adjustment costs, then both players have strategic power, but this power is asymmetric across players as $\alpha_1 \neq \alpha_2$. A second case we analyze is one in which the dynamic elements are symmetric across players ($\alpha_1 = \alpha_2 = \alpha$). In this case, both players operate with identical adjustment cost functions and identical strategic power.

III.A: Asymmetric Dynamics

We seek to determine the effects on steady state choices and returns of small adjustment costs facing only one player. Letting $\alpha_2 = 0$, we differentiate steady state choices given by equation (17) and returns given by (18) with respect to α_1 and evaluate these derivatives at $\alpha_1 = 0$. The signs of these derivatives indicate the qualitative effects on steady state MPE of dynamic elements facing a single player in the infinite horizon game.

Let a superscript o denote the value of a variable evaluated at $\alpha_1 = \alpha_2 = 0$. Then, differentiating the steady state levels given by equation (17) with respect to α_1 and evaluating the derivative at $\alpha_1 = 0$ yields $\forall j = 1, 2$:

$$\frac{\partial \bar{u}_j^o}{\partial \alpha_1} = G_1^o \frac{\partial M_{j1}^o}{\partial \alpha_1} + G_2^o \frac{\partial M_{j2}^o}{\partial \alpha_1} + M_{j1}^o \frac{\partial G_1^o}{\partial \alpha_1} + M_{j2}^o \frac{\partial G_2^o}{\partial \alpha_1}. \tag{20}$$

Equation (20) suggests that we must evaluate the derivatives of G and M with respect to to α_1 evaluated at $\alpha_1 = 0$. The full derivation is contained in the appendix for general quadratic dynamic function $P^j(u\ \tilde{u})$.

In the case when $P^j(u\ \tilde{u})$ is restricted to the generalized adjustment costs form as in (2) above, we show in the appendix that the derivatives can be written as:

$$\frac{\partial \bar{u}_1^o}{\partial \alpha_1} = \left(\frac{-\beta R_{11}}{\Delta^2}\right) R_2^o R_{12} [P_{13} + P_{14}] \tag{21.1}$$

and

$$\frac{\partial \bar{u}_2^o}{\partial \alpha_1} = \left(\frac{\beta}{\Delta^2}\right) R_2^o [(R_{12})^2 P_{13} + (R_{11})^2 P_{14}] \qquad (21.2),$$

where $\Delta = (R_{11})^2 - (R_{12})^2 > 0$. Note that we have adopted the following notation under symmetry of $R^j(\cdot)$ and $P^j(\cdot)$,

$$R_2^o = \left.\frac{\partial R^j}{\partial u_i}\right|_{(\bar{u}_1\ \bar{u}_2)} \qquad R_{11} = \frac{\partial^2 R^j}{\partial u_j^2} \qquad R_{12} = \frac{\partial^2 R^j}{\partial u_j \partial u_i}$$

$$P_{13} = \frac{\partial^2 P^j}{\partial u_j \partial \tilde{u}_j} \qquad P_{14} = \frac{\partial^2 P^j}{\partial u_j \partial \tilde{u}_i}.$$

Comparing equations (9.1) and (9.2) with equations (21.1) and (21.2), it is evident that the classification given by Table 2 in the two-period model extends to a classification of steady state MPE in the infinite horizon model. Here, we adapt the Fudenberg and Tirole terminology for the infinite horizon game as presented in Table 4.1.

Table 4.1: Infinite Horizon Asymmetric Dynamics - Player 1

	(Investment Makes Player 1)	
	(Tough)	(Soft)
	$R_2^o(P_{13}+P_{14}) < 0$	$R_2^o(P_{13}+P_{14}) > 0$
$R_{12} > 0$	$(\partial \bar{u}_1^o / \partial \alpha_1) < 0$	$(\partial \bar{u}_1^o / \partial \alpha_1) > 0$
(Strategic Complements)	*(Markov Puppy Dog)*	*(Markov Fat Cat)*
$R_{12} < 0$	$(\partial \bar{u}_1^o / \partial \alpha_1) > 0$	$(\partial \bar{u}_1^o / \partial \alpha_1) < 0$
(Strategic Substitutes)	*(Markov Top Dog)*	*(Markov Lean and Hungry)*

This classification scheme affords a straightforward analogy with the two-period games analyzed in Section II in the FT classification. The term $R_2^o(P_{13}+P_{14})$ is the same as the corresponding term in Table 2 for the FT taxonomy, and by analogy, we label it "investment makes player 1 tough or soft"[7] and R_{12} determines whether strategies are strategic complements or substitutes. The combination of these effects determines the appropriate behavior for firm 1 in the MPE steady state so as to induce a softer behavior by firm 2 in the steady state. Table 4.2 provides a complete classification of player 2's behavior in a steady state MPE.

[7] An alternate definition of "investment makes you tough/soft" for the infinite horizon dynamic game is $\frac{\partial V^j(\cdot)}{\partial u_i} >/< 0$. However, we cannot explicitly evaluate this term, and since we are focusing here on the parallels with two stage competition, we have adopted the usage described in the text. Slade's empirical papers (1990) and (1992) also include a discussion and interpretation of "investment makes you tough" in Markov dynamic games.

Table 4.2: Infinite Horizon Asymmetric Dynamics - Player 2

	$R_2^o < 0$	$R_2^o > 0$
$((R_{12})^2 P_{13} + (R_{11})^2 P_{14}) > 0$	$(\partial \bar{u}_2^o / \partial \alpha_1) < 0$	$(\partial \bar{u}_2^o / \partial \alpha_1) > 0$
$((R_{12})^2 P_{13} + (R_{11})^2 P_{14}) < 0$	$(\partial \bar{u}_2^o / \partial \alpha_1) > 0$	$(\partial \bar{u}_2^o / \partial \alpha_1) < 0$

What is perhaps surprising is that the effect of strategic behavior in this infinite horizon dynamic game is exactly the same both qualitatively and quantitatively as in the two stage game which we studied at the beginning of the paper. In the infinite horizon game, a change in the current period action affects not only current payoffs and payoffs in the subsequent period (as in a two stage game) but payoffs in *all* future periods. The latter effect is captured through the value function. What our analysis shows is that locally around zero adjustment costs, the strategic effects from the two different games are identical.

The effect of asymmetric dynamics on the value of the game for the two players can be obtained from (18) and the first order conditions:

$$\frac{\partial \bar{V}^j(\bar{u}^o)}{\partial \alpha_1} = \left(\frac{1}{1-\beta}\right) R_2^o \frac{\partial \bar{u}_i^o}{\partial \alpha_1} \qquad for\ i \neq j. \tag{22}$$

Hence, whether or not a player is better off in the presence of asymmetric adjustment costs relative to the repeated game depends on how his rival responds in the steady state to adjustment costs and how that response affects the player's one-shot payoffs.

III.B: Symmetric Dynamics

We now turn to the case where both players face identical adjustment costs; $\alpha_1 = \alpha_2 = \alpha$. We evaluate the sign of the derivatives of steady state MPE with respect to α at $\alpha = 0$ to examine the qualitative effects of small adjustment costs facing both players. Differentiating the steady state levels given by equation (17) with respect to α and evaluting the derivative at $\alpha = 0$ gives

$$\frac{\partial \bar{u}_j^o}{\partial \alpha} = G_1^o \frac{\partial M_{j1}^o}{\partial \alpha} + G_2^o \frac{\partial M_{j2}^o}{\partial \alpha} + M_{j1}^o \frac{\partial G_1^o}{\partial \alpha} + M_{j2}^o \frac{\partial G_2^o}{\partial \alpha}. \tag{23}$$

The full derivation of the derivatives of the matrices on the right-hand side of equation (23) is contained in the appendix for general quadratic dynamic function $P^j(u\ \tilde{u})$.

In the case when $P^j(u\ \tilde{u})$ is restricted to the generalized adjustment costs form as in (2) above, we show in the appendix that the derivatives can be written as follows for $j = 1, 2$:

$$\frac{\partial \bar{u}_j^o}{\partial \alpha} = \left(\frac{\beta(R_{11} - R_{12})}{\Delta^2}\right) R_2^o (R_{11}P_{14} - R_{12}P_{13}) \tag{24}$$

where Δ is defined above.

Comparing equation (11) with (24), it is evident that Table 3 extends to a classification of steady state MPE in the infinite horizon game. We again adapt the Fudenberg and Tirole terminology to the symmetric game as presented in Table 5.

Table 5: Infinite Horizon Symmetric Dynamics

	$R_2^o < 0$	$R_2^o > 0$
$(R_{12}P_{13} - R_{11}P_{14}) > 0$	$(\partial \bar{u}_j^o / \partial \alpha) < 0$ *(Markov Puppy Dog)*	$(\partial \bar{u}_j^o / \partial \alpha) > 0$ *(Markov Fat Cat)*
$(R_{12}P_{13} - R_{11}P_{14}) < 0$	$(\partial \bar{u}_j^o / \partial \alpha) > 0$ *(Markov Top Dog)*	$(\partial \bar{u}_j^o / \partial \alpha) < 0$ *(Markov Lean and Hungry)*

From (19), it is clear that the effect of symmetric dynamics on the value of the game for the two players is again given by equation (22) with α_1 replaced by α:

$$\frac{\partial \bar{V}^j(\bar{u}^o)}{\partial \alpha} = \left(\frac{1}{1-\beta}\right) R_2^o \frac{\partial \bar{u}_i^o}{\partial \alpha} \qquad \textit{for } i \neq j. \tag{22'}$$

Hence, the same factors affect whether players are better or worse off in the presence of symmetric adjustment costs.

IV. Applications

In this section we present examples of infinite horizon dynamic games and classify each game according to Tables 4-5. In addition to these qualitative results, we compute the relevant derivatives for each game and suggest quantitative effects of small adjustment costs.

Dynamic Cournot with Quantity Adjustment Costs

Consider a standard Cournot duopoly with linear inverse demand:

$$p = a - b(q_1 + q_2)$$

and constant marginal costs $w < a$. Suppose each firm faces a quadratic adjustment cost to changing its quantities of the form:

$$\alpha_j (q_j - \tilde{q}_j)^2.$$

Then, the functions $R^j(\cdot)$ and $P^j(\cdot)$ are given by

$$R^j(q) = (a - b(q_i + q_j) - w)q_j$$

$$P^j(q\,\tilde{q}) = -(q_j - \tilde{q}_j)^2.$$

Taking derivatives and evaluating yields:

$$R_{11} = -2b \qquad R_{12} = -b \qquad P_{13} = 2 \qquad P_{14} = 0$$

$$R_2^o = -b\bar{q}_j = \frac{(w-a)}{3} < 0$$

In both the asymmetric and symmetric dynamics cases, Tables 4 and 5 indicate that these are Markov Top Dog games and steady state MPE in the presence of adjustment costs will differ from equilibria of the repeated game as those tables suggest. In the asymmetric dynamics case, substituting the above derivatives into (21.1) and (21.2) and simplifying yields the derivatives

$$\frac{\partial \bar{q}_1}{\partial \alpha_1} = \frac{4\beta(a-w)}{27b^2} > 0 \qquad \frac{\partial \bar{q}_2}{\partial \alpha_1} = -2\beta\frac{(a-w)}{27b^2} < 0. \tag{25}$$

Thus, the presence of asymmetric adjustment costs leads to a steady state equilibrium with equilibrium quantity higher for firm 1, and lower for firm 2, than in the repeated game without adjustment costs. Note that this is true although in the steady state no adjustment costs are actually incurred. Their existence confers a strategic advantage to firm 1 which allows it to increase production in the direction of the one shot Stackelberg quantity. From (22), it is clear that because of this effect, the firm with adjustment costs is better off in the presence of such costs while its competitor is worse off.

The analogy with the asymmetric two period game is close, and the forces driving the results are similar. Investment in this game consists simply of increasing production which, because of adjustment costs, makes producing *less* next period more expensive. The $R_2^o < 0$ term indicates that increases in production by firm 1 reduce profits for firm 2. The term $R_{12} < 0$ shows that the game is one of strategic substitutes. Hence, if firm 1 seeks to induce a softer behavior from its rival through it production strategy, the firm should overproduce relative to the repeated game outcome. Because quantities are strategic substitutes, firm 2 will underproduce relative to the repeated outcome. Equation (25) also implies that the firm with adjustment costs will increase her quantity more than her rival will decrease his quantity in the MPE steady state. Hence, total output will be larger in the presence of small asymmetric adjustment costs and prices will be lower.

The existence of symmetric adjustment costs leads to a steady state equilibrium with both quantities exceeding the equilibrium levels of the repeated game. From (22′), it is clear that this overproduction makes both firms worse off than if adjustment costs were zero, and the equilibrium was simply static Cournot, infinitely repeated. In a differential game with "sticky prices", Fershtman and Kamien (1987) derive a similar result in which firms overproduce relative to the static Cournot equilibrium in which prices adjust instantaneously. Driskill and McCafferty (1989) examine a differential game analog to the game presented here and also derive the overproduction result in the closed-loop equilibrium. In our discrete time game, however, in contrast to the these models, the closed-loop equilibrium *does* approach the static Cournot equilibrium as adjustment costs approach zero.

Dynamic Cournot with Price Adjustment Costs

Suppose that each firm faces a quadratic adjustment cost in price, rather than quantity, changes. One interpretation of this model is that there are "menu costs." In this case $P(\cdot)$ takes the form:

$$\begin{aligned} P^j(q\,\tilde{q}) &= -(p_j(q) - \tilde{p}_j(\tilde{q}))^2 \\ &= -(-b(q_1+q_2) + b(\tilde{q}_1+\tilde{q}_2))^2. \end{aligned}$$

Taking derivatives and evaluating yields:

$$P_{13} = P_{14} = 2b^2,$$

with the derivatives of $R(\cdot)$ as in the previous game.

In the asymmetric case, Table 4 indicates that this is again a Markov Top Dog game. From (21.1) and (21.2), we have

$$\frac{\partial \bar{q}_1}{\partial \alpha_1} = \frac{8\beta(a-w)}{27} > 0 \qquad \frac{\partial \bar{q}_2}{\partial \alpha_1} = \frac{-10\beta(a-w)}{27} < 0 \tag{26}$$

In contrast to the Cournot game with quantity adjustment costs, the game with price adjustment costs is characterized by lower total output and higher prices than in the repeated game without adjustment costs.

In the symmetric dynamics case, Table 5 indicates that this is a Markov Puppy Dog game. The existence of symmetric adjustment costs on prices leads to a steady state equilibrium with both quantities *lower* than the equilibrium levels of the repeated game. From (22′), it is clear that this underproduction makes both firms better off than if adjustment costs on prices were zero. The existence of price based adjustment costs allows both firms to move toward the collusive outcome; effectively, adjustment costs operate as joint commitment device to reduce output.

Dynamic Bertrand with Price Adjustment Costs

Suppose that we simply switch the role of prices and quantities in the above games so that direct demand functions are given by

$$q_j = a - bp_j + cp_i$$

where products j and i are now imperfect substitutes, and $c < b$. Such demand functions can be derived from quadratic utility functions defined on the two products, or from a spatial model of differentiated products on a Hotelling line. Adjustment costs on prices are given by:

$$\alpha_j(p_j - \tilde{p}_j)^2.$$

Again this can be interpreted as a "menu costs" game. In this case, we have

$$R^j(p) = (a - bp_j + cp_i)p_j$$

$$P^j(p\,\tilde{p}) = -(p_j - \tilde{p}_j)^2$$

So that, evaluating the relevant derivatives gives:

$$R_{11} = -2b \qquad R_{12} = c$$

$$P_{13} = 2 \qquad P_{14} = 0$$

$$R_2^o = c\bar{p}_j = \frac{ac}{2b - c} > 0$$

Tables 4 and 5 indicate that this is a Markov Fat Cat game in both the asymmetric and symmetric cases. Hence, both firms price higher in the presence of asymmetric adjustment costs than in the repeated game. For quantitative effects, substitution into (21.1) and (21.2) yields the following derivatives

$$\frac{\partial \bar{p}_1^o}{\partial \alpha_1} = \frac{4\beta abc^2}{(2b - c)^3(c + 2b)^2} > 0 \qquad \frac{\partial \bar{p}_2}{\partial \alpha_1} = \frac{2\beta ac^3}{(2b - c)^3(c + 2b)^2} > 0. \tag{27}$$

Equation (27) indicates that the firm with adjustment costs increases her price by more than her rival. It can be shown that these effects result in a lower quantity of good 1 and a higher quantity of good 2 in the MPE steady state. From (22), we see that both firms are better off in the presence of asymmetric menu costs. Because prices are strategic complements, even firm 2, which is a passive player following a strategy of a one shot Nash best response in each period, is still made better off by the adjustment cost function of its rival. The same results are found in the analogous two period game.

When adjustment costs are symmetric across firms, steady state equilibrium prices are higher than those in the repeated static game. Each firm raises price in the current period to commit to a higher price next period and hence induce the rival to set a higher price next period. Output of both goods will be lower than in the repeated game. Finally, as in the asymmetric case, both firms are better off in the presence of symmetric menu costs.

Dynamic Bertrand with Quantity Adjustment Costs

We maintain the model of price competition between differentiated products, but consider adjustment costs based on the change in quantities between the current period and the previous period. The adjustment costs can be considered to be incurred by the firms directly, because of the nature of capacity; or, the model can be thought of as one of consumer lock-in, in which consumers incur costs of switching their choice from period to period.[8] In either case, adjustment costs are based on quantities, not on prices.

In this case, the $P^j(\cdot)$ functions are

$$\begin{aligned} P^j(p\,\tilde{p}) &= -(q_j(p) - \tilde{q}_j(\tilde{p}))^2 \\ &= -(-bp_j + cp_i + b\tilde{p}_j - c\tilde{p}_i)^2 \end{aligned}$$

The derivatives off $R^j(\cdot)$ are unchanged, but the relevant derivatives of $P^j(\cdot)$ are now:

$$P_{13} = 2b^2 \qquad\qquad P_{14} = -2bc$$

and

$$R_{12}P_{13} - R_{11}P_{14} = -2b^2c < 0$$

Table 4 indicates that the asymmetric game is again a Markov Fat Cat game, whereas Table 5 indicates that the symmetric game is a Markov Lean and Hungry game. For the quantitative effects in the asymmetric case, substitution into (21.1) and (21.2) yields the derivatives:

$$\frac{\partial \bar{p}_1}{\partial \alpha_1} = \frac{4\beta ab^3c^2}{(c-2b)^3(c+2b)^2} > 0 \qquad\qquad \frac{\partial \bar{p}_2}{\partial \alpha_1} = \frac{2\beta ab^2c^3}{(c-2b)^3(c+2b)^2} > 0 \qquad (28)$$

[8] For a completely specified model of switching costs which utilizes a Markov perfect equilibrium, see Beggs and Klemperer (1992).

Again, these asymmetric effects suggest that output of firm 1 will be lower in the MPE steady state while output of firm 2 will be higher than in the repeated game. It is also immediate from (22) that both firms are made better off.

In the symmetric case, the strategic dynamics lead to a steady state equilibrium in which prices are *lower* than in the non-strategic case. From (22′) both firms are made worse off, in contrast to the asymmetric case. The intuition behind the symmetric result is as follows. A commitment by a firm to a low price in the current period works, through the substitutes effect, to reduce the current quantity of the rival. That, in turn, implies a higher current price for the rival, and a lower quantity, which increases the cost of the rival increasing quantity next period, and effectively commits her to a high price next period. With one firm acting alone, this would increase profits, but as usual in symmetric games, the prisoner's dilemma effect leads to an equilibrium in which both firms are worse off with lower steady state equilibrium prices.

The above analysis suggests that information about static payoff functions and adjustment costs allows us to predict how MPE steady states will differ from the equilibrium of the repeated game. In particular, steady state equilibria need not be explicitly calculated to qualitatively analyze the effects of adjustment costs in strategic environments. Although our analysis is restricted to adjustment costs close to zero, in most applications, steady state MPE are monotonic functions of adjustment costs.[9] We now examine links between steady state MPE and equilibria of static conjectural variations games.

V. Conjectural Variations and Markov Perfect Equilibrium

Conjectural Variations equilibria have been employed in Industrial Organization for a long time, since well before more rigorous game theoretic models of oligopoly became popular. A symmetric conjectural variations (cv) equilibrium to a duopoly model is characterized, for example, by first order conditions of the following type:

$$R^j_j(u) + \lambda R^j_i(u) = 0 \tag{29.1}$$

$$R^i_i(u) + \lambda R^i_j(u) = 0 \tag{29.2}$$

[9] We have verified this through simulation.

where $R^j(u)$ is the same quadratic payoff function used elsewhere in the paper and $\lambda = \partial u_i / \partial u_j = \partial u_j / \partial u_i$ is called the coefficient of conjectural variation and is specified exogenously. For example, if the above game were in quantities with standard payoff functions, then $\lambda = 0$ yields the Cournot equilibrium, $\lambda = -1$ yields the competitive equilibrium, and $\lambda = +1$ yields the cartel (perfectly collusive) outcome.

Although the idea of conjectural variations games is conceptually flawed,[10] it has proved very useful in empirical work. Where λ is estimated, as part of a structural system such as (29) above, it is a way of parameterizing the degree of competition in a particular market.[11]

Suppose we write the solution to (29.1) and (29.2) as

$$u^*(\lambda) = L\lambda + K \tag{30}$$

where L, K are 2x1 vectors of coefficients and constants. Then it is clear that we can match the cv equilibrium to the steady state MPE of our infinite horizon dynamic game. By equating (17) and (30), we could solve for the value of λ that would make the steady state MPE value and the cv equilibrium value the same. The following equation, then, implicitly defines a cv parameter as a function of economy parameters, $\lambda(\alpha, \beta)$:

$$L\lambda(\alpha,\beta) + K = M(\alpha,\beta)^{-1}G(\alpha,\beta) \equiv \bar{u}(\alpha,\beta), \tag{31}$$

where the matrices M and G have been written to emphasize the dependence of the steady state MPE on the level of adjustment costs and on the discount factor. Provided that $\lambda(\alpha, \beta)$ is a well behaved function, every steady state MPE consists of infinite repetition of a cv equilibrium for an appropriate value of λ. In a differential game, Dockner (1992) analyzes a similar function and more fully characterizes relationships between economy parameters and the magnitude of the cv parameter.

The importance of this result is not a theoretical one; rather it provides a different, a perhaps sounder theoretical underpinning for the use of cv equilibrium in empirical work.

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[10] A cv equilibrium is not a well specified game; see for example Shapiro (1989).

[11] For an excellent discussion of the theoretical problems and empirical usefulness of the cv concept, see Bresnahan(1989).

If the market under study can be regarded sensibly as a dynamic game, then the estimated cv parameter captures a well defined property of the strategic interaction, which can be matched to a MPE of a fully specified game.

The sign of λ can be used as another way of classifying equilibria to the dynamic game. Equations (30) and (31) can be rewritten as:

$$u^*(\lambda(\alpha,\beta)) = \bar{u}(\alpha,\beta). \tag{32}$$

Noting that $\lambda(0,\beta)=0$ and differentiating (32) around this value yields

$$\left.\frac{\partial \bar{u}}{\partial \alpha}\right|_{\alpha=0} = \left.\frac{\partial u^*}{\partial \lambda}\right|_{\lambda=0} \left.\frac{\partial \lambda}{\partial \alpha}\right|_{\lambda=\alpha=0}.$$

From (29.1) and (29.2) we can compute

$$\left.\frac{\partial u^*}{\partial \lambda}\right|_{\lambda=0} = \left(\frac{R_{12}-R_{11}}{\Delta}\right) R_2$$

where the derivatives of $R(\cdot)$ are evaluated at $\lambda=0$ and $\Delta=(R_{11})^2-(R_{12})^2>0$. Finally, since $R_{12}-R_{11}>0$ by stability, we have

$$Sign \left.\frac{\partial \bar{u}}{\partial \alpha}\right|_{\alpha=0} = Sign R_2 \left.\frac{\partial \lambda}{\partial \alpha}\right|_{\lambda=\alpha=0}. \tag{33}$$

Recalling that $\lambda(0,\beta)=0$, the preceding analysis suggests that the steady state MPE of the symmetric game can be classified according to the sign of λ. Letting $\hat{\lambda}$ be an empirical estimate of λ for a particular industry, then we can classify the steady state MPE of the equivalent infinite horizon dynamic game according to Table 6.

Table 6: CV Parameter Classification

	$R_2 < 0$	$R_2 > 0$
$\hat{\lambda} > 0$	$(\partial \bar{u}_j^o/\partial\alpha) < 0$ *(Markov Puppy Dog)*	$(\partial \bar{u}_j^o/\partial\alpha) > 0$ *(Markov Fat Cat)*
$\hat{\lambda} < 0$	$(\partial \bar{u}_j^o/\partial\alpha) > 0$ *(Markov Top Dog)*	$(\partial \bar{u}_j^o/\partial\alpha) < 0$ *(Markov Lean and Hungry)*

Thus, to classify oligopolies in this way requires both an estimate of λ and knowledge of R_2. The latter might be obtained either as part of a prior hypothesis, or as a result of a joint estimation procedure.

VI. Conclusions

This paper has illustrated that the presence of adjustment costs in an infinite horizon dynamic game introduces a strategic incentive to deviate from the equilibrium of the repeated static game in a steady state Markov perfect equilibrium. This occurs even though no adjustment costs are paid in the steady state and such deviations will not arise in perfectly competitive or monopolistic environments. In addition, the limit equilibrium as adjustment costs approach zero is the same as the static Nash equilibrium. This result appears to be particular to discrete time games and is in contrast to discontinuities found in continuous time games.

A classification scheme for steady state Markov perfect equilibria, based on properties of payoff functions, was shown to be identical to that of analogous two stage games. This classification is useful in that it implies that steady state equilibria need not be explicitly calculated to qualitatively analyze the effects of adjustment costs in strategic environments. Finally, it is argued that estimated conjectural variations parameters may capture a well defined property of strategic interaction in a dynamic game.

APPENDIX

Given the symmetric and quadratic structure of payoffs, we may write the first partial derivatives of $R(\cdot)$ and $P(\cdot)$ as follows where u_j denotes own choices and u_i denotes rival's choices:

$$R_1 = B_1 + R_{11}u_j + R_{12}u_i$$

$$R_2 = B_2 + R_{12}u_j + R_{22}u_i$$

$$P_1 = D_1 + P_{11}u_j + P_{12}u_i + P_{13}\tilde{u}_j + P_{14}\tilde{u}_i$$

$$P_2 = D_2 + P_{12}u_j + P_{22}u_i + P_{23}\tilde{u}_j + P_{24}\tilde{u}_i \qquad (A1)$$

$$P_3 = D_3 + P_{13}u_j + P_{23}u_i + P_{33}\tilde{u}_j + P_{34}\tilde{u}_i$$

$$P_4 = D_4 + P_{14}u_j + P_{24}u_i + P_{34}\tilde{u}_j + P_{44}\tilde{u}_i$$

Here B_k and D_k for $k = 1, 2$ are the coefficients on u_k in $R(\cdot)$ and $P(\cdot)$ respectively, and D_k for $k = 3, 4$ are the coefficients on $\tilde{u}_{k-2}$ in $P(\cdot)$.

Solving the first order conditions given by equation (14) in the text with the derivatives of the value functions given by equations (15.1) and (15.2) determines Nash equilibrium current choices conditioned on the state vector $\tilde{u}$:

$$\hat{u} = F\tilde{u} + G \qquad (16)$$

where

$$F = H\begin{bmatrix} \alpha_1 P_{13} & \alpha_1 P_{14} \\ \alpha_2 P_{14} & \alpha_2 P_{13} \end{bmatrix} \qquad G = H\begin{bmatrix} B_1 + \alpha_1 D_1 + \beta a_{11} \\ B_1 + \alpha_2 D_1 + \beta a_{12} \end{bmatrix}$$

and

$$H = \frac{1}{\Psi}\begin{bmatrix} -(R_{11} + \alpha_2 P_{11} + \beta b_{12}) & R_{12} + \alpha_1 P_{12} + \beta c_{11} \\ R_{12} + \alpha_2 P_{12} + \beta c_{12} & -(R_{11} + \alpha_1 P_{11} + \beta b_{11}) \end{bmatrix} \qquad (A2)$$

and

$$\Psi = (R_{11} + \alpha_1 P_{11} + \beta b_{11})(R_{11} + \alpha_2 P_{11} + \beta b_{12})$$

$$-(R_{12} + \alpha_1 P_{12} + \beta c_{11})(R_{12} + \alpha_2 P_{12} + \beta c_{12}).$$

Using equations (15), (19), and (A1), we can write the following for $j = 1, 2$, $i \neq j$:

$$a_{1j} + b_{1j}\tilde{u}_j + c_{1j}\tilde{u}_i = \begin{bmatrix} B_2 + R_{12}\hat{u}_j + R_{22}\hat{u}_i + \\ +\alpha_j(D_2 + P_{12}\hat{u}_j + P_{22}\hat{u}_i + P_{23}\tilde{u}_j + P_{24}\tilde{u}_i) \\ +\beta(a_{2j} + b_{2j}\hat{u}_i + c_{1j}\hat{u}_j) \end{bmatrix} F_{ij}$$
$$+\alpha_j(D_3 + P_{13}\hat{u}_j + P_{23}\hat{u}_i + P_{33}\tilde{u}_j + P_{34}\tilde{u}_i)$$

and

$$a_{2j} + b_{2j}\tilde{u}_i + c_{1j}\tilde{u}_j = \begin{bmatrix} B_2 + R_{12}\hat{u}_j + R_{22}\hat{u}_i + \\ +\alpha_j(D_2 + P_{12}\hat{u}_j + P_{22}\hat{u}_i + P_{23}\tilde{u}_j + P_{24}\tilde{u}_i) \\ +\beta(a_{2j} + b_{2j}\hat{u}_i + c_{1j}\hat{u}_i) \end{bmatrix} F_{ii}$$
$$+\alpha_j(D_4 + P_{14}\hat{u}_j + P_{24}\hat{u}_i + P_{34}\tilde{u}_j + P_{44}\tilde{u}_i)$$

Substituting for $\hat{u}_1$ and $\hat{u}_2$ from equation (16), rearranging, and matching coefficients gives the following for j=1,2; $i \neq j$:

$$\begin{aligned} a_{1j} &= G_1\gamma_{1j} + G_2\gamma_{2j} + (B_2 + \alpha_j D_2 + \beta a_{2j})F_{ij} + \alpha_j D_3 \\ a_{2j} &= G_1\delta_{1j} + G_2\delta_{2j} + (B_2 + \alpha_j D_2 + \beta a_{2j})F_{ii} + \alpha_j D_4 \\ b_{1j} &= F_{jj}\gamma_{1j} + F_{ij}\gamma_{2j} + \alpha_j(P_{23}F_{ij} + P_{33}) \\ b_{2j} &= F_{ji}\delta_{1j} + F_{ii}\delta_{2j} + \alpha_j(P_{24}F_{ii} + P_{44}) \\ c_{1j} &= F_{ji}\gamma_{1j} + F_{ii}\gamma_{2j} + \alpha_j(P_{24}F_{ij} + P_{34}) \end{aligned} \tag{A3}$$

where

$$\gamma_{1j} = (R_{12} + \alpha_j P_{12} + \beta c_{1j})F_{ij} + \alpha_j P_{13} \qquad \gamma_{2j} = (R_{22} + \alpha_j P_{22} + \beta b_{2j})F_{ij} + \alpha_j P_{23}$$

$$\delta_{1j} = (R_{12} + \alpha_j P_{12} + \beta c_{1j})F_{ii} + \alpha_j P_{14} \qquad \delta_{2j} = (R_{22} + \alpha_j P_{22} + \beta b_{2j})F_{ii} + \alpha_j P_{24}$$

(A3) represents a non-linear system of ten equations in the ten unknown coefficients contained in the derivatives of the value functions given by equation (15). If these equations can be solved for explicit solutions, those solutions can be substituted into (A2) and (16) to solve for Markov perfect equilibria.

We first show that a solution to the system of equations given by (A3) exists in a neighborhood of $\alpha_1 = \alpha_2 = 0$. Let superscript o denote the value of a variable evaluated at $\alpha_1 = \alpha_2 = 0$. From equations (A2)-(A3),

$$F^o_{ij} = a^o_{ij} = b^o_{ij} = c^o_{1j} = 0 \qquad \forall i, j$$

$$G_1^o = G_2^o = \bar{u}_1^o = \bar{u}_2^o = \frac{-B_1}{(R_{11} + R_{12})}$$

$$\Psi^o \equiv \Delta = (R_{11})^2 - (R_{12})^2$$

and

$$H^o = \frac{1}{\Delta}\begin{bmatrix} -R_{11} & R_{12} \\ R_{12} & -R_{11} \end{bmatrix}.$$

(Note that H^o is symmetric.)

In addition, for $x \in \{a_{1j}, a_{2j}, b_{1j}, b_{2j}, c_{1j}\}$, $j = 1, 2$,

$$\frac{\partial F_{ij}^o}{\partial x} = 0 \qquad \forall i, j$$

and

$$\frac{\partial \gamma_{1j}^o}{\partial x} = \frac{\partial \gamma_{2j}^o}{\partial x} = \frac{\partial \delta_{1j}^o}{\partial x} = \frac{\partial \delta_{2j}^o}{\partial x} = 0 \qquad \forall j$$

Given these relationships, it is easy to show that the Jacobian of the system of equations given by (A3) evaluated at $\alpha_1 = \alpha_2 = 0$ is a 10x10 identity matrix and is nonsingular. The implicit function theorem, then, implies that the coeffcients of the derivatives of the value functions are continuously differentiable functions of α_1 and α_2 in a neighborhood of $\alpha_1 = \alpha_2 = 0$. Furthermore, steady state equilibria given by equation (17) exist and are continuously differentiable functions of α_1 and α_2 in a neighborhood of zero adjustment costs.

Asymmetric Dynamics

Equations (21.1) and (21.2) are derived in this section. Let $\alpha_2 = 0$. Differentiating the steady state levels given by equation (17) with respect to α_1 and evaluating the derivative at $\alpha_1 = 0$ gives

$$\frac{\partial \bar{u}_1^o}{\partial \alpha_1} = G_1^o\left(\frac{\partial F_{11}^o}{\partial \alpha_1} + \frac{\partial F_{12}^o}{\partial \alpha_1}\right) + \frac{\partial G_1^o}{\partial \alpha_1} \tag{A4.1}$$

and

$$\frac{\partial \bar{u}_2^o}{\partial \alpha_1} = G_1^o\left(\frac{\partial F_{21}^o}{\partial \alpha_1} + \frac{\partial F_{22}^o}{\partial \alpha_1}\right) + \frac{\partial G_2^o}{\partial \alpha_1}. \tag{A4.2}$$

From equation (A2),

$$\frac{\partial F_{11}^o}{\partial \alpha_1} = P_{13} H_{11}^o \qquad \frac{\partial F_{12}^o}{\partial \alpha_1} = P_{14} H_{11}^o \tag{A5}$$

$$\frac{\partial F_{21}^o}{\partial \alpha_1} = P_{13} H_{12}^o \qquad \frac{\partial F_{22}^o}{\partial \alpha_1} = P_{14} H_{12}^o$$

Note that

$$\frac{\partial F_{2j}^o}{\partial \alpha_1} = \frac{-R_{12}}{R_{11}} \frac{\partial F_{1j}^o}{\partial \alpha_1} \qquad for\ j = 1,2 \tag{A6}$$

Also from equation (A2),

$$\frac{\partial G_1^o}{\partial \alpha_1} = B_1\left(\frac{\partial H_{11}^o}{\partial \alpha_1} + \frac{\partial H_{12}^o}{\partial \alpha_1}\right) + H_{11}^o\left(D_1 + \beta\frac{\partial a_{11}^o}{\partial \alpha_1}\right) + \beta H_{12}^o \frac{\partial a_{12}^o}{\partial \alpha_1} \tag{A7.1}$$

and

$$\frac{\partial G_2^o}{\partial \alpha_1} = B_1\left(\frac{\partial H_{21}^o}{\partial \alpha_1} + \frac{\partial H_{22}^o}{\partial \alpha_1}\right) + H_{21}^o\left(D_1 + \beta\frac{\partial a_{11}^o}{\partial \alpha_1}\right) + \beta H_{22}^o \frac{\partial a_{12}^o}{\partial \alpha_1}. \tag{A7.2}$$

Noting that $R_2^o = B_2 + G_1^o(R_{12} + R_{22})$, and using equations (A3) and (A5), we have

$$\frac{\partial a_{11}^o}{\partial \alpha_1} = P_{13} H_{12}^o R_2^o + G_1^o(P_{13} + P_{23}) + D_3 \tag{A8.1}$$

$$\frac{\partial a_{12}^o}{\partial \alpha_1} = P_{14} H_{11}^o R_2^o \tag{A8.2}$$

$$\frac{\partial b_{11}^o}{\partial \alpha_1} = P_{33} \qquad \frac{\partial b_{21}^o}{\partial \alpha_1} = P_{44} \qquad \frac{\partial c_{11}^o}{\partial \alpha_1} = P_{34} \tag{A8.3}$$

$$\frac{\partial b_{12}^o}{\partial \alpha_1} = \frac{\partial b_{22}^o}{\partial \alpha_1} = \frac{\partial c_{12}^o}{\partial \alpha_1} = 0 \tag{A8.4}$$

$$\frac{\partial \Psi}{\partial \alpha_1} = R_{11}(P_{11} + \beta P_{33}) - R_{12}(P_{12} + \beta P_{34}) \tag{A8.5}$$

$$\frac{\partial H_{11}^o}{\partial \alpha_1} = \frac{-H_{11}^o \frac{\partial \Psi}{\partial \alpha_1}}{\Delta} \qquad \frac{\partial H_{12}^o}{\partial \alpha_1} = \frac{P_{12} + \beta P_{34} - H_{12}^o \frac{\partial \Psi}{\partial \alpha_1}}{\Delta} \tag{A8.6}$$

$$\frac{\partial H_{21}^o}{\partial \alpha_1} = \frac{-H_{12}^o \frac{\partial \Psi}{\partial \alpha_1}}{\Delta} \qquad \frac{\partial H_{22}^o}{\partial \alpha_1} = \frac{-(P_{11} + \beta P_{33} + H_{11}^o \frac{\partial \Psi}{\partial \alpha_1})}{\Delta} \tag{A8.7}$$

Substituting equation (A8.5) into (A8.6)-(A8.7) and adding gives

$$\frac{\partial H_{11}^o}{\partial \alpha_1} + \frac{\partial H_{12}^o}{\partial \alpha_1} = H_{11}^o(H_{11}^o + H_{12}^o)[P_{11} + P_{12} + \beta(P_{33} + P_{34})] \qquad (A9.1)$$

and

$$\frac{\partial H_{21}^o}{\partial \alpha_1} + \frac{\partial H_{22}^o}{\partial \alpha_1} = H_{12}^o(H_{11}^o + H_{12}^o)[P_{11} + P_{12} + \beta(P_{33} + P_{34})]. \qquad (A9.2)$$

Substituting equations (A5), (A8.1), (A8.2), and (A9) into equations (A7.1) and (A7.2) gives the following:

$$\frac{\partial G_1^o}{\partial \alpha_1} = B_1 H_{11}^o(H_{11}^o + H_{12}^o)[P_{11} + P_{12} + \beta(P_{33} + P_{34})]$$
$$+H_{11}^o[D_1 + \beta P_{13} H_{12}^o R_2^o + \beta G_1^o(P_{13} + P_{23}) + \beta D_3)] + \beta H_{11}^o H_{12}^o R_2^o P_{14}$$

and

$$\frac{\partial G_2^o}{\partial \alpha_1} = B_1 H_{12}^o(H_{11}^o + H_{12}^o)[P_{11} + P_{12} + \beta(P_{33} + P_{34})]$$
$$+H_{12}^o[D_1 + \beta P_{13} H_{12}^o R_2^o + \beta G_1^o(P_{13} + P_{23}) + \beta D_3)] + \beta (H_{11}^o)^2 R_2^o P_{14}$$

Substituting this and equation (A5) into the expression for the derivative of the steady state choice for each player given by equation (A4) and combining terms gives

$$\frac{\partial \bar{u}_1^o}{\partial \alpha_1} = H_{11}^o(D_1 + \beta D_3) + B_1 H_{11}^o(H_{11}^o + H_{12}^o)\left[\sum_{j=1}^{4}(P_{1j} + \beta P_{3j})\right]$$
$$+\beta H_{11}^o H_{12}^o R_2^o[P_{13} + P_{14}] \qquad (A10.1)$$

$$\frac{\partial \bar{u}_2^o}{\partial \alpha_1} = H_{12}^o(D_1 + \beta D_3) + B_1 H_{12}^o(H_{11}^o + H_{12}^o)\left[\sum_{j=1}^{4}(P_{1j} + \beta P_{3j})\right]$$
$$+\beta R_2^o[(H_{12}^o)^2 P_{13} + (H_{11}^o)^2 P_{14}] \qquad (A10.2)$$

In a game with quadratic adjustment costs on a linear function of the choice variables of the form $g(u) = a + bu_1 + cu_2$, $P^j(\,)$ takes the form

$$P^j(u\ \tilde{u}) = -(g(u) - g(\tilde{u}))^2$$

or

$$P^j(u\ \tilde{u}) = -[b^2(u_1^2 + \tilde{u}_1^2 - 2u_1\tilde{u}_1) + c^2(u_2^2 + \tilde{u}_2^2 - 2u_2\tilde{u}_2) + 2bc(u_1u_2 - u_1\tilde{u}_2 - u_2\tilde{u}_1 + \tilde{u}_1\tilde{u}_2)]$$

By inspection,

$$D_1 = D_2 = D_3 = D_4 = 0$$

$$P_{11} = P_{33} = -P_{13} = -2b^2$$

$$P_{22} = P_{44} = -P_{24} = -2c^2$$

$$P_{12} = -P_{14} = -P_{23} = P_{34} = -2bc$$

Hence, the first two terms in equations (A10.1) and (A10.2) equal zero and the expressions reduce to:

$$\frac{\partial \bar{u}_1^o}{\partial \alpha_1} = \frac{-\beta R_{11}}{\Delta^2} R_{12} R_2^o [P_{13} + P_{14}] \tag{21.1}$$

and

$$\frac{\partial \bar{u}_2^o}{\partial \alpha_1} = \frac{\beta}{\Delta^2} R_2^o [(R_{12})^2 P_{13} + (R_{11})^2 P_{14}] \tag{21.2}$$

These are equations (21.1) and (21.2) in the text.

Symmetric Dynamics

Equation (24) is derived in this section. Differentiating the steady state levels given by equation (17) with respect to α and evaluating the derivative at $\alpha = 0$ gives

$$\frac{\partial \bar{u}_1^o}{\partial \alpha} = G_1^o \left(\frac{\partial F_{11}^o}{\partial \alpha} + \frac{\partial F_{12}^o}{\partial \alpha} \right) + \frac{\partial G_1^o}{\partial \alpha} \tag{A11.1}$$

and

$$\frac{\partial \bar{u}_2^o}{\partial \alpha} = G_1^o \left(\frac{\partial F_{21}^o}{\partial \alpha} + \frac{\partial F_{22}^o}{\partial \alpha} \right) + \frac{\partial G_2^o}{\partial \alpha} \tag{A11.2}$$

From equation (A2),

$$\frac{\partial F_{11}^o}{\partial \alpha} = \frac{\partial F_{22}^o}{\partial \alpha} = (P_{13} H_{11}^o + P_{14} H_{12}^o) \tag{A12.1}$$

$$\frac{\partial F_{12}^o}{\partial \alpha} = \frac{\partial F_{21}^o}{\partial \alpha} = (P_{14} H_{11}^o + P_{13} H_{12}^o) \tag{A12.2}$$

and

$$\frac{\partial G_1^o}{\partial \alpha} = B_1 \left(\frac{\partial H_{11}^o}{\partial \alpha} + \frac{\partial H_{12}^o}{\partial \alpha} \right) + H_{11}^o \left(D_1 + \beta \frac{\partial a_{11}^o}{\partial \alpha} \right) + H_{12}^o \left(D_1 + \beta \frac{\partial a_{12}^o}{\partial \alpha} \right) \tag{A13.1}$$

$$\frac{\partial G_2^o}{\partial \alpha} = B_1\left(\frac{\partial H_{21}^o}{\partial \alpha} + \frac{\partial H_{22}^o}{\partial \alpha}\right) + H_{11}^o\left(D_1 + \beta\frac{\partial a_{12}^o}{\partial \alpha}\right) + H_{12}^o\left(D_1 + \beta\frac{\partial a_{11}^o}{\partial \alpha}\right) \qquad (A13.2)$$

From equation (A3), for j=1,2; $i \neq j$:

$$\frac{\partial a_{1j}^o}{\partial \alpha} = R_2^o\frac{\partial F_{ij}}{\partial \alpha} + G_1^o(P_{13} + P_{23}) + D_3 \qquad (A14.1)$$

and using equation (A12),

$$\frac{\partial a_{11}^o}{\partial \alpha} = \frac{\partial a_{12}^o}{\partial \alpha}.$$

In addition, for j=1,2:

$$\frac{\partial b_{1j}^o}{\partial \alpha} = P_{33} \qquad \frac{\partial b_{2j}^o}{\partial \alpha} = P_{44} \qquad \frac{\partial c_{1j}^o}{\partial \alpha} = P_{34} \qquad (A14.2)$$

$$\frac{\partial \Psi}{\partial \alpha} = 2R_{11}(P_{11} + \beta P_{33}) - 2R_{12}(P_{12} + \beta P_{34}) \qquad (A14.3)$$

$$\frac{\partial H_{11}^o}{\partial \alpha} = \frac{\partial H_{22}^o}{\partial \alpha} = \frac{-(P_{11} + \beta P_{33} + H_{11}^o\frac{\partial \Psi}{\partial \alpha})}{\Delta} \qquad (A14.4)$$

$$\frac{\partial H_{12}^o}{\partial \alpha} = \frac{\partial H_{21}^o}{\partial \alpha} = \frac{P_{12} + \beta P_{34} - H_{12}^o\frac{\partial \Psi}{\partial \alpha}}{\Delta} \qquad (A14.5)$$

Therefore, equation (A13) implies

$$\frac{\partial G_1^o}{\partial \alpha} = \frac{\partial G_2^o}{\partial \alpha}$$

and from equation (A11),

$$\frac{\partial \bar{u}_1^o}{\partial \alpha} = \frac{\partial \bar{u}_2^o}{\partial \alpha}.$$

Substituting from equations (A12) and (A14) into (A13), gives the following:

$$\frac{\partial G_1^o}{\partial \alpha} = G_1^o(H_{11}^o + H_{12}^o)[P_{11} + P_{12} + \beta(P_{33} + P_{34})]$$

$$+[H_{11}^o + H_{12}^o][D_1 + \beta R_2^o(H_{11}^o P_{14} + H_{12}^o P_{13}) + \beta G_1^o(P_{13} + P_{23}) + \beta D_3]$$

Substituting this and equation (A12) into the expression for the derivative of the steady state choice for player j given by equation (A11) and combining terms gives

$$\frac{\partial \bar{u}_j}{\partial \alpha} = (H_{11}^o + H_{12}^o)(D_1 + \beta D_3) + G_1^o(H_{11}^o + H_{12}^o)\left[\sum_{j=1}^{4}(P_{1j} + \beta P_{3j})\right]$$

$$+\beta(H_{11}^o + H_{12}^o)R_2^o(P_{14}H_{11}^o + P_{13}H_{12}^o) \qquad (A15)$$

As discussed in the asymmetric game, in an adjustment cost game on a linear function of the choice variables, the first two terms in equation (A15) equal zero and the expression reduces to

$$\frac{\partial \bar{u}_j}{\partial \alpha} = \frac{\beta(R_{11} - R_{12})}{\Delta^2} R_2^o(R_{11}P_{14} - R_{12}P_{13}) \qquad (24)$$

This is equation (24) in the text.

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