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Incomplete Diversification and Asset Pricing

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1. Introduction

Asset pricing theories explain risk premia on financial assets as compensating investors for risk exposures or risks that investors cannot diversify. The theories differ in their specification of these undiversifiable or systematic risks. In the Sharpe [1964], and Lintner [1965] Capital Asset Pricing Model investors are only exposed to the risks of the market portfolio. The Arbitrage Pricing Theory of Ross [1976] has investors exposed to a finite set of factor risks. While in the consumption beta models of Merton [1973] and Breeden [1979] investors face real consumption risk. In these theories investors reduce risk by diversifying their portfolios across the universe of assets.

This paper focuses on another dimension of diversification and the resulting asset pricing model, more akin to insurance. By aggregating across the risk exposures of a large number of investors we derive an asset pricing model that averages out many investor specific risk concerns. Hence even if investors have to take positions in many specific risks and diversification is incomplete at the individual investor level, many of these nondiversifed risks need have no impact on the market prices of assets. For emphasis and exactness we model an economy with *infinitely many investors*, in which each single investor is insignificant.

Formally, we define, in a general and abstract setting, the concept of *investor specific risk exposures* in equilibrium in terms of measurability with respect to an appropriate investor specific σ-algebra of events. We then identify these investor specific risk exposures and relate them to *personalized asset pricing models*. An asset pricing model for the economy or a *market asset pricing model* is obtained by aggregating personalized asset variables defined on the probability space $(\Omega,\mathcal{F},P)$, where Ω is a set of events, $\mathcal{F}$ is a σ-algebra of events and P is a probability measure. For generality we suppose that preferences are defined over an attainable convex set $X_i \subset V$.

Each investor $i \in I$ is supposed to have a monotone increasing, continuous and quasi-concave utility function u_i defined on X_i.[1] Cash flows at time 1 are obtained by holding assets at time 0. There are a finite set J of assets indexed by j, with claims to time 1 state contingent cash flows $Z_j \in V$ for all $j=1,\ldots,J$.[2] Let the vector a_i denote investor i's holdings of the J assets, the associated time 1 cash flow is given by the linear operator $Z[a]=\sum_j Z_j a_{ji}$ that maps $\mathbb{R}^J$ into V. Each investor also has an initial endowment of assets $\bar{a}_i$.

It is easily shown that the set of feasible asset portfolios for i, $A_i = Z^{-1}(X_i \cap Z[\mathbb{R}^J])$, is convex. Define induced preferences on A_i by $u_i^*(a_i)=u_i(Z[a_i])$. These induced preferences inherit the properties of being continuous and quasi-concave from u_i and the linearity and continuity of the operator Z. Furthermore, we also suppose nonsatiation of u_i^* or the absence of bliss points.

Consider an economy with a countable infinity of investors.[3] One may therefore suppose, without loss of generality, that I, the index set for the investors, is the set of all natural numbers or positive integers. Since we wish to model individual investors as insignificant in the infinite economy, we follow Aumann [1964] and Ostroy [1984], by modeling investors as having zero measure. Accordingly we take the space of investors to be a finitely additive non-atomic measure space $(I,\mathcal{A},\mu)$, where I is the set of positive integers, $\mathcal{A}$ is the algebra of all subsets of I and μ is a finitely additive

So for example the function $h(i)=1/i$ is strictly positive and null. Null perturbations have no effect on the limits of average allocations taken over a sequence of economies with a population tending to infinity and the weighting of single investors approaching zero. It is precisely for this reason, that from the perspective of the limit economy, such perturbations are admissible without disturbing the limit equilibrium.

The definition of equilibrium used by Weiss [1981] is in terms of these equivalence classes for allocations. Equilibria have the property that investors may deviate from their utility maximizing allocations by a null function without disturbing the market clearing condition of the limit economy.

A competitive equilibrium for the asset exchange economy over the infinite set of investors I is defined as follows:

Definition: An *attainable allocation* is a μ integrable function $a:I \longrightarrow \mathbb{R}^J$ such that $a_i \in A_i$ for all i and

$$\int_I a_i d\mu(i) = \int_I \bar{a}_i d\mu(i) .$$

Definition: An attainable allocation is *budget feasible for the price system* $p \in \mathbb{R}^J$ if there exists a subset $A \subseteq I$ with $\mu(A)=\mu(I)$ and a null function $h:I \longrightarrow \mathbb{R}^J$ such that

$$p^T(a_i - h_i) = p \cdot \bar{a}_i \quad \text{for all } i \in A.$$

where the superscript T denotes transposition. The definition of budget feasibility permits individual exceptions to the budget constraint for a null set of investors and for a non-null set by a null aggregate.

Definition: A *competitive equilibrium* is an attainable allocation a^* and a price system p^* such that a^* is budget feasible for p^* and for some subset $A \subseteq I$, $\mu(A)=\mu(I)$ and null functions $h^*:I \longrightarrow \mathbb{R}^J$, $k^*:I \longrightarrow \mathbb{R}^1$

marginal utilities.

Lemma 1. For each $i \in I$, there exist random variables $\psi^i(\omega)$, $\psi_0^i(\omega) \in V^i$; $\psi^i, \psi_0^i \geq 0$ a.e. in ω with respect to P, such that,

(1) $$\partial u_i^*(a_i^*)/\partial a_{ij} = \int \psi^i(\omega) Z_j(\omega) P(d\omega)$$

(2) $$\partial u_i^*(a_i^0)/\partial a_{ij} = \int \psi_0^i(\omega) Z_j(\omega) P(d\omega)$$

Proof. See Appendix.

The random variables ψ^i and ψ_0^i are the marginal utilities of state contingent cash flows evaluated at the cash flows arising from the equilibrium and optimal asset holdings a_i^* and a_i^0 respectively.

Theorem 2. For all i, the market price of traded assets p_j^* satisfies,

(3) $$p_j^* = E^P[\lambda_0^i Z_j] \ .$$

Proof. Since u_i^* is maximized for all i with respect to the budget constraint, the first order condition implies that

(4) $$\partial u_i^*(a_i^0)/\partial a_{ij} = \gamma_0^i p_j^* \ .$$

The result follows from (1) defining $\lambda_0^i = \psi_0^i/\gamma_0^i$.■

Define the linear valuation operators, $\Phi_0^i[x]$, $\Phi^i[x]$ by

(5) $$\Phi_0^i[x] = E^P[\lambda_0^i x] \qquad \text{for } x \in V$$

and

(6) $$\Phi^i[x] = E^P[\lambda^i x] \qquad \text{for } x \in V$$

where $\lambda^i = \psi^i/\gamma_0^i$.

The random variables λ_0^i and λ^i are state price functions (Duffie [1988]) and define the state contingent discount to be applied to future or time 1 cash flows in determining their contribution to current values. The linear operator Φ^i defined by (6) provides a personalized valuation of x thousand dollars in these four states. If the investor's equilibrium state price function turns out to be insensitive to weather conditions in the distant country but responsive to her state of health, with the state price function taking on for example the values .1, .1, .3, and .3 in the states GC, GS, BC, and BS then the weather in the distant country is not a risk concern while her state of health is. The personal valuation of x, $\Phi^i[x]$ is the same as the personal valuation of $E^P[x|\text{state of health}]$, or the cash flow 1.5, 1.5, 3.5 and 3.5. The investor may be thoght of as first averaging out events with respect to which no risk adjustment turns out to be necessary in equilibrium, and then prices the resulting cash flow, taking account of personally required risk compensations. Note in this context that even if the utility function is insensitive to weather in the distant country, the state price function may be sensitive to such events if the traded cash flows Z_j are responsive to such events.

Definition: The σ-algebra $\mathcal{G}^i$ defines *investor i's risk exposure in equilibrium* if $\mathcal{G}^i$ is the smallest σ-algebra satisfying

$$\Phi^i[x] = \Phi^i[\ E^P[x|\mathcal{G}^i]\ ].$$

If the value of x to i, at the margin equals the value to i of the expectation of x conditional on $\mathcal{G}^i$, then investor i is marginally, $\mathcal{G}^i$ conditionally, risk neutral. Hence investor i's risk concerns or relevant risk exposures are captured in the σ-algebra $\mathcal{G}^i$.

The example motivating this definition suggests that $\mathcal{G}^i$ is related to the sensitivity of equilibrium marginal rates of substitution to events. This suggestion is confirmed in Theorem 3 below. Specifically, let $\mathcal{F}^i = \sigma(\lambda^i)$ be the smallest σ algebra with respect to which λ^i is measurable.

let $Q=(q_k, k=1,\ldots,\infty)$ be a countable orthonormal basis for V. Since V is self dual, λ^i is in the closed linear span of Q and we may write that

(7) $$\lambda^i = \sum_{k=1}^{\infty} \phi_{i,k} q_k \ .$$

Define

$$Q^i = \{ q_k \mid \phi_{i,k} \neq 0 \}$$

as the set of basis elements that is actually required to span λ^i. For purposes of simplification or empirical approximation we may suppose that Q^i is finite. Standard arguments now enable us to derive the personalized asset pricing model

(8) $$\mu \approx \gamma_0^i + \beta^i \gamma_1^i$$

where μ is the vector of mean returns on the traded assets, β^i is the matrix of asset beta's with respect to elements of Q^i and γ_0^i and γ_1^i are constants. Expression (8) is written as an approximation for this economy on two counts. First, asset prices are approximately given by the operators Φ^i, with the difference being arbitrarily small for all but finitely investors, and second, an approximation may be involved in getting Q^i to be finite.

The number of factors involved in the linear representation (7) may be unduly large if λ^i is infact a nonlinear function of a few factors, say

(9) $$\lambda^i = \lambda(S_1,\ldots,S_{K(i)})$$

where $S_1,\ldots,S_{K(i)}$ are the K(i) factors needed to describe nonlinearly the variations or measurability of λ^i. Equation (8) provides us with a K(i) dimensional nonlinear representation of λ^i.[6] This may be further reduced to a linear model by introducing as separate factors the products of powers of the the primary factors in the nonlinear representation. The representation (9) clearly subsumes (7) and allows for more powerful dimensional reductions of λ^i

of the insured, by essentially an application of the law of large numbers.

It is first established that the average of personalized values equals market prices. This is done by showing that the operator $\Phi^i - \Phi^i_0$ is a null operator in that for all x, $\Phi^i[x] - \Phi^i_0[x]$ is a null function of i. For this theorem we employ a condition on the norm boundedness of the first and second Fréchet differentials of the u_i's.

Assumption 2. Suppose that u_i is twice Fréchet differentiable and that there exists a constant C such that, $\|Du_i[x;\cdot]\|$ and $\|D_2u_i[x;\cdot]\|$ is uniformly bounded by C for all x and i.

<u>Theorem 4.</u> Assumption 2 implies that $\Phi^i - \Phi^i_m$ and $\Phi^i_0 - \Phi^i_m$ are a null operators.

<u>Proof.</u> (See Appendix).

Suppose Assumption 2 and let Φ be the average of the operators Φ^i, more precisely

$$\Phi[x] = \int_I \Phi^i[x]d\mu(i).$$

The norm boundedness of Φ^i under Assumption 2 implies that Φ is a continuous linear functional on V and hence there exists λ such that

$$\Phi[x] = E^P[\ \lambda x\ ]. \tag{10}$$

Define $\mathcal{M}$ as the smallest σ-algebra with respect to which λ is measurable. We will show that, unlike the operators Φ^i, Φ agrees with market prices for traded assets. Furthermore there is a precise relationship between the σ-algebra $\mathcal{M}$ and the σ-algebras $(\mathcal{F}^i = \mathcal{G}^i,\ i \in I)$ whereby $\mathcal{M}$ is considerably smaller than the union of the $\mathcal{F}^i$'s. Hence $\mathcal{M}$ is a candidate for a relatively parsimonious specification of market risk exposures or systematic risks.

The tail algebra can be considerably smaller than the union of individual σ-algebras $\sigma(\lambda^i)$. Hence many risk factors relevant to individual investors need not be important in the market place for pricing assets. A sufficient condition useful in providing examples where $\mathcal{M}$ is considerably smaller is given by the following theorem.

Theorem 7. If $\mathcal{D} \subseteq \mathcal{F}^i$, for all i and the sequence of σ-algebras $\mathcal{F}^i$ are conditionally independent, conditional on $\mathcal{D}$ then $\mathcal{M} = \mathcal{D}$.

Proof. This is a consequence of the conditional zero-one law (See Appendix). The variables defining $\mathcal{D}$ measurability can be likened to risks accounted for in determining insurance premiums, the additional variables needed to define $\mathcal{F}^i$ measurability are personal risks that the insurer avoids through aggregating across the pool of insurers. Hence life insurance premiums may vary with smoking habits as this item has been isolated as an important part of $\mathcal{D}$, while many other factors affecting personal life risks, elements of $\mathcal{F}^i$, are ignored for the purposes of setting life insurance premia.

If we define by Q^M the basis elements needed to span λ in equation (11), then by standard arguments we may derive the *exact* asset pricing model

(13) $$\mu = \gamma_0 + \beta\gamma_1$$

where β is now the matrix of asset beta's with respect to the elements of Q^M. Unlike expression (8), equation (13) is exact as the operator (11) gives asset prices exactly. Since $\mathcal{M}$ is contained in the tail algebra of the $\mathcal{F}^i$'s, the number of factors represented in (13) is expected to be considerably smaller than the union of all the factors represented in the personalized asset pricing models.

$$Z[a] = \alpha^T a + S^T B a + u^T a$$

where α is the vector of coefficients α_j and B is a matrix with K rows $\beta=(\beta_j, j\in J)$.

The single investor's utility function may now be written as

$$U_i = u_i(\ \alpha^T a_i + S^T Ba_i + u^T a_i + y_i,\ S,\ v^i).$$

It follows from the specification of u_i and the Frechet differentiability of u_i with respect to the traded time 1 cash flow w that the Frechet differential of u_i, $\delta u_i(w,h)$ takes the form

$$\delta u_i(w,h) = \int_\Omega \psi_i(w,S,v^i)h(\omega)P(d\omega)$$

from which it follows that investor i's state price function has the form

$$\lambda^i = \Lambda^i(\ \alpha^T a_i + S^T Ba_i + u^T a_i + y_i\ ,\ S,\ v^i\ )$$

where $\Lambda^i = \psi_i/\gamma_0^i$.

The risk factors priced by investor i in equilibrium are therefore given by

$$\mathcal{F}^i = \sigma(\ \lambda^i\ ) \subseteq \sigma(S, v^i, u^T a^i + y_i) = \mathcal{K}^i$$

where $\sigma(X)$ for a vector of random variables X also denotes the smallest σ-algebra with respect to which the vector X is measurable. Within this general framework we can discuss a number of special cases that have received attention in the literature.

First consider models in which both v^i and y_i are absent. For example, Ross [1976], Connor[1984], Milne [1988] discuss the diversification of the idiosyncratic components $u^T a^i$ by essentially setting out conditions under which $u^T a^i$ is zero for each i. The factors then reduce to S with no necessity of invoking a law of large numbers. The associated conditions on preferences and asset returns are however quite strong. Milne [1988] also discusses approximate asset pricing models with $u^T a^i$ approaching zero as the number of

$$L_i = \chi_i + \zeta_i^T S + y_i$$
$$p_i = \kappa_i + \xi_i^T S + v_i$$

Substituting back into (19) we obtain

(20) $$\lambda^i = \Pi^i(Z[a_i] + \chi_i + \zeta_i^T S + y_i, \kappa_i + \xi_i^T S + v_i).$$

Now perform the regression (16) and substitute into (20) to obtain the form (17). To derive (18) we require conditional independence of $(u^T a_i + y_i, v_i)$, conditional on S. This might require us to expand S to include portfolios that are useful in predicting Z_j in the regression (16) even though they may not be significant in explaining L_i or p_i. Under multivariate normality of (S, u, y, v) the conditional independence follows from the orthogonality of (u,y,v) and S obtained on the three regressions for Z_j, L_i, and p_i on S.

The factors relevant for asset pricing suggested by our model of an asset exchange economy include those factors that explain the cross sectional variation across investors of effects on marginal utilities or the investor specific duals λ^i. This may usefully be contrasted with the more traditional approach of focusing solely on explaining the cross sectional variation across assets of asset returns. The important insight into asset pricing gained from our analysis is precisely the proposition that empirical work on asset pricing needs to focus on factors relevant in explaining the investor specific pricing duals λ^i across i in addition to identifying factors explaining Z_j across the set of assets.

Once we have established the validity of (18) for some set of factors S, a traditional K factor approximate asset pricing model may be derived by invoking a first order approximation to the function Λ using familiar arguments (See Breeden [1979], Grossman and Shiller [1982], Madan [1988], Milne [1988] and Back [1990]).

that arise from incomplete diversification of personal risks across the space of assets. Personalized investor specific asset pricing models reflect the multitude of these risks. By averaging across the pool of investors, in a manner akin to how insurers average risks across the pool of the insured, market risk exposures and asset pricing models are derived. It is observed on invoking a law of large numbers applied to an infinite population of investors that many personally relevant risk considerations can be eliminated from the market asset pricing model.

Examples illustrating the effects of undiversified labor income and taste specific price indices are provided. An important insight into asset pricing gained from our analysis is the proposition that work on asset pricing needs to focus on identifying and explaining investor specific risk exposures cross sectionally across the pool of investors in addition to explaining the variation of asset cash flows. In this sense the approach outlined here is jointly focused on both the pricing dual and the primal aspects of asset cash flows.

that is the limit of measures relevant for finite economies and reflects the limits of averages.

We first define a sequence of finitely additive measures μ_n on the set of all subsets of I as follows:

$$\mu_n(A) = |A\cap L(n)|/n$$

where $L(n) = \{k|1\le k\le n\}$, $|X|$ denotes the cardinality of the set X, and μ_n is the proportion of elements less than or equal to n that belong to A. It is clear that μ_n is a finitely additive measure on the set of all subsets of I. Since μ_n is a function from $\mathcal{A}$ the set of all subsets of I into the unit interval I we may think of μ_n as an element of the set $I^{\mathcal{A}}$. If we endow $I^{\mathcal{A}}$ with the product topology of the Euclidean topology on I then $I^{\mathcal{A}}$ is a compact set by Tychonoff's theorem. Therefore the set $\{\mu_n|n\ge 1\}$ has an accumulation point μ. Note that $\mu(A)=\lim_n\mu_n(A)$ whenever this limit exists. Hence, since for all finite sets A, $\lim_n\mu_n(A)=0$ the μ measure of all finite sets is zero.

For the finite additivity of μ, suppose that A_1 and A_2 are two disjoint sets with $A=A_1\cup A_2$. Since μ is an accumulation point there exists a subsequence μ_{n_k} such that $\lim_k\mu_{n_k}(A_1)=\mu(A_1)$, $\lim_k\mu_{n_k}(A_2)=\mu(A_2)$ and also $\lim_k\mu_{n_k}(A)=\mu(A)$. Now, by the finite additivity of the μ_n's, we have that for all k, $\mu_{n_k}(A_1)+\mu_{n_k}(A_2)=\mu_{n_k}(A)$, and it follows on taking limits that

$$\mu(A)=\mu(A_1)+\mu(A_2).$$

To observe that μ is non-atomic, observe that for each m we may define sets $C_1, C_2,\ldots, C_m$ such that $k\in C_i$ just if $i = 1 + k \bmod m$. For each i and n equal to mN, $\mu_n(C_i)=1/m$, while for n exceeding mN, we have that

$$N/(mN + m-1) \le \mu_n(C_i) \le (N+1)/(mN + 1)$$

Since as n and N tend to infinity these upper and lower bounds converge to 1/m, it follows that $\mu_n(C_i)$ converges to 1/m and so $\mu(C_i)$ equals 1/m for

Definition. An allocation a^B is budget feasible for the group B for prices p, if there exists a real valued null function h such that, for almost all $i \in B, pa_i^B + h_i \leq p\bar{a}_i$

Definition. An allocation a^B is preference maximal for B if,

a) a^B is budget feasible,

b) for every allocation c^B of B, if $c^B >_B a^B$, then there exists $S \subseteq B$, $\mu(S)>0$, such that the restriction of c^B to S is not budget feasible for S.

Definition. A Weiss competitive equilibrium (WCE) for an Asset Exchange Economy is a price vector $p^* \in R^J$, and an allocation a^* such that:

1) for all $B \subseteq I$, $\mu(B)>0$, the restriction of a^* to B is preference maximal for B;

2) $\int a_i^* d\mu(i) = \int \bar{a}_i d\mu(i) = \bar{a}$.

The existence of competitive equilibrium for such an economy can be established using a modification of the arguments in Weiss to account for short sales along the lines of Milne [1976].

We now establish the equivalence between a competitive equilibrium and a WCE under assumption 1.

Suppose first that we have a competitive equilibrium. Therefore there exists $A \subseteq I$, $\mu(A)=\mu(I)$ and h_i^*, k_i null functions satisfying

$$\text{i)} \quad p^*(a_i^* - h_i^*) = p^*\bar{a}_i$$

and

$$\text{ii)} \quad u_i^*(a_i^* - h_i^*) \geq u_i^*(a_i^0) - k_i$$

Define the real valued function $h_i = p^* h_i^*$, and note that as h_i^* is null, so also is h_i. It follows from property i) that

$$p^* a_i^* - h_i \leq p^* \bar{a}_i$$

for all $i \in A \cap B$, which is almost everywhere in B for all B of positive measure. Hence a^* is budget feasible for all B, $\mu(B)>0$.

clauses for entry into Γ. For any such h, let k be defined by

$$k_i = u_i^*(a_i^0) - u_i^*(a_i^* - h_i)$$

We wish to show that if k is not null then a^0 contradicts preference maximality of a^* for some set of positive measure.

Suppose that k is not null. Since $a_i^* - h_i$ is budget feasible for i, k_i is non-negative. k not null, implies that there exists a set of positive measure B such that k_i exceeds a constant c for all i in B. Consider now the restrictions to B of a^0 and $(a^* - h)$, $a^0|_B$ and $(a^* - h)|_B$ respectively. For all $i \in B$, $u_i^*(a_i^0) > u_i^*(a_i^* - h) + c$. By Assumption 1, choose δ such that $|a - b| \leq \delta$ implies $|u_i^*(a) - u_i^*(b)| \leq c/4$. Since for null functions s_i and t_i the norms are almost everywhere less than δ, we have that $u_i^*(a_i^0 - s_i) > u_i^*(a_i^* - h + t_i)$ for almost all $i \in B$. Equivalently,

$$a^0 \succ_B (a^* - h)$$

As the points a^* and a^*-h are in the same equivalence class modulo null functions this implies that

$$a^0 \succ_B a^*$$

However, $a^0|_B$ is budget feasible for all subsets S of B, and so we have a contradiction of a^* being preference maximal for B. Therefore k must be null.

<u>3. Proof of Lemma 1.</u>

Since $u_i^* = u_i(Z[a_i])$, the differential of u_i^* with respect to a_i is the Fréchet differential of u_i evaluated at $x_i^* = Z[a_i^*]$ applied to the differential of Z with respect to a_i, which is Z_j. The Fréchet differential of u_i evaluated at x_i^* is a linear operator which by the self duality of V is gievn by an element of V that we denote $\lambda^i(\omega)$, with the application to Z_j being as described in (1). Nonnegativity of λ^i follows from A.4. The construction of $\lambda_0^i(\omega)$ is similar, except that we now work with allocation a^0

Let $a'_i = a^*_i - h^*_i$ with a^* and h^* satisfying the conditions of theorem 3. Since $u^*_i(a^0_i) - u^*_i(a'_i) \leq k_i$ it follows that

$$\Psi_i(d_i(a'_i)) \leq u^*_i(a^0_i) - u^*_i(a'_i) \leq k_i$$

Now $d_i(a'_i) \geq \alpha$ implies that $\Psi_i(d_i(a'_i)) \geq \Psi_i(\alpha)$ as Ψ_i is monotone increasing. It follows then that $k_i \geq \Psi_i(\alpha)$. Since $\Psi_i(\alpha)$ is positive for positive α, $d_i(a'_i)$ not null implies k_i not null. But as by theorem k_i is null, we must have that $d_i(a'_i)$ is null.

Now choose a^0_i in Δ^i so that $\|a'_i - a^0_i\| \leq d_i(a'_i) + 1/i$ and theorem 1 holds for a^0_i with $a'_i - a^0_i$ being a null function. This implies that $Z[a'_i] - Z[a^0_i]$ is a null function and by an argument similar to that used for $\Phi^i - \Phi^i_m$, we have that $\Phi^i_0 - \Phi^i_m$ is a null operator.

5. The Conditional Zero One Law.

Consider a complete probability $(\Omega, \mathcal{F}\ P)$ and a complete σ-field $\mathcal{D} \subseteq \mathcal{F}$. Suppose $X_1, X_2, \ldots$ are random variables on $(\Omega, \mathcal{F}, P)$ which are conditionally independent given $\mathcal{D}$. Write for $n>m$,

$$\mathcal{F}^n_m = \sigma(X_m, X_{m+1}, \ldots, X_n)$$

and

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{F}^{\infty}_n$$

<u>THEOREM</u> $\mathcal{C} = \mathcal{D}$

<u>Proof</u> Suppose $A \in \mathcal{C}$. Then $A \in \mathcal{F}^{\infty}_1 = \bigcup_{n=1}^{\infty} \mathcal{F}^n_1$. By the monotone class theorem there exist sets $A_n \in \mathcal{F}^n_1$ such that $P(A \Delta A_n | \mathcal{D}) \longrightarrow 0$ as $n \longrightarrow \infty$. That is ,

$$\lim_n P(A_n | \mathcal{D}) = P(A | \mathcal{D})$$

and

$$\lim_n P(A \cap A_n | \mathcal{D}) = P(A | \mathcal{D})$$

But $A \in \mathcal{C} \subseteq \mathcal{F}^{\infty}_{n+1}$, and so A and A_n are conditionally independent given $\mathcal{D}$.

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FOOTNOTES

[1]The utility function could represent the utility of consuming the entire cash flow at time 1 or it could represent the optimized utility of a dynamic program beginning at time 1. The utility function could also be used to represent the immediate one period objectives of institutional investors, firms or other members of the investing community.

[2]Extending the results of this paper to the case of infinitely many assets is an interesting and useful research problem. There are however technical difficulties associated with the double infinity of assets and investors.

[3]We restrict to a countable infinity of individuals since the law of large numbers does not hold for a continuum, (See Judd [1983], Feldman and Gilles [1985]).

[4]The measure space is atomic if some subset of positive measure cannot be split into two sets of strictly lower measure. We shall take our measure space of individuals to be non-atomic, and hence single individuals must have zero measure.

[5]We are indebted to Ravi Bansal, Wayne Ferson and Mark Weinstein for discussions on these aspects of representing linear pricing rules.

[6]For a recent empirical implementation of such a nonlinear representation for a linear pricing rule the reader is referred to Bansal and Vishwanathan (1992).