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a Differential R & D Duopoly Game

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Abstract

At each date, the two players play an R & D investment game "followed" by a Cournot quantity setting game. Each player's R & D investment augments the common stock of technical knowledge and lowers goods production costs for each player. Profits gross of R & D investment expenditures are quadratic in the state (knowledge here) for each player. R & D investment costs are assumed quadratic in each player's investment. The Nash feedback and Nash open-loop solutions differ in general with the feedback solutions being "more competitive", i.e., yielding lower production costs in the steady state.

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Introduction

We extend the two stage duopoly R & D game to one in continuous time with repeated forward looking play. At each date the two players play an R & D investment game and the outcome of this play yields a technology for goods production. Goods production "follows" in a Cournot game in quantities, given the current technology at the same date. The two stages, R & D investment by i followed by the quantity of output decision by i, of Spencer-Brander [1983] and d'Aspremont-Jacquemin [1988] occur at each date of the dynamic duopoly game. There are spillovers in R & D investment. Though each player invests in R & D, there is only one stock of R & D at each date and this stock defines each player's current production cost. In other words at each date, a dynamic R & D duopoly investment game is followed by a static duopoly production game. With a linear market demand schedule, we have a gross profit function quadratic in the state variable for each player. Investment costs are treated as quadratic, leaving us with a quadratic objective function for each player. This leads to the open-loop Nash and feedback Nash solutions differing. We establish that the stock of R & D is larger under the feedback Nash solution. It can be viewed as "more competitive".

The Model

At each date t, the two players play an R & D investment game and realize profits from a simultaneous "downstream" Cournot quantity "game" in which current production costs depend directly on the outcome of the R & D investment game. With a linear market demand curve, each player's current profits gross of R & D expenditure are <u>quadratic</u> in the R & D stock or state of technology for production. We have then the inverted market demand schedule $p(t) = \alpha - q_1(t) - q_2(t)$, $\alpha > 0$ where $q_i(t)$ is the current output of firm i and $p(t)$ is current market price of output. Firm i's gross profit is

$$\pi_i = p(t)q_i(t) - Kq_i(t) \qquad i = 1,2$$

where K is the unit cost for producing output. (K will be lowered by investment in R & D in the R & D investment game.) In a Cournot duopoly "game" in quantities, each firm maximizes π_i given q_j parametric. This yields equilibrium quantities

$$q_i = \frac{\alpha-K}{3}, \qquad \alpha > K \qquad i = 1,2$$

which in turn yields

$$\pi_i = [K^2 - 2\alpha K + \alpha^2]/9 \qquad i = 1,2 \tag{1}$$

(which yields $\pi_i = q_i^2$).

At each date, firm i expends $cI_i + \frac{1}{2}I_i^2$ on R & D where c is a non-negative parameter. Each firm then maximizes

$$F_i = \pi_i - cI_i - \tfrac{1}{2}I_i^2 \qquad i = 1,2 \tag{2}$$

by choice of R & D intensity I_i. These investments in new knowledge cause the state of technical knowledge to improve or production costs Kq_i with state K to decline in

$$\dot{K} = s[A - I_1 - I_2 - K] \tag{3}$$

where s is a parameter representing the speed of decline in K in response to investments and A is a positive parameter. K can be thought of as the inverse of the technical knowledge stock and I_i is a flow of investment in improved technical knowledge. Note that there is net and gross investment in technical knowledge. Thus in a steady state ($\dot{K}$=0), "replacement" investment $I_1 + I_2 = A - K$ is required. Investment in excess of this replacement level is cost reducing for production. Given initial K_0, investments $\{I_1\}$, $\{I_2\}$ bring K(t) down to its steady state level. At that level we assume I_1^∞, I_2^∞ positive. We assume positive spillovers from i's investment in R & D to j via the commonality of stock K. We thus build in imperfect appropriability as intrinsic, and symmetric in the sense that a player's current investment of a unit of I_i has the same effect on K as the investment of a unit of I_j.

The Open Loop Case

In this case, the current value Hamiltonian for player i is

$$H^i = \pi_i(K) - cI_i - \tfrac{1}{2}I_i^2 + m_i s[A - I_1 - I_2 - K] \qquad i = 1,2$$

Necessary conditions are

$$H^i_{I_i} = -c - I_i - m_i s = 0 \tag{4}$$

$$-\dot{m}_i(t) = -rm + \frac{2K}{9} - \frac{2\alpha}{9} - ms \tag{5}$$

$$\text{and } \lim_{t\to\infty} e^{-rt} m_i(t) = 0 \tag{6}$$

It follows that

$$m_i(t) = \int_t^\infty e^{-(s+r)(\tau-t)}\left(\frac{2K(\tau)}{9} - \frac{2\alpha}{9}\right)d\tau \tag{7}$$

From (7), we observe that $m_1(t) = m_2(t)$ since each has the same right hand side. Thus $I_1(t) = I_2(t) = I(t)$ from (4). We can differentiate (4) to get $\dot{m}s$ and substitute in (5). This yields

$$\dot{I} = -(r+s)(-c-I) + \frac{2sK}{9} - \frac{2\alpha s}{9} \tag{8}$$

Stationary strategies involve $\dot{I} = \dot{K} = 0$. In Appendix I, we sketch a result on convergence of K_0 to K^*. From (3) and (8) we obtain

$$I^* = \frac{-(r+s)c - \frac{2}{9}s(A-\alpha)}{r + \frac{5}{9}s} \tag{9}$$

$$K^* = \frac{(r+s)(A+2c) - \frac{4\alpha s}{9}}{r + \frac{5}{9}s} \tag{10}$$

As $s \to \infty$, the speed of response of K to investments in R & D approaches infinity (instant response). In this case (10) becomes

$$K^L = \frac{A + 2c - \frac{4\alpha}{9}}{\frac{5}{9}} \tag{11}$$

This turns out to be the same value of K as would emerge in a static, Cournot R & D game between our two players. That is, each player substitutes $K = A - I_1 - I_2$ from (3) into $\pi_i(K)$ and then maximizes $\pi_i - cI_i - \frac{1}{2}I_i^2$ with I_j treated as parametric. The solution value for K is the same as that in (11). Thus the limiting value for K in (11) corresponds to the solution to a static Cournot duopoly game in I_1 and I_2. This conforms with our intuition.[1] Of central interest is the comparison of the steady states under feedback strategies with the open loop case above.

Feedback Nash Strategies

To find feedback strategies $I_i(K(t), t)$ we form the value function for each player and solve by dynamic programming:

$$rV^i(K) = \max_{I_i} \left\{ \frac{K^2 - 2\alpha K + \alpha^2}{9} - cI_i - \tfrac{1}{2}I_i^2 + sV_K^i[A - I_i - I_j - K] \right\}$$

$$i = 1,\ 2;\ i \neq j \tag{12}$$

where V_K^i refers to the derivative of $V^i(\ )$ with respect to K. Maximization in (12) with respect to I_i yields

$$I_i(K) = -[c + sV_K^i] \qquad i = 1,2 \tag{13}$$

When we substitute from (13) in (12), we obtain

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[1] There is an "opposite" static duopoly game in which each player treats K as parametric. In this case $I_1 = I_2 = -c$ and $K = A + 2c$. In (10) when $\lim_{r \to \infty}$ is taken, we obtain this value of K.

$$rV^i = \frac{K^2-2\alpha K+\alpha^2}{9} + (c + sV_K^i)c - \frac{(c+sV_K^i)^2}{2}$$
$$+ sV_K^i[A - K + 2c + sV_K^i + sV_K^j] \qquad i=1,2;\ i\neq j \qquad (14)$$

Note that both V_K^i and V_K^j appear in (14). These are the intrinsic interdependencies between each player's optimization.

We propose "candidate" solutions $V^i(K)$ for (14) and proceed to solve. Since our "candidate" solutions turn out to "work", we obtain solutions to (12) which yield steady state Nash feedback values for I_i, I_2, and K. Our "candidate" for $V^i(K)$ is

$$V^i(K) = \delta_i - \phi_i K + \mu_i \frac{K^2}{2} \qquad i=1,2 \qquad (15)$$

which yields

$$V_K^i = \mu_i K - \phi_i \qquad i=1,2 \qquad (16)$$

Substitution from (15) and (16) in (14) yields, after some simplification (see Appendix II),

$$r\delta - r\phi_i K + r\mu_i \frac{K^2}{2} = \frac{K^2-2\alpha K+\alpha^2}{9} + [\tfrac{1}{2}s^2\mu_i - s\mu_i + s^2\mu_i\mu_j]K^2$$
$$+ \left[s\phi_i - s^2\phi_i\mu_i - s^2\phi_i\mu_j - s^2\phi_j\mu_i + s\mu_i[A+2c]\right]K$$
$$+ \tfrac{1}{2}c^2 + \left(\tfrac{1}{2}s\phi_i + s\phi_j - (A + 2c)\right)s\phi_i$$
$$i=1,2;\ i\neq j \qquad (17)$$

A solution requires that the left and right hand sides of (17) must be equal for all values of K. In particular, the coefficients on K^2 on each side of (17) must be equal, and the coefficients on K, and the constants. These relationships (requirements) on parameters in (17) allow us to solve for values of δ_i, ϕ_i, and μ_i (i=1,2) which satisfy (17) and in turn define the value functions in (15) exactly. The (13) yields solution values for I_1 and I_2. Using (17) we obtain (Appendix III):

$$\mu_i = \frac{(2s+r)\pm\left[(2s+r)^2-\frac{8}{3}s^2\right]^{\frac{1}{2}}}{6s^2} \qquad i=1,2 \qquad (18)$$

and

$$\phi_i = \frac{-\mu_i s(A+2c)+\frac{2\alpha}{9}}{s-3\mu_i s^2+r} \qquad i=1,2 \tag{19}$$

We also show that $\mu_1 = \mu_2$ and that "the minus case" in (18) is valid for our problem.

Substitution from (16) in (3) and solving for the case of $\dot{K} = 0$, yields

$$K = \frac{A+2c-2s\phi}{1-2\mu s} \tag{20}$$

where we have now removed subscripts on ϕ and μ. Clearly in light of (20), $s\phi$ and $s\mu$ are of interest. Now from (18)

$$s\mu = \frac{(2s+r)-\left[(2s+r)^2-\frac{8}{3}s^2\right]^{\frac{1}{2}}}{6s} \tag{21}$$

and[2] $\lim\limits_{s\to\infty} s\mu = \frac{1}{3}\left[\frac{\sqrt{3}-1}{\sqrt{3}}\right]$

which we denote by β. From (19) we obtain

$$s\phi = \frac{-\mu s(A+2c) + \frac{2\alpha}{9}}{1 - 3s\mu + \frac{r}{s}} \tag{22}$$

and

$$\lim_{s\to\infty} s\phi = \frac{-\beta(A+2c) + \frac{2\alpha}{9}}{1-3\beta}$$

which we denote by γ. Then

$$\lim_{s\to\infty} K = \frac{A+2c-2\gamma}{1-2\beta} = \frac{(A+2c)(1-\beta) - \frac{4\alpha}{9}}{(1-3\beta)(1-2\beta)}$$

$$= \left[(A + 2c) - \frac{4\alpha}{9}\left(\frac{9}{6 + \sqrt{3}}\right)\right]\left(\frac{6 + \sqrt{3}}{\sqrt{3} + 2}\right) \tag{23}$$

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[2] There is the problem of K going negative as $s \to \infty$. (Kamien and Fershtman [1987] suffers from this problem also.) E. Dockner has proposed restricting attention to limiting cases of $s \to 0$. We will report on this in the future in a somewhat different model.

We can label the right side of (23) K^F where F is for feedback. K^F is to be compared with its open loop counterpart K^L in (11). There,

$$K^L = \left[(A + 2c) - \frac{4\alpha}{9}\right]\frac{9}{5}.$$

There are non-negativity requirements on K^L and K^F as well as on the corresponding values for I in $A - 2I = K$. For $K^L \geq 0$, $(A + 2c)$ must be greater than or equal to $\frac{4\alpha}{9}$. For the corresponding $I \geq 0$, we require $K^L \leq A$ or $\alpha \geq A + \frac{9}{2}c$. The two non-negativity requirements yield

$$A + \frac{9}{2}c \leq \alpha \leq \frac{9}{4}A + \frac{9}{2}c. \tag{24}$$

There is a corresponding relation on A, c and α for $K^F \geq 0$ and the related I. However, (24) will suffice in our analysis here of K^L and K^F. We divide the K^L and K^F relations by A + 2c to get

$$\frac{K^L}{A+2c} = \frac{9}{5} - \frac{4}{5}\left(\frac{\alpha}{A+2c}\right) \tag{25}$$

and

$$\frac{K^F}{A+2c} = \left[\frac{6+\sqrt{3}}{2+\sqrt{3}}\right] - \left[\frac{4}{2+\sqrt{3}}\right]\left(\frac{\alpha}{A+2c}\right) \tag{26}$$

We sketch (25) and (26) in Figure 1.

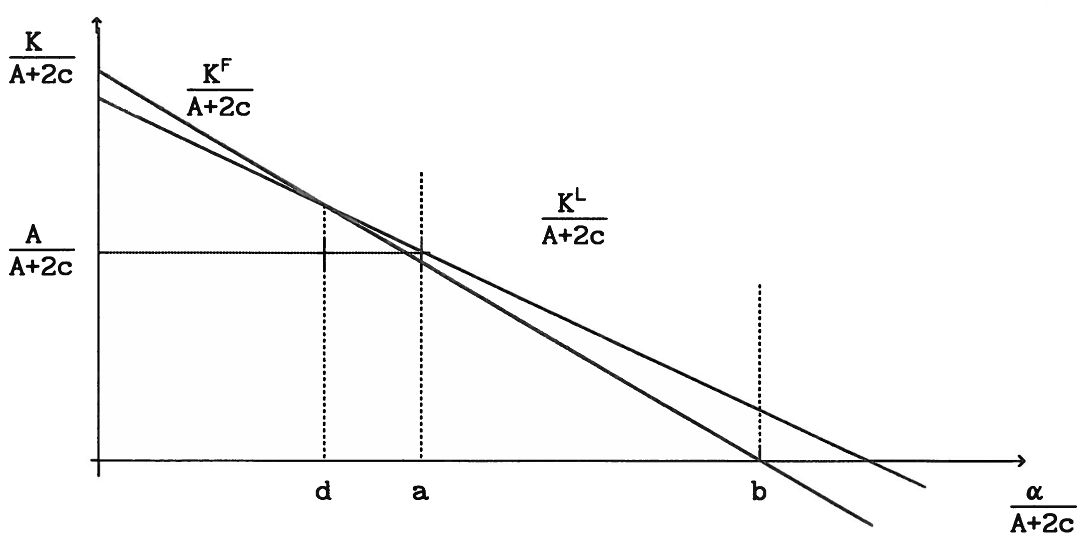

Figure 1

The non-negativity condition on K^F defines point b in Figure 1. Inspection of the relations defining K^F and K^L yields $K^L > 0$ at b. Point a in Figure 1 is defined by the non-negativity condition on I corresponding to K^L. At a one can compute K^F equal to $A - \left(\frac{6 - 2\sqrt{3}}{2 + \sqrt{3}}\right)c$ which is less than $K^L = A$ at that point. At d, $\left(\frac{\alpha}{A+2c}\right) = \frac{5}{16}\left(\frac{18 - 5\sqrt{3}}{3 - \sqrt{3}}\right)$. Hence parameters A, c, and α must be such that $\alpha/(A+2c)$ falls between a and b in Figure 1 in order that K^F, K^L and the corresponding I's are non-negative solutions to the appropriate open-loop and feedback problems. Hence $K^L > K^F$ in two corresponding solutions (with same parameters). This is our main result. Roughly speaking the feedback duopoly relative to the open loop duopoly is more competitive in the sense that more technical knowledge is produced (K^F smaller than K^L).

The Price of K

K is a stock in our model. m_i is the shadow price of a unit of this stock in the open loop case. From (4) we observe $m_i s = -I_i - c < 0$. The price is negative because more K implies a higher cost of production. Here, a lower stock is more profitable at the margin. From (3) we obtain $I = \frac{A-K}{2}$ in the steady state and hence in the limit as $s \to \infty$, we have $sm_i = -\frac{A+K^L}{2} - c$. V_K^i is the corresponding shadow price for K in the feedback case. In (13) we have $sV_K^i = -I_i - c$ which upon substituting for I as above, yields $sV_K^i = -\frac{A+K^F}{2} - c$. It follows that $m_i s - V_K^i s = \frac{1}{2}[K^L - K^F] > 0$. But recall, each price is negative, Hence $|V_K^i| > |m_i|$. The larger negative price in the feedback case implies that a unit "disposal" of K is more valuable. A unit of K^F commands a higher negative price because in the feedback equilibrium K^F

has been brought lower (a desirable outcome) than K^L. The negativity of these asset prices is not the novelty. Rather they are asset prices which reflect the commonality of asset K to each player's individualistic investment in K.

Reformulation and Extension

E. Dockner has proposed a reformulation of the R & D investment-cost reduction relationship. He introduces knowledge stock N which depreciates (evaporates at rate λ). Then $\dot{N} = I_1 + I_2 - \gamma N$. Our cost parameter K is inversely related to N(t). Let $\dot{K} = -\lambda\dot{N}$. Then we obtain $\dot{K} = \lambda\left[\frac{A\gamma}{\lambda} - I_1 - I_2 - \frac{\gamma K}{\lambda}\right]$. The limiting case of interest becomes that for which $\lambda \to 0$. One can then establish that the feedback case is more competitive than the open loop case. Other interesting results on efficiency and free-riding can also be obtained.

E. Dockner also proposes investigating a differential game of output market sharing by the duopolists but no R & D spillovers. The basic structure parallels that in Dockner [1992].

Appendix I: The dynamics of K(t) in the Open Loop Solution

Differentiate (3) with respect of time to obtain

$$\ddot{K} + s\dot{I}_1 + s\dot{I}_2 + \dot{K} = 0 \quad \text{or} \quad \ddot{K} + 2s\dot{I} + \dot{K} = 0.$$

Differentiate (4) and combine with (5) and (3) to obtain

$$2s\dot{I} = -2s^2rm + \frac{4s^2K}{9} - \frac{4s^2\lambda}{9} - 2s^3m$$

$$= -2sr\left[-c - \frac{sA}{2s} + \frac{sK}{2s}\frac{\dot{K}}{2s}\right] + \frac{4s^2K}{9} - \frac{4s^2\lambda}{9} - 2s^2\left[-c - \frac{sA}{2s} + \frac{sK}{2s} + \frac{\dot{k}}{2s}\right]$$

Then our differential equation in K, $\dot{K}$ and $\ddot{K}$ becomes

$$\ddot{K} + (1-r-s)\dot{K} + \left(-rs - \frac{5}{9}s^2\right)K = (r+s)[-2sc - sA] + \frac{4s^2\lambda}{9}$$

The particular solution of this second order linear differential equation is K^* expressed in (10) above. The homogeneous part has the characteristic equation $x^2 + Bx + C = 0$ where $B \equiv (1-r-s)$ and $C \equiv \left(-rs - \frac{5}{9}s^2\right)$. Since $C < 0$, the solutions $\bar{x}$, $\underline{x}$ are both real. Hence convergence to K^* from K(0) is monotonic and we assumed $K(0) > K^*$. Convergence requires that the root of the quadratic is negative. That

is $x = \dfrac{-B-\sqrt{B^2-4C}}{2} < 0.$

Appendix II (Obtaining Equation (17))

We substitute expressions for $V^i(K)$ (equation (15)) and $V^i_K(K)$ (equation (16)) in (14) to obtain

$$r\delta_i - r\phi_i K + r\mu_i \frac{K^2}{2} = \frac{K^2-2\alpha K+\alpha^2}{9} - (-c+s\phi_i - s\mu_i K)c - \tfrac{1}{2}(-c+s\phi_i - s\mu_i K)^2$$
$$+ (s\mu_i K - s\phi_i)[A-K-(-2c+s\phi_i - s\mu_i K + s\phi_j - s\mu_j K)] \qquad i=1,2;\ i\neq j \qquad \text{(A1)}$$

The principal simplification in (A1) occurs when one discovers that

$$-c(-c+s\phi_i-\mu_i sK) - \tfrac{1}{2}(-c+s\phi_i-s\mu_i K)^2 = -\tfrac{1}{2}(s\phi_i - s\mu_i K)^2 + \frac{c^2}{2}$$

Given this, (17) in the text follows by direct methods.

Appendix III (μ_i and its properties)[3]

In (17), we set the coefficient for K^2 on the left hand side equal to the corresponding coefficient on the right hand side. This yields

$$\frac{1}{9} + \frac{1}{2} s^2\mu_i^2 - s\mu_i + s^2\mu_i\mu_j = \frac{r\mu_i}{2}$$

which becomes

$$s^2\mu_i^2 + (2s^2\mu_j - 2s - r)\mu_i + \frac{2}{9} = 0 \qquad i=1,2;\ i\neq j \qquad \text{(A2)}$$

We wish to prove that $\mu_1 = \mu_2$. It will follow that $\phi_1 = \phi_2$ and $\delta_1 = \delta_2$ and then that $I_1^*(K) = I_2^*(K)$ or that the solutions are symmetric. To establish $\mu_1 = \mu_2$, we subtract the two equations in (A2). This yields

$$(\mu_1 - \mu_2)[s^2(\mu_1 + \mu_2) - (2s + r)] = 0 \qquad \text{(A3)}$$

Thus either $\mu_1 = \mu_2$ or $s^2(\mu_1 + \mu_2) = 2s + r$. We now show that the condition $s^2(\mu_1 + \mu_2) = 2s + r$ contradicts the condition for convergence of K(t) to its steady state value. Equations (3), (13) and (16) yield.

$$\dot{K} - sK[s(\mu_1 + \mu_2) - 1] = s[A + 2c - s(\phi_1 + \phi_2)]$$

a first order differential equation with constant coefficients. For K(t) to not explode, the coefficient $s^2(\mu_1 + \mu_2) - s$ must be negative. That is $s^2(\mu_1 + \mu_2) < s$. This contradicts $s^2(\mu_1 + \mu_2) = 2s + r$ above. Hence we conclude $\mu_1 - \mu_2 = 0$ or $\mu_1 = \mu_2 = \mu$.

Now we wish to establish that the smaller value for μ in (18) is "admissible" because this root is compatible with convergence (non-explosiveness) of K(t). First we observe that with r = 0, the larger root is $\mu^+ = \dfrac{3 + \sqrt{3}}{9s}$. With $\mu_1 = \mu_2 = \mu$, the convergence criterion is $2\mu s - 1 < 0$ or $\mu < \frac{1}{2s}$. Since $\mu^+ > \frac{1}{2s}$, it violates the convergence criterion and is thus not "admissible". With r = 0, the smaller root is

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[3] The strategy of proof here draws on Kamien and Schwartz [1990; pp. 279-280] which in turn is a textbook report on Fershtman and Kamien [1987].

$\mu^- = \dfrac{3 - \sqrt{3}}{9s}$ which does satisfy the convergence criterion. Also $(d\mu^-/dr)_{r=0} = -2/(15s^2) < 0$ or μ^- is a maximum at $r = 0$. Hence μ^- is the "admissible" root and we indicate it in the text simply by μ.

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