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The Tragedy of the Commons Revisited

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Abstract

We formulate the Malthus-Hardin tragedy of the commons as a special case of dynamic game between "tribes". At each date a member of a tribe desires more newborns of her/his type and more current consumption, harvested from the commons. Equilibrium in the dynamic game yields steady state level of per capita consumption for each person and a steady population level for each tribe (births equal deaths). We obtain four outcomes, depending on our assumptions about the discount rates of members of a tribe and about the mode of competition for "large populations" for each tribe. We compare Nash open loop and Nash feedback solutions with a quadratic current utility function and observe a larger population under the feedback solution. The classic tragedy of the commons solution obtains as a special case of the open loop solution, one with an infinite rate of discount for players.

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Introduction

T.R. Malthus [1798] envisaged population growth on Earth at an exponential rate pushing living standards to subsistence because arable land was in a finite amount. Output per person from land, average product, would decline to subsistence. Malthus dressed up this idea in his pseudo scientific phrasing: output grows arithmetically and population geometrically.[1] At subsistence, there will be a steady state: births equal deaths. Presumably most of the dying will be infants if parents continue to procreate at a high rate. Hardin [1968] recast Malthus' scenario by viewing newborns as entrants to an earthly "commons". Access to land and other finite productive natural resources is not rationed. The commons becomes over-grazed and productivity per person declines to a minimum sustainable level. (Other key papers analyzing common property issues are Gordon [1954], Weitzman [1974], and Eswaran and Lewis [1984]).

The common "mechanism" in the Malthus-Hardin argument is the decision of a couple to reproduce being independent of anticipated population levels. Parents are myopic and fail to see that their "extra" children will cause living standards for that next generation to decline. The current population is treated as constant in the future in their implicit decision to have children. If the parents took the future effects of their "extra" children into account, they would presumably have fewer children. This raises the vexing issue of parent i saying if I restrict my family size, will parent j follow suit? This is a standard game between "independent" decision

[1]Hartwick [1988] is an inquiry into the historical origins of this idea of Malthus.

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Appendix III: The value of ${}^{FB}B$ around $\varepsilon=1$.

From (27) in the text we have

$$ {}^{FB}B = \left(\frac{\alpha}{2(1-\sigma K)+\beta\Delta}\right)\left\{1-\sigma K-\sigma K\left[\frac{2\beta\sigma\Delta}{\beta r+\sqrt{Z}}\right]\right\} $$

where $Z \equiv (2\beta\sigma\Delta + 4\sigma + \beta r)^2 - 12\sigma^2(1-\varepsilon)$

and $K = (2\beta\sigma\Delta + 4\sigma + \beta r - \sqrt{Z})/6\sigma^2$

Now

$$ \frac{d\,{}^{FB}B}{d\varepsilon} = \frac{\alpha\sigma Z^{1/2}}{2(1-\sigma K)+\beta\Delta}\left[\frac{-2}{(2(1-\sigma K)+\beta\Delta)}\{\,\cdot\,\} + \left(1 + [\,\cdot\,] + \frac{6\sigma^3 K 2\beta\Delta}{(\beta r+\sqrt{Z})^2}\right)\right] $$

We make use of the fact that

$$ \frac{dK}{d\varepsilon} = -Z^{-1/2} \quad \textit{and} \quad \frac{dZ^{1/2}}{d\varepsilon} = \frac{1}{2}Z^{-1/2}\,12\sigma^2 $$

Note that $dK/d\varepsilon$ and $dZ^{1/2}/d\varepsilon$ have the same sign at $\varepsilon=1$ and $\varepsilon = 0.99999$ and $\varepsilon = 1.00001$. Also $d^{FB}B/d\varepsilon$ does not change sign at $\varepsilon=1$ and does not exhibit jumps. Since $K=0$ at $\varepsilon=1$ and $Z^{1/2} = (2\beta\sigma\Delta+4\sigma+\beta r)$, we obtain

$$ \left.\frac{d\,{}^{FB}B}{d\varepsilon}\right|_{\varepsilon=1} = \frac{\alpha\sigma}{(2+\beta\Delta)(2\beta\sigma\Delta+4\sigma+\beta r)}\left[1 - \frac{2}{2+\beta\Delta} + \frac{\sigma(2+\beta\Delta)}{\beta r+\beta\Delta\sigma+2\sigma}\right] $$

$$ = \frac{\alpha\beta\sigma\Delta}{(2+\beta\Delta)^2(\beta\sigma\Delta+2\sigma+\beta r)} > 0 $$

makers. This is the topic of this paper. We cast the decision of a pair of parents of type i to have a family of size B_i as a game on the commons involving parents of type j. Current population size aggregated over different types of people determines per capita living standards. Average product declines with a larger total population (strong diminishing returns). Game theory allows us to incorporate alternative assumptions about anticipations by players. The Malthus-Hardin view becomes a special case -very myopic. The alternative static view is the Cournot case in which each player treats the current family size of her "opponent" as exogenous. We then turn to explicitly intertemporal theory and consider two dynamic differential game formulations - Nash open loop and Nash feedback. This latter builds in the property of sub-game perfection and we find that the steady state is "more competitive" than is the corresponding open loop case - more competitive in the sense that each player produces a larger family in the steady state than in the open loop case.

The "price" of an extra child becomes its value today to the parents relative to its "cost" in the future in terms of lower living standards. In equilibrium current benefits to the parents equal current costs assessed in terms of future lower living standards for the future generations. The mode of competition among "opposing" parents implies a distinct price per newborn. Thus, broadly speaking, it is the parents attitude to the living standards of future generations and to the actions, current and future, of "rival" parents which determines the current sizes of families. Market "prices" for newborns reflect attitudes to the current state and the future. Each generation inherits current world population and per capita living standards; each player controls current family size, preferring more of their own children to less. There is an exogenously given death rate for adults. In a steady state births form all players equal deaths in each "period". We perform the analysis in continuous time.

A standard argument for parents to have children is that parents wish to insure themselves against unforeseen adversity such as destitution due to illness, injury or decrepitude. This is the "pension motive". One needs an explicit over-lapping generations model to formalize this line of thought. See Zimmerman [1989] for articles and references. Some suggest that parents have children because they "like children" and children provide a repository for bequests (see Kemp, Leonard, Long (1984)). A more general argument subsumes the pension argument and says parents have children for both investment and consumption purposes. We follow this latter approach. Our current cost of another child to the parents only shows up as declining marginal utility to parents of children.

The Model

P is per capita output and consumption. If N is the current world population, then f(N) is total output and $f(N)/N \equiv P$ is per capita output and consumption. We assume

$$P = A - \lambda N \tag{1}$$

or $\dot{P} = -\lambda\dot{N}$. A and λ are positive and P must remain non-negative. Current adult population N(t) changes as

$$\dot{N} = s\cdot[B_1+B_2] - \delta N(t) \tag{2}$$

where δ is the death rate of adults and B_i is births of children from parents of type i. s is a parameter relating newborns to adults. The interpretation of s=1 is that each birth "translates" into an adult in N(t). s less than 1 implies that only a fraction of births "translate" into adults.

Appendix II: The value of ${}^{OL}B$ around $\varepsilon=1$

From equation (12) in the text we have

$${}^{OL}B = \frac{\alpha(\beta\sigma\Delta+\sigma\varepsilon+\beta r)}{2(\beta\sigma\Delta+\sigma\varepsilon+\beta r)+\beta\Delta(\sigma\Delta\beta+\sigma+\beta r)}$$

for $\varepsilon = \pi\beta \leq 1$. Differentiation yields

$$\frac{d\,{}^{OL}B}{d\varepsilon} = \frac{\alpha\sigma}{2(\beta\sigma\Delta+\sigma\varepsilon+\beta r)+\beta\Delta(\sigma\Delta\beta+\sigma+\beta r)}$$

$$\left\{1 - \frac{2(\beta\sigma\Delta+\sigma\varepsilon+\beta r)}{2(\beta\sigma\Delta+\sigma\varepsilon+\beta r)+\beta\Delta(\sigma\Delta\beta+\sigma+\beta r)}\right\}$$

Note that $d^{OL}B/d\varepsilon$ exhibits no jumps or changes in sign in the neighborhood of $\varepsilon=1$. Now

$$\left.\frac{d\,{}^{OL}B}{d\varepsilon}\right|_{\varepsilon=1} = \frac{\alpha\sigma}{(2+\beta\Delta)(\beta\sigma\Delta+\sigma+\beta r)}\left\{1 - \frac{2}{2+\beta\Delta}\right\}$$

$$= \frac{\alpha\beta\sigma\Delta}{(2+\beta\Delta)^2(\beta\sigma\Delta+\sigma+\sigma\beta r)} > 0$$

Appenidix I: The dynamics of P(t) in the open loop solution

We differentiate (3) with respect to time to obtain

$$\ddot{P} + \sigma\dot{B}_1 + \sigma\dot{B}_2 + \Delta\dot{P} = 0 \quad or \quad \ddot{P} + 2\sigma\dot{B} + \Delta\dot{P} = 0.$$

Differentiate (6) to get $-\sigma\dot{m}_i = \beta\dot{B}_i - \dot{P}$. Then from (6) and (7) we obtain $\beta\dot{B}_i - \dot{P} = \sigma B_i - \sigma m_i(r+\sigma\Delta) - \sigma\pi P$. Use (6) to eliminate m_i and (3) to eliminate each B_i. Then

$$2\sigma\dot{B}_i = [-2\sigma^2\pi - 3\sigma^2\Delta - 2\sigma r - \sigma\Delta\beta r - \sigma^2\Delta^2\beta](P/\beta) + [\sigma - \beta r - \beta\sigma\Delta](\dot{P}/\beta) + [\sigma+\beta(r+\sigma\Delta)](\sigma\alpha/\beta)$$

Substituting above yields the second order linear differential equation

$$\ddot{P} + \phi\dot{P} + \Omega P = -[\sigma+\beta(r+\sigma\Delta)](\sigma\alpha/\beta)$$

where $\phi \equiv [\Delta\beta+\sigma-\beta r-\beta\sigma\Delta]/\beta$

and $\Omega \equiv [-2\sigma\pi-3\sigma\Delta-2r-\Delta\beta r-\Delta^2\sigma\beta]\sigma/\beta$

The particular solution to our differential equation is

$$-[\sigma+\beta(r+\sigma\Delta](\sigma\alpha/\beta)/\Omega.$$

which is ${}^{OL}P$ in the text. The homogenous part has the characteristic equation $x^2+\phi x+\Omega=0$. Since $\Omega<0$, both roots $\bar{x},\underline{x}$ are real and there will be no cycles in P(t). Also convergence requires $x<0$. Hence

$$\underline{x} = \frac{-\phi-\sqrt{\phi^2-4\Omega}}{2}$$

is the required root to the quadratic and for P(0) to converge to ${}^{OL}P$.

(The extra births represent infants who fail to reach adulthood. With explicit uncertainty and risk aversion on the part of parents, one could have a perpetually growing population given some exogenous rate of infant mortality.) We make no explicit use of the particular specification of s and it can be taken as unity below. The combining of (1) and (2) yields

$$\dot{P} = -\lambda s\cdot[B_1+B_2] + \lambda\delta[\frac{A-P}{\lambda}] \quad or$$

$$\dot{P} = \sigma\cdot[\alpha-B_1-B_2-\Delta P] \qquad (3)$$

where $\sigma = \lambda s$, $\alpha = \delta A/\lambda s$, and $\Delta = \delta/\lambda s$. It follows that $\alpha = \Delta A$. In a steady state, $\dot{P}=0$ and more births imply a lower per capita consumption. A central result of this paper is that in the feedback solution, $B_1 + B_2$ is larger than in the open loop solution. Hence steady state per capita consumption is lower in the feedback solution.

The current utility of parents of type i is increasing in newborns B_i and per capita consumption P. Tractability invites us to make current utility quadratic in B_i and P as in

$$U_i = PB_i - \frac{\pi}{2}P^2 - \frac{\beta}{2}B_i^2 \qquad (4)$$

where π and β are positive parameters. A <u>necessary</u> condition for $\partial U_i/\partial B_i > 0$ and $\partial U_i/\partial P > 0$ is $\pi\beta < 1$. We assume $\pi\beta < 1$. We return to this condition below. Clearly $\partial^2U_i/\partial^2B_i < 0$ and $\partial^2U_i/\partial P^2 < 0$. There is diminishing marginal utility to more newborns and a higher consumption standard. We are endeavoring to treat current family size and material well-being of adults as relatively symmetric. One could imagine $\pi\approx0$ and $\pi\beta\approx1$. This would imply a higher weight on per capita consumption relative to current family size, B_i.

It makes most sense to view U_i as the aggregate utility of a generation of like people or of a tribe. The members of the tribe wish for more of their own type in the future, a higher B_i

and for more current consumption P_i.[2] Observe that current births have no cost in terms of current consumption levels, only in terms of future consumption levels. Tribe i maximizes $\int_0^\infty e^{-rt} U_i(t)dt$ by choice of $\{B_i\}$ subject to $N(0) = N_0$ (implying $P(0) = A-\lambda N_0$) and (3). Tribe j maximizes $\int_0^\infty U_j(t)dt$ subject to the same constraints. We limit ourselves to two tribes. It seems reasonable to assume P(0) is small and population grows to the steady state level. We discuss convergence of P(t) below.

The Open Loop Nash Solution

In this case, player i treats the other's time path $\{B_j(t)\}$ as parametric. We form the current value Hamiltonian

$$H_i(t) = PB_i - \frac{\pi}{2}P^2 - \frac{\beta}{2}B_i^2 + m_i(t)\sigma[\alpha - B_1 - B_2 - \Delta P] \qquad i=1,2 \qquad (5)$$

where $m_i(t)$ is the co-state variable (shadow price on P(t)). The necessary conditions for a maximum are

$$\frac{\partial H_i}{\partial B_i} = 0 \qquad or \qquad P - \beta B_i = m_i(t)\sigma \qquad i=1,2 \qquad (6)$$

and

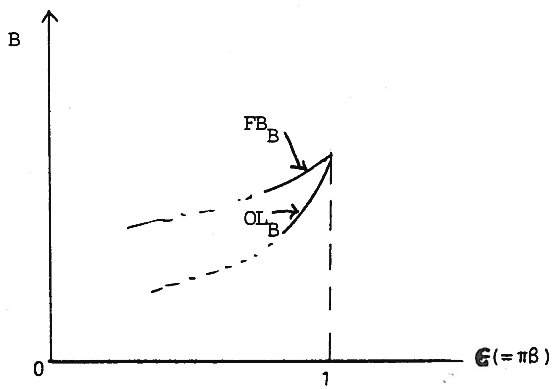
[2]In a steady state the birth level remains constant and it is easy to argue that each of N people in the population is associated with $B_1 + B_2$ births. The birth rate is constant. Our utility function is defined on total births of type i and adult per capita consumption P. It thus represents group characteristics (total births of a type) and individual characteristics (per capita consumption). It is thus an indicator of group instantaneous welfare. By normalizing current births as in $B_i/(N/2)$ we would have per capita births, assuming equal populations of each type. Then we would be obliged to keep track of the time path of N(t) as well as $B_i(t)$, a more complicated exercise. Hence the use of our utility function for a group or tribe. It is as if social norms or group decision-making were generating a target level of births of type i and a per capita consumption level P.

capita consumption higher. This is the "bellies over babies" solution. Each of these solutions turns out to be limiting cases of a Nash open loop game with a positive discount rate. The open loop game formulation is roughly speaking infected by dynamic inconsistency. Each player who forms a strategy at time zero can generally do better to revise her strategy at a later date in an update or revision. The feedback Nash formulation circumvents this problem. Intuitively it is "more competitive". We observe this explicitly in our model. The steady state birth levels are higher in the feedback solution. Our analysis has cast the tragedy of the commons as a dynamic game and exhibited the role of the discount rate and the mode of competition on the levels of population and per capita consumption in the commons.

In the open loop problem, m_i is the shadow price in current utils of an extra unit of per capita consumption P. This in turn defines the current marginal utility (util price) of an extra birth of type i to people of type i. Thus the choice of P and B_i at each date generates a corresponding implicit price for a unit of P and B_i. The same is true in the feedback problem where V_P^i is the implicit price to a person of type i of an extra unit of P. The essence of over-population from the tragedy of the commons persepctive, is that too low a price is charged to access the commons - entry is free. Entry is achieved by being born. Restraint on population is the implicit rationing of entry to the commons by charging a price for entry. In our model, each steady state population level has an associated implicit price to each group for a newborn - an entrant to the commons.

Figure 1.

Figure 1



This result implies that the feedback solution is more competitive in the sense that each tribe maintains a higher birth level than it does in the open loop case. This conforms with our intuition. A priori, the feedback solution captures the effect of intense competition between players at every date into the future whereas the open loop case models competition as intense at date t_0 but as "stand pat" at future dates, given the policies at t_0.

Concluding Remarks

We have seen that the classic tragedy of the commons is a special case of a general game of intertemporal competition among "players" keen at each date on per capita consumption and more children (offspring). In the classic tragedy of the commons, each player slights the effect its offspring have on the next generation's per capita consumption. Players exhibit very high discount rates. With no discounting, the birth levels end up lower in the steady state and per

$$-\frac{\partial H_i}{\partial P} = \dot{m}(t) - rm_i(t) \qquad or \qquad -\dot{m}(t) = B_i - m_i(r+\sigma\Delta)-\pi P \qquad i=1,2 \tag{7}$$

Substitute for B_i in (7) from (6) and we obtain

$$-\dot{m}_i = \frac{P}{\beta} - \frac{m_i\sigma}{\beta} - m_i(r+\sigma\Delta) - \pi P \tag{8}$$

Given transversality conditions

$$\lim_{t\to\infty} m_i(t)P(t) \to 0 \qquad (i=1,2)$$

we can solve (8) as

$$m_i(t) = \int_t^\infty e^{(r+\sigma\Delta+\frac{\sigma}{\beta})(s-t)}(\frac{1}{\beta} - \pi)P(s)ds \qquad (i=1,2) \tag{9}$$

This implies $m_1(t) = m_2(t)$ and hence $B_1(t) = B_2(t)$. The solution is symmetric in the players' actions. Thus in the steady state we have $2B = \alpha-\Delta P$ where $B = B_1 = B_2$. From (6) we obtain $\dot{m}_i = (\dot{P}-\beta\dot{B}_i)/\sigma$. Substitution in (8) yields

$$\dot{B} = [(\beta r + \beta\sigma\alpha - \sigma)/\beta]\, B - [(r + 2\sigma\Delta + \pi\sigma)/\beta]P + \alpha\sigma/\beta \tag{10}$$

and

$$\dot{P} = \sigma\alpha - 2B\sigma - \sigma\Delta P = 0 \tag{11}$$

In the steady state $\dot{B}=0$ in (10). (The dynamics of P(t) are reported on in Appendix I.) We then solve for

$$^{OL}B = \frac{\alpha}{2 + \frac{\Delta(\sigma+\beta r+\sigma\Delta\beta)}{\sigma\Delta+\sigma\pi+r}} \tag{12}$$

It follows that $^{OL}P = (\alpha-2B)/\Delta$.

Two special cases merit attention here.

i) Classical Tragedy of the Commons.

Suppose each tribe assumes that their offspring will have no deleterious effect on per capita consumption in the future. We restrict attention to steady states. Thus $P = (\alpha - B_1 - B_2)/\Delta$. Tribe i maximizes

$$U_i = PB_i - \pi/2\, P^2 - \beta/2\, B_i^2 \qquad (i=1,2)$$

by choice of B_i, treating P as parametric.

$$\frac{dU_1}{dB_1} = 0 \Rightarrow P - \beta B_1 = 0$$

$$\textit{and} \qquad (\alpha - B_1 - B_2)/\Delta - \beta B_1 = 0$$

For the other tribe we have

$$(\alpha - B_1 - B_2)/\Delta - \beta B_2 = 0$$

Then

$$^{TC}B = B_1 = B_2 = \frac{\alpha}{2+\Delta\beta} \tag{13}$$

and

$$P = \frac{\alpha\beta}{2+\Delta\beta} \tag{14}$$

Of interest is the solution in (12) yields that in (13) for the case of $r \rightarrow \infty$. Thus the classical tragedy of the commons outcome results when the current generation has no concern for the future (an infinite discount rate). Since $\pi\beta = \varepsilon$ or $\pi = \varepsilon/\beta$, $0 < \varepsilon < 1$, one observes that $^{OL}B < {}^{TC}B$. The extreme myopia in the classical tragedy of the commons solution implies more children (newborns) in the steady state.

Per capita consumption declines to zero for population $N = A/\lambda$. The value of P in (14) implies $N = A/(\lambda + \delta\beta/2)$. (We set s=1.) Clearly for $\beta \approx 0$, the tragedy of the commons

We know $0 < 1-\beta\pi < 1$. Thus $(2\beta\sigma\Delta + 4\sigma + \beta r)^2 - 12\sigma^2(1-\beta\pi) > 0$. Clearly if $+\surd$ in (22) were the relevant solution value for K, our above argument following equation (23) would apply. Hence in (22) the relevant solution is

$$^{FB}K = {}^{FB}K_1 = {}^{FB}K_2 = \left[2\beta\sigma\Delta + 4\sigma + \beta r - \sqrt{(2\beta\sigma\Delta + 4\sigma + \beta r)^2 - 12\sigma^2(1-\beta\pi)}\right]/6\sigma^2. \tag{23}$$

Given $K_1 = K_2$, it follows that $E_1 = E_2$. To see this return to (19) and observe that we require coefficients in P to sum to zero. That is

$$-rE_1 = -\frac{\sigma^2 K E_1}{\beta} - \frac{\sigma^2 K E_2}{\beta} - \frac{\sigma^2 K E_1}{\beta} + \frac{2\sigma E_1}{\beta} + \sigma\Delta E_1 + \alpha\sigma K \tag{24}$$

We also require for i=2

$$-rE_2 = -\frac{\sigma^2 K E_2}{\beta} - \frac{\sigma^2 K E_1}{\beta} - \frac{\sigma^2 K E_2}{\beta} + \frac{2\sigma E_2}{\beta} + \sigma\Delta E_2 + \alpha\sigma K \tag{25}$$

(24) and (25) imply $E_1 = E_2$. Given $E_1 = E_2$, from (24) we obtain

$$E_1 = E_2 = \frac{-2\beta\sigma\alpha K}{2\beta r - 6\sigma^2 K + 4\sigma + 2\beta\sigma\Delta} \tag{26}$$

We substitute from (23) and (26) into (18) and then into (16). We use the steady state relation $\Delta P = (\alpha - 2B)$ also to obtain

$$^{FB}B = {}^{FB}B_1 = {}^{FB}B_2 = \left(\frac{\alpha}{2(1-\sigma K)+\beta\Delta}\right) \times \left\{(1-\sigma K) - \sigma K\left[\frac{2\beta\sigma\Delta}{\beta r + \sqrt{(2\beta\sigma\Delta + 4\sigma + \beta r)^2 - 12\sigma^2(1-\beta\pi)}}\right]\right\} \tag{27}$$

Our principal result is that the birth level ^{FB}B in (27) exceeds that for the open loop solution ^{OL}B in (12). To establish this result we observe (i) when $\pi\beta = 1$, $^{FB}K = 0$ and $^{FB}B = {}^{OL}B$. This follows from inspection of ^{OL}B in (12) and ^{FB}B in (27). We then observe (ii) that $d^{FB}B/d\varepsilon > 0$ and $d^{OL}B/d\varepsilon > 0$ for $\varepsilon = \pi\beta$ and $0 < \varepsilon \leq 1$. See Appendices II and III. We observe (iii) that $d^{FB}B/d\varepsilon < d^{OL}B/d\varepsilon$ at $\varepsilon = 1$. See Appendices II and III. Hence $^{FB}B > {}^{OL}B$. See

P and on those constituting the constant in (19) must be zero. This gives us six equations in K_1, K_2, E_1, E_2 and g_1 and g_2. We proceed to solve for E_1 and K_1 which will yield a value for B_1 for the steady state.

From (19), the coefficient on P^2 set equal to zero yields

$$\sigma^2 K_1^2 + (2\sigma^2 K_2 - 2\beta\sigma\Delta - 4\sigma - \beta r)K_1 + (1-\beta\pi) = 0 \qquad (20)$$

Also

$$\sigma^2 K_2^2 + (2\sigma^2 K_1 - 2\beta\sigma\Delta - 4\sigma - \beta r)K_2 + (1-\beta\pi) = 0 \qquad (21)$$

Equation (21) follows from the "companion" equation to (19) with i=2. From (20) and (21) we observe that

$$(K_1 - K_2)\{(K_1 + K_2)\sigma^2 - (2\beta\sigma\Delta + 4\sigma + \beta r)\} = 0 \qquad (22)$$

From (22) we have either $K_1 = K_2$ or

$$\sigma^2(K_1 + K_2) = 2\beta\sigma\Delta + 4\sigma + \beta r \qquad (23)$$

We can show that (23) cannot hold for our model (i.e. $K_1=K_2$ must be true). Our proof is by contradiction (and draws on the argument in Kamien and Schwartz [1990; p.279]). From (16) and (18) we have

$$B_i = \frac{(1-\sigma K_i)P+\sigma E_i}{\beta} \qquad i=1,2$$

and from (3) and (23)

$$\dot{P} = \sigma\left[\alpha - \frac{\sigma(E_1+E_2)}{\beta}\right] + [2\beta\sigma\Delta+2\sigma+\beta r]P/\beta$$

Clearly if $\dot{P}$ is to converge from P(0), the coefficient on P must be negative which it is not, given (23). Hence by contradiction, we rule out (23) and infer $K_1 = K_2$.

Given $K_1 = K_2$ in the solution of (19), we proceed to solve for K in (20) for $K = K_1 = K_2$. That is, we have

$$\bar{K},\underline{K} = \left[2\beta\sigma\Delta+4\sigma+\beta r \pm \sqrt{(2\beta\sigma\Delta+4\sigma+\beta r)^2 - 12\sigma^2(1-\beta\pi)}\right]/6\sigma^2 \qquad (22)$$

population is very close to the sustainable population with $P \approx 0$. β small can be associated with a low cost of having children. Thus for β small, the tragedy of the commons population is approximately at its maximum level. We have implicitly treated subsistence per capita consumption as zero in this discussion. If there is an exogenous $\bar{P}$ for subsistence, then the maximum sustainable population would be $(A - \bar{P})/\lambda$. At this population a steady state involves $\delta/2\lambda\ (A-\bar{P})$ births per period for each tribe. Malthus' notion was that parents would choose to have newborns in excess of the above steady state level and this would require premature deaths as numbers of people pushed consumption levels below $\bar{P}$. We have not built an explicit subsistence level of consumption into our model. In our classical tragedy of the commons solution, in the steady state, parents choose numbers of newborns which yield steady state population levels above $N = A/\lambda$ where P=0. We have also obtained the result that 2λ must exceed $\delta\beta$ in order that the population for the classical tragedy of the commons solution remains positive.

ii) No Discounting Case

Consider again steady states, but now each tribe treats the other's level of newborns as parametric. Again $P = (\alpha - B_1 - B_2)/\Delta$. U_i is maximized by choice of B_i, with B_j treated as parametric. That is

$$\frac{dU_i}{dB_i} = P + B_i\frac{dP}{dB_i} - \pi P\frac{dP}{dB_i} - \beta B_i = 0 \qquad (i=1,2)$$

In this case

$$^{ZD}B = B_1 = B_2 = \frac{\alpha}{2 + \Delta\left(\frac{1+\beta\Delta}{\pi+\Delta}\right)} \qquad (15)$$

Observe that this corresponds to the open loop case for r=0. Since $1 + \beta\Delta > \pi\beta + \beta\Delta$, $^{ZD}B > {}^{TC}B$. It also follows that $^{OL}B < {}^{ZD}B$ because

$$\frac{1+\beta\Delta}{\beta\pi+\beta\Delta} > \frac{\beta r+\sigma+\sigma\Delta\beta}{\beta r+\sigma\beta\pi+\sigma\Delta\beta}$$

Thus the open loop case yields a steady state level of newborns for each tribe between the levels with zero discounting and discounting at an infinite rate. The zero discounting case is most prudent or most foresighted. Roughly speaking, parents take full account of the effects of their newborns on per capita consumption in the future.

<u>The Feedback Solution</u>

In this approach each player re-optimizes at each date, working backwards from the "final date". Thus at each date it is not possible for player i to "update" her action and gain extra payoff from a superior program. Each player has roughly speaking done all the "updating" possible, given the remaining competition to the end. The open loop solution is not "proofed" against re-optimization as time passes. We say that the open loop solution does not possess the property of <u>perfection</u>. We turn to determining the feedback solution. Given that it satisfies the property of optimality for each player in a backward recursion, we use dynamic programming. We have the Bellman equation

$$rV^i(P) = \max_{B_i} \left\{ PB_i - \frac{\beta}{2} B_i^2 - \frac{\pi}{2} P^2 + V_p^i(P)\sigma[\alpha - B_i - B_j - \Delta P] \right\} \quad i=1,2 \quad (15)$$

where $V^i(P)$ is the value function defined as

$$\int_t^\infty e^{-r(s-t)} U_i^*(s) ds$$

where $U_i^*(s)$ is the optimal value of $U_i(s)$. $V_P^i(P)$ is the derivative of $V^i(P)$ with respect to P. We have omitted date t in (15). All variables are defined at date t. Dynamic programming is in large part the search for the $V^i(P)$ function in terms of the state variable P which satisfies (15) at each date. This then characterizes the optimal time path of the state variable P(t).

Maximization of (15) with respect to B_i implies

$$B_i = \frac{P - V_P^i(P)\sigma}{\beta} \quad (16)$$

for an interior solution. Note that time does not appear explicitly in (16). Substitution from (16) into (15) yields

$$rV^1(P) = \left(\frac{P - V_P^1(P)\sigma}{\beta}\right)P - \left(\frac{P - V_P^1(P)\sigma}{\beta}\right)^2 \frac{\beta}{2} - \frac{\pi}{2}P^2 + \sigma V_P^1(P)\left[\alpha - \Delta P - \frac{2P}{\beta} + \frac{V_P^1(P)\sigma}{\beta} + \frac{V_P^2(P)\sigma}{\beta}\right] \quad (17)$$

There is an analogous equation to (17) for i=2. We assume that the solution $B_i(P) > 0$ so that we are searching over solution candidates which are interior. We hypothesize that solution function $V^i(P)$ is quadratic in P. That is we try

$$V^i(P) = g_i - E_i P + K_i \frac{P^2}{2} \qquad (i=1,2) \quad (18)$$

This yields $V_P^i(P) = -E_i + K_i P$. We substitute into (17) to obtain

$$\begin{aligned} \frac{1}{2} r K_1 P^2 - rE_1 P + rg_1 &= (P - \sigma K_1 P + \sigma E_1)P/\beta - (P - \sigma K_1 P + \sigma E_1)^2/2\beta - \pi \frac{P^2}{2} \\ &\quad + (\sigma K_1 P - \sigma E_1)\left\{\alpha - \Delta P - 2P/\beta + [(K_1 + K_2)P - E_1 - E_2]\frac{\sigma}{\beta}\right\} \\ &= \frac{P^2(1-\beta\pi)}{2\beta} + \frac{\sigma^2 K_1^2 P^2}{2\beta} + \frac{\sigma^2 K_1 K_2 P^2}{\beta} - \sigma K_1 \Delta P^2 \\ &\quad - \frac{\sigma K_1 2P^2}{\beta} - \frac{\sigma^2 K_1 E_1 P}{\beta} - \frac{\sigma^2 K_1 E_2 P}{\beta} - \frac{\sigma^2 K_2 E_1 P}{\beta} \\ &\quad + \alpha\sigma K_1 P + \frac{2\sigma E_1 P}{\beta} + \sigma\Delta E_1 P + \frac{\sigma^2 E_1^2}{2\beta} \\ &\quad + \frac{\sigma^2 E_1 E_2}{\beta} - \alpha\sigma E_1 \end{aligned} \quad (19)$$

There is an analogous expression for i=2. Equation (19) must be satisfied for values of P(t) satisfying an optimal path (i.e. for many different values of P). Hence the coefficients on P^2 and