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Anonymous Sequential Games with General State Space

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Abstract

In this paper we consider Anonymous Sequential Games with Aggregate Uncertainty. We prove existence of equilibrium when there is a general state space representing aggregate uncertainty. When the economy is stationary and the underlying process governing aggregate uncertainty Markov, we provide Markov representations of the equilibria.

1 Introduction

In this paper we provide a set of results on the existence of equilibrium in a class of dynamic games: anonymous sequential games. This class of multistage games features a continuum of agents and is characterized by the "anonymity" property that an agent's payoff in any period depends on what other agents do *only* through the aggregate distributions over agent types and their actions. Economic problems in which agents are "small", for example competitive economies, are modelled quite naturally as anonymous games. The anonymous sequential game framework with aggregate uncertainty is quite general and presents an attractive alternative to representative agent macroeconomic models. The framework permits one to address economic problems where individual stochastic heterogeneity is important, *as is its interaction with aggregate stochastic variables.*

Heterogeneity helps explain both the individual allocation of resources, as well as the evolution of aggregate economic variables. Agents who differ in their abilities, endowments or preferences may make different employment decisions, hold different portfolios or purchase different goods; firms which differ in their costs may make different investment or R&D decisions. For each agent, the dynamic evolution of such characteristics is invariably stochastic in nature: how successful was a firm's R&D investment? what was the return on an asset? what was the worker-firm match quality? what was a firm's cost shock?, etc. In some cases, questions of this sort can be addressed in a model which has no aggregate uncertainty (see, for instance, Jovanovic (1982), Hopenhayn (1990), Jovanovic and MacDonald (1988)). However, for many economic problems it is too restrictive to impose the requirement that the aggregate distribution of agent types evolve nonstochastically. For instance, modeling of business cycles demands consideration of aggregate demand shocks which affect all firms directly. Government policy choices, such as the rate of money growth, which are random from the perspective of individual agents are aggregate in nature. Technology shocks reflecting global innovations such as computers, are aggregate in nature, as are the so-called "oil" shocks. In such cases the stochastic evolution of the economy-wide aggregates is an important determinant of agent decisions and hence of economic behavior. The anonymous sequential game framework with aggregate uncertainty allows one to model such phenomena. For instance, Bergin and Bernhardt (1990) employ this framework to examine entry, exit, investment and R&D decisions of firms whose costs evolve stochastically and who face aggregate business cycle demand shocks.

Anonymous sequential games are formally defined in Jovanovic and Rosenthal (1988) for the case where there is no aggregate uncertainty, and an existence theorem is given there. Here we extend Jovanovic and Rosenthal (1988) and Bergin and Bernhardt (1991) to allow for aggregate uncertainty with general state spaces. We also provide two results on the existence of Markovian equilibria when the model is stationary and the underlying stochastic process governing the economy is Markov.

To understand the nature of the results, it is necessary first to develop the notions of "aggregate uncertainty" and "no aggregate uncertainty" within the context of our formulation. If the set of agent types in our economy is Λ and the action space A (common to all agents), then the aggregate distribution over agent types and actions is some distribution τ on $Y = \Lambda \times A$.[1] The anonymity assumption says that the behaviour of other agents affects agent α's utility (only) through τ. Such a joint distribution τ is called a distributional strategy.[2] Agents' characteristics (e.g. technology quality) can evolve stochastically over time, so that a particular $\alpha \in \Lambda$ (the characteristics space) is not identified with "the same" player over time. At time t, if agent $\alpha \in \Lambda$ takes action $a \in A$, and the distributional strategy is τ_t, then he obtains utility $u_t(\alpha, a, \tau_t)$. Given α, τ_t and a, the player then draws a new characteristic ξ_{t+1} (reflecting idiosyncratic risk) from a distribution[3] $\mathbf{P}_{\xi_{t+1}}(\bullet; \tau_t, \alpha, a)$, which determines his type in $t+1$. In turn, in period $t+1$, the player obtains a new characteristic, drawn from a distribution $\mathbf{P}_{\xi_{t+2}}(\bullet; \tau_{t+1}, \xi_{t+1}, a)$, when action a is taken, the period $t+1$ distributional strategy is τ_{t+1}, and so on. Thus, idiosyncratic risk arises because a player's payoff depends on the random evolution of his characteristic in Λ space.

"No aggregate uncertainty" is formulated in this model as the non-stochastic evolution of a sequence of joint distributions, $\{\mu_t\}$, on the characteristics space Λ. Given the aggregate distribution, τ_t, on $\Lambda \times A$, the "no aggregate uncertainty" hypothesis means that next period's distribution over characteristics space is given by

$$\mu_{t+1}(\bullet) = \int \mathbf{P}_{\xi_{t+1}}(\bullet; \tau_t, \alpha, a)\tau_t(d\alpha \times da).$$

Even though each agent faces individual uncertainty through $\mathbf{P}_{\xi_{t+1}}$, this uncertainty at the individual level "washes out" in the aggregate so that μ_{t+1} is non-stochastic. This "washing out" of individual risk is intimately related to the fact that the model has a continuum of agents. A detailed discussion on this matter is given in Feldman and Gilles (1985). The hypothesis of no aggregate uncertainty has proved very useful in proving existence of equilibrium and for the analysis and economic characterization of these models. Jovanovic and Rosenthal (1988), and Bergin and Bernhardt (1991) provide further details and discussion of these issues.

To illustrate the potential difficulties in deriving economic restrictions were the aggregate distribution conditionally stochastic, consider a situation where α indexes a firm's technology and where $\tilde{\xi}_\alpha$ is firm α's technology next period. One might anticipate that, from an economic perspective, agent α is better off "drawing" a good technology (a high value of $\tilde{\xi}_\alpha$). However, with a stochastic aggregate distribution and the unavoidable correlation of draws across the

[1] We identify an agent with his type in the sense that we will refer to both agent type α and agent α interchangeably – e.g. firm α has technology type α.

[2] The properties and use of such strategies are discussed further in MasColell (1984), Jovanovic and Rosenthal (1988) and the references cited there.

[3] A "$\bullet$" will sometimes be used as an argument of a measure to denote an arbitrary measurable set in the relevant space. Given two measures, μ and φ on some sigma field $\mathcal{B}$, the expression $\mu(\bullet) = \varphi(\bullet)$ means $\mu(B) = \varphi(B)$, $\forall B \in \mathcal{B}$. Given a metric space X, $\mathcal{B}_X$ is the associated sigma field.

α's, conditional on a high value of $\tilde{\xi}_\alpha$, the distribution of technologies may be more likely to be concentrated on good technologies. Thus, in a competitive situation, given that $\tilde{\xi}_\alpha$ is high (more efficient), other firms are more likely to be more efficient and the "gain" to $\tilde{\xi}_\alpha$ of being more efficient may be offset by the fact that the competition is stiffer. Thus the expected result - that the expected payoff given greater efficiency is higher - may be reversed. Bergin and Bernhardt (1991) also discuss technical problems which arise.

The structure underlying the development of our model of aggregate and no aggregate uncertainty is the following. Underlying the transition function $\mathbf{P}_\xi(\bullet;\tau,\alpha,a)$ (and ignoring time subscripts for now) is a probability space $(N,\mathcal{B}_N,\mathbf{p})$. The process governing the evolution of individual characteristics is $\xi(\eta,\alpha,a,\tau)$: if $\eta\in N$ is drawn, agent α takes action a, and the current joint distribution on actions and agent characteristics is τ, then agent α's characteristic next period is $\xi(\eta,\alpha,a,\tau)$. The transition function is determined by this process according to:

$$\mathbf{P}_\xi(B;\tau,\alpha,a)=\mathbf{p}(\{\eta\mid\xi(\eta,\alpha,a,\tau)\in B\}),\forall B\in\mathcal{B}_\Lambda.$$

For any given η and τ, the aggregate distribution next period is given by

$$\mu^\eta(B)=\tau(\{(\alpha,a)\mid\xi(\eta,\alpha,a,\tau)\in B\}),\forall B\in\mathcal{B}_\Lambda.$$

In general, $\mu^\eta(\bullet)$ is a random measure. Letting $\mathcal{M}(\Lambda)$ denote the set of probability measures on Λ and $\mathcal{B}_{\mathcal{M}(\Lambda)}$ denote the Borel field on $\mathcal{M}(\Lambda)$, the distribution of μ^η is given by:

$$\psi_\mu(Q)\equiv\mathbf{p}(\{\eta\mid\mu^\eta\in Q\}),Q\in\mathcal{B}_{\mathcal{M}(\Lambda)}.$$

Similarly, the joint distribution on $\mathcal{M}(\Lambda)\times\Lambda$ is given by:

$$\psi(Q)\equiv\mathbf{p}(\{\eta\mid(\mu^\eta,\xi(\eta,\alpha,a,\tau))\in Q\}),Q\in\mathcal{B}_{\mathcal{M}(\Lambda)}\otimes\Lambda.$$

The hypothesis of "no aggregate uncertainty" is the hypothesis that the distribution of this random measure, ψ_μ, is degenerate: $\exists\mu^*\in\mathcal{M}(\Lambda)$, $\mu^\eta=\mu^*$, $\mathbf{p}$ a.e. η. Aggregate uncertainty may be defined (by default) as the case where μ^η has a nondegenerate distribution (ψ_μ).

Bergin and Bernhardt (1991) develop a useful decomposition of uncertainty into aggregate and idiosyncratic components. Aggregate uncertainty is introduced by having a random variable $\theta\in\Theta$ represent an aggregate "shock" to both payoffs and the transition function governing individual risk. Idiosyncratic uncertainty is represented by a second stochastic component. In the present notation, such a procedure is equivalent to writing $\eta=(\omega,\theta)\in(\Omega,\Theta)$, where θ represents aggregate uncertainty and ω embodies "idiosyncratic risk". In this formulation, the underlying probability space has the form $(\Omega\times\Theta,\mathcal{B}_\Omega\otimes\mathcal{B}_\Theta,\varrho\otimes\nu)=(N,\mathcal{B}_N,\mathbf{p})$ and $\xi(\eta,\alpha,a,\tau)=\xi((\omega,\theta),\alpha,a,\tau)$. Thus, if the "aggregate shock" is θ, the aggregate distribution τ, and agent α takes action a, then agent α's characteristic next period is drawn from the distribution $\mathbf{P}_\xi(\bullet;\tau,\theta,\alpha,a)$. As before, the aggregate distribution is a random variable, but if we impose the "no aggregate uncertainty" hypothesis conditional on θ, then next period's aggregate distribution is non-stochastic, conditional on θ. The aggregate distribution is defined (for a given (ω, θ)) as:

$$\mu_{(\omega,\theta)}(B) = \tau(\{(\alpha, a) \mid \xi((\omega, \theta), \alpha, a, \tau) \in B\}), \forall B \in \mathcal{B}_{\Lambda}.$$

No aggregate uncertainty conditional on θ is the requirement that, given θ, $\exists \mu^*$, such that $\mu_{(\omega,\theta)} = \mu^*$, ϱ a.e. ω. At the same time, individual agents face individual uncertainty through ω because the distribution over agent α's characteristics is given by:

$$\mathbf{P}_{\xi}(B; \tau, \theta, \alpha, a) = \varrho(\{\omega \mid \xi((\omega, \theta), \alpha, a, \tau) \in B\}).$$

No aggregate uncertainty conditional on θ, implies that the aggregate distribution next period, μ^*, can be computed according to:

$$\mu^*(B) = \int_Y \mathbf{P}_{\xi}(B; \tau, \theta, \alpha, a)\tau(d\alpha \times da), \forall B \in \mathcal{B}_{\Lambda}.$$

In this formulation the aggregate shock θ enters as an argument of the transition function, affecting each agent and represents the aggregate uncertainty facing every agent. Agents' actions can be conditioned on the aggregate shock, so that the aggregate shock can also affect the transition to future states indirectly through current agents' actions. Finally, θ can enter payoffs directly. For instance, θ may be an aggregate demand shock or an aggregate inflation shock which affects all firms (directly through their profits and indirectly through their actions and the future evolution of their costs). Motivation behind this formulation of aggregate uncertainty is discussed at some length in Bergin and Bernhardt (1991), where an existence theorem is given in the case where the state space of aggregate uncertainty is countable (i.e., Θ is countable).

Here, we first extend this formulation of aggregate uncertainty to a general setting, providing an equilibrium existence theorem in a model with an abstract state space representing aggregate uncertainty. The extension to a more general state spaces for aggregate uncertainty is important because uncountable state spaces arise in a natural way in many applications. For instance, an aggregate demand shock θ facing firms may be drawn from some continuous distribution. The proof of existence of equilibrium in the general case is of independent interest because the approach used in the countable case does not carry over. The mathematical arguments developed to prove existence with the countable state space do not extend to more general state spaces because the construction involves selecting each finite history of aggregate shocks and developing "pointwise" arguments there. The extension is achieved at a slight cost in that we require each agent's action set be a fixed set A over all times and histories. In the countable case the possibility that a particular agent's actions could depend on the history of aggregate shocks or any other relevant variable was allowed.

Finally, we assume that the model is stationary and provide two results on the existence of Markov equilibria. The Markov representations provide an alternative way of viewing equilibria, and the additional structure facilitates the study of equilibria, simplifying the interpretation and analysis of equilibrium behaviour. The results here are closely related to some of the literature in stochastic games. In a stochastic game, a state space S is specified. There are a finite number of players, with action space $A_i(s)$, for player i, $i = 1, \ldots, n$, which is typically state dependent. Let $A(s) = \times_{i=1}^n A_i(s)$. If at time t in state s agents choose an action vector $a \in A(s)$, then the payoff to agent i is $u_i(s, a)$. Following the choice of $a \in A(s)$, a new state is drawn from some distribution $\mathbf{p}(d\tilde{s} \mid s, a)$. At time t, when agents select actions from $A(s_t)$, where s_t is the state at time t, they observe the history of states as well as the current state, $(s_1, s_2, \ldots, s_t)$, and the history of actions $(a_1, a_2, \ldots, a_{t-1})$, where $a_\tau \in A(s_\tau)$. Payoffs are discounted over time at the rate δ, so that the present value of i's payoff at time t is $(1-\delta)\delta^{t-1}u_i(s_t, a_t)$, where $s_t \in S$ and $a_t \in A(s_t)$. Thus, a strategy for i, $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{it}, \ldots)$, is a collection of functions with $\sigma_{it}(s_1, \ldots, s_t, a_1, \ldots, a_{t-1}) \in A_i(s_t)$. A strategy, σ_i, is called *Markov* if for all t, $\sigma_{it}(s_1, \ldots, s_t, a_1, \ldots, a_{t-1}) = \sigma^*_{it}(s_t)$, for all $(s_1, \ldots, s_t, a_1, \ldots, a_{t-1})$. If, in addition, the functions σ^*_{it} and $\sigma^*_{i\tau}$ agree, $\forall t, \tau$, then the strategy is called a *stationary Markov strategy.* In this model, a proof of existence of equilibrium is very difficult (when the state space S is not finite). Such a proof is given in Mertens and Parthasarathy (1988), who also discuss some of the difficulties involved in obtaining Markov equilibrium strategies. Duffie, Geanakopolos, MasColell and MacLennan (1989) also discuss stochastic games as an application of a general result on existence of equilibrium. They prove existence of a stationary ergodic Markov equilibrium on an enlarged state space which includes payoffs. This circumvents some of the difficulties involved in obtaining Markov results on the S state space.

The first result we give on Markov equilibria assumes that the stochastic process governing the θ process is Markov. The "natural state space" here is $\mathcal{M}(\Lambda) \times \Theta$, where an element of this space, (μ, θ) represents the aggregate distribution on agents' characteristics (μ) and the aggregate uncertainty parameter θ. As the game proceeds, a random vector $(\mu_1, \theta_1, \mu_2, \theta_2, \ldots, \mu_t, \theta_t)$ evolves. The distribution of θ_{t+1} depends only on θ_t (by the Markov property), and the value of μ_{t+1} depends only on (μ_t, θ_t) and the current strategies of agents. In this result we enlarge the state space, $\mathcal{M}(\Lambda) \times \Theta$, to include payoffs and provide a Markov characterization of equilibrium strategies. This approach is analogous to that in Duffie, Geanakopolos, MasColell and MacLennan (1989) (henceforth DGMM) who also include payoffs in the state space: the "state" at time t includes the present value of future payoffs (i.e. the term "Markov" is with respect to the enlarged state space). In a sense, this representation has a natural interpretation as a type of rational expectations equilibrium. It is worth stressing that we show that *every* equilibrium payoff in the game arises as the payoff to an equilibrium of this form. In this result, the transition functions are assumed to satisfy a form of weak* continuity whereas DGMM assume a stronger form of continuity (that the transition functions converge on Borel sets). In addition, we require no assumptions concerning absolute continuity of the transitions functions either relative to each other or relative to any fixed measure.

The enlarged state space can make it difficult to "pin down" behavior. In order to provide a Markov result on the "natural" state space we drop the conditional no aggregate uncertainty hypothesis and return to a general model of aggregate uncertainty as a random measure μ^η. In this model, where aggregate shocks are not explicitly formulated, the "natural" state space is $\mathcal{M}(\Lambda)$. Given an underlying stationary Markov stochastic environment we demonstrate that a Markovian equilibrium exists on the standard (i.e. not enlarged) state space. This result requires stronger, but still standard, continuity assumptions on the transition functions which are similar to those in DGMM. We now turn to a description of the game and presentation of the results. Proofs and notation are given in the appendix.

2 The Model

The set of agents is denoted Λ with representative element α, where Λ is assumed to be a compact metric space. Λ is the "characteristics" space. Similarly, the set of actions available to any agent α is a compact metric space A. Let $Y \equiv \Lambda \times A$. An aggregate distribution (on agents' characteristics) is a measure μ on Λ. Given a metric space X, the set of probability measures on X is denoted $\mathcal{M}(X)$, the set of continuous functions on X is written $\mathcal{C}(X)$, and the family of Borel sets of X is given by B_X. The t-fold product of the set X is denoted $X^t = \times_{s=1}^t X$ and the Borel field on X^t is denoted $\mathcal{B}_X^t$. We assume that the initial measure of agents is 1 ($\mu(\Lambda) = 1$, where μ is the initial measure on Λ) and consider a model with an infinite number of periods.

The state space representing aggregate uncertainty each period is a metric space Θ, with $\theta \in \Theta$. In the infinite period model the state space is $\Theta^\infty \equiv \times_{t=1}^\infty \Theta$, with representative element $\theta^\infty = (\theta_1, \theta_2, \ldots \theta_t, \ldots)$. Denote the vector $(\theta_1, \theta_2, \ldots, \theta_t)$ by $\theta^t \in \Theta^t = \times_{s=1}^t \Theta$. θ^t is the history of aggregate shocks up to the end of time t. Fix an exogenously given distribution ν on Θ^∞. Denote its marginal distribution on Θ^t by ν_t, its conditional distribution on Θ^∞ given the first t elements of θ^∞ by $\nu(\bullet \mid \theta^t)$, and the conditional distribution on Θ^t given $\theta^s, s < t$, by $\nu_t(\bullet \mid \theta^s)$.

The set of measurable functions from Θ^t to $\mathcal{M}(\Lambda \times A)$ is given by $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$. A period distributional strategy at time t, τ_t, is a measurable function from the space of aggregate shock histories Θ^t to $\mathcal{M}(\Lambda \times A)$, so that $\tau_t \in \mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$. A distributional strategy for the infinite period model is a vector $\tau = (\tau_1, \tau_2, \ldots, \tau_t, \ldots)$ of the period distributional strategies.

Now introduce a process ξ_t, with corresponding transition function $\mathbf{P}_{\xi_{t+1}}(\bullet, \bar{\tau}_t, \bar{\theta}^t, y)$ which, conditional on (1) the aggregate shock history $\bar{\theta}^t \in \Theta^t$, (2) the time t distributional strategy $\bar{\tau}_t \equiv \tau_t(\bullet, \bullet; \bar{\theta}^t) \in \mathcal{M}(\Lambda \times A)$, and (3) $y = (\alpha, a)$, gives the distribution of agent α's "type", ξ_{t+1}, in period $t+1$. At time t, if $\bar{\tau}_t$ is the period distributional strategy on $Y = \Lambda \times A$, and $\bar{\theta}^t$ is the aggregate shock history then, ignoring for the moment that $\bar{\tau}_t$

is related to $\bar{\theta}^t$, the aggregate distribution at time $t+1$ is given by

$$\mu_{t+1}(\bullet) = \int_Y \mathbf{P}_{\xi_{t+1}}(\bullet, \bar{\tau}_t, \bar{\theta}^t, y)\bar{\tau}_t(dy).$$

This connects the distribution on characteristics intertemporally: the distributions $\bar{\tau}_t$ and $\bar{\tau}_{t-1}$ are not independent. The marginal distribution of $\bar{\tau}_t$ on Λ is a distribution on characteristics which must agree with the distribution implied by the transition process: the measure of agents in a given set in Λ at time t must equal the measure of agents entering that set from the previous period. We return to this issue if consistency below.

Utility at time t is a function from $\Lambda \times A \times \mathcal{M}(\Lambda \times A) \times \Theta^t$ to $\mathcal{R}$. If an agent of type α takes action a given distributional strategy τ_t and aggregate shock history θ^t then the agent's payoff is $u_t(\alpha, a, \tau_t, \theta^t)$. Utility at θ^t is dependent on the aggregate distribution over $\Lambda \times A$, *conditional* on θ^t. Throughout, we take u_t to be continuous on $\Lambda \times A$. In addition, we assume that

$$sup_{(t,\alpha,a,\tau_t,\theta^t)\in(\mathcal{Z}_+\times\Lambda\times\mathcal{M}(\Lambda\times A)\times\Theta^t)} \mid u_t(\alpha, a, \tau_t, \theta^t) \mid \leq K' < \infty,$$

so that without loss of generality we may take

$$0 \leq u_t(\alpha, a, \tau_t, \theta^t) \leq K < \infty, \forall(t, \alpha, a, \tau_t, \theta^t).$$

The discount rate at time t is δ_t (with $\delta_1 = 1$ and $sup_{t\geq 2}\delta_t = \delta < 1$), so that the present value of time t payoffs is $(\Pi_{s=1}^t \delta_s)u_t$ (In the discussion of the stationary model we set $\delta_t = (1-\delta)\delta^{t-1}$).

The sequence of events at time t is the following. First the period t aggregate shock, $\bar{\theta}_t$, is realized. Then agent α, observing τ and the history of aggregate shocks (including the current shock), $\bar{\theta}^t$, picks an action $a \in A$, and receives utility $u_t(\alpha, a, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}_t)$. The agent's characteristics for period $t+1$, ξ_{t+1}, are then realized. The function τ_t gives a measure $\mathcal{M}(\Lambda \times A)$ from which the transition distribution on Λ is determined: $\mathbf{P}_{\xi_{t+1}}(\bullet, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}^t, y)$. Given a continuous function f on Λ ($f \in \mathcal{C}(\Lambda)$), $\int_\Lambda f(\xi)\mathbf{P}_{\xi t+1}(d\xi, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}^t, y)$ is denoted by $\mathbf{P}_{\xi_{t+1}}(f, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}^t, y)$.

Given a normed space $(X, \| \ \|)$, let $\mathbf{L}_1(\Theta^t, X, \nu_t)$ represent the set of measurable functions from Θ^t to X with norm $\int_{\Theta^t} \| f(\theta^t) - g(\theta^t) \| \nu_t(d\theta^t)$. For $f, g \in \mathcal{C}(Y)$, let $\| f - g \| \equiv sup_y \mid f(y) - g(y) \mid$. If $f, g \in \mathbf{L}_1(\Theta^t, \mathcal{C}(Y), \nu_t)$, then $f(y, \bar{\theta}^t)$ and $g(y, \bar{\theta}^t)$ are functions on Y for each $\bar{\theta}^t$, and $\| f - g \| \equiv sup_y \mid f(y, \bar{\theta}^t) - g(y, \bar{\theta}^t) \mid$ (*or* $\| f - g \|_y$ $(\bar{\theta}^t)$ to make the dependence on $\bar{\theta}^t$ explicit). The $\mathbf{L}_1(\Theta^t, \mathcal{C}(Y), \nu_t)$ norm topology is determined by the metric $\int_{\Theta^t} \| f - g \|_y \nu_t(d\theta^t)$.

Define a topology on $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$ according to the following convergence criterion[4]: Say that $\tau_t^n \to \tau_t$ if and only if for all $f \in \mathcal{C}(Y)$ and $g \in \mathbf{L}_1(\theta^t, \mathcal{R}, \nu_t)$,

$$\int f(y)g(\theta^t)\tau_t^n(dy; \theta^t)\nu_t(d\theta^t) \to \int f(y)g(\theta^t)\tau_t(dy; \theta^t)\nu_t(d\theta^t).$$

[4]Thanks are due to J-F. Mertens for suggesting this topology.

This is the coarsest topology on $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$ for which $\int f(y)g(\theta^t)\tau_t(dy;\theta^t)\nu_t(d\theta^t)$ is continuous in τ_t. With this topology, $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$ is compact (Mertens (1986)).

We assume that the transition distribution $\mathbf{P}_{\xi_{t+1}}(f, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}^t, y)$ is continuous in y for each $f \in \mathcal{C}(\Lambda)$. In addition, the following continuity conditions relative to τ are imposed. For fixed f, and $\tau_t \in \mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$, both $\mathbf{P}_{\xi_{t+1}}$ and u_t may be viewed as continuous real valued functions on Y for each $\bar{\theta}^t$ and hence as elements of $\mathbf{L}_1(\Theta^t, \mathcal{C}(Y), \nu_t)$. To simplify notation, write $\mathbf{P}_{\xi t+1}(f, \tau_t, \bar{\theta}^t, y)$ for $\mathbf{P}_{\xi_{t+1}}(f, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}^t, y)$ and $u_t(\alpha, a, \tau_t, \bar{\theta}_t)$ for $u_t(\alpha, a, \tau_t(\bullet, \bullet; \bar{\theta}^t), \bar{\theta}_t)$, where τ_t is understood to be an element of $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$. We assume that $\mathbf{P}_{\xi_{t+1}}(f, \tau_t, \bar{\theta}^t, y)$ and $u_t(\alpha, a, \tau_t, \bar{\theta}_t)$ are norm continuous in τ_t:

$$\int_{\Theta^t} sup_y \mid \mathbf{P}_{\xi_{t+1}}(f, \tau_t^n, \bar{\theta}^t, y) - \mathbf{P}_{\xi_{t+1}}(f, \tau_t, \bar{\theta}^t, y) \mid \nu_t(d\theta^t) \overset{\tau_t^n \to \tau_t}{\longrightarrow} 0$$

and

$$\int_{\Theta^t} sup_y \mid u_t(y, \tau_t^n, \bar{\theta}_t) - u_t(y, \tau_t, \bar{\theta}_t) \mid \nu_t(d\theta^t) \overset{\tau_t^n \to \tau_t}{\longrightarrow} 0.$$

3 Equilibrium

The first result attaches a value function to each collection of $(\alpha, a, \tau, \theta^t)$. This value function, $V_t(\alpha, a, \tau, \theta^t)$ gives the payoff to agent α at time t, given that α takes action a, the aggregate shock history is θ^t and the distributional strategy τ. The proof of the existence of a value function does not require that the time t utility function, u_t, be continuous in the aggregate shock θ_t.

Theorem 1 *For each t, there exist value functions $V_t(\alpha, a, \tau, \theta^t)$ and $W_t(\alpha, \tau, \theta^t)$, which are continuous in (α, a) and α respectively, norm continuous in τ and satisfy $W_t(\alpha, \tau, \theta^t) = max_a V_t(\alpha, a, \tau, \theta^t)$.*

In the proof we first consider a truncated n-period version of the game and families of value functions $V_t^n(\alpha, a, \tau, \theta^t)_{t=1}^n$ and $W_t^n(\alpha, \tau, \theta^t)_{t=1}^n$, where, for example, $W_t^n(\alpha, \tau, \theta^t)$ is the expected payoff in the truncated game (from period t on) to player α given history θ^t and τ, when α plays optimally from period t to the end of the n period game. We show that for each t, these functions are continuous in (α, a) and α respectively, and norm continuous in τ. We then demonstrate that these functions converge uniformly as $n \to \infty$, so that the limiting value functions also have these properties.

We now formulate the appropriate *consistency conditions* on the distributional strategy sequences $\tau = \{\tau_t\}$ (in terms of the distribution over characteristics). Clearly, if τ is an equilibrium distributional strategy, then the measure of agents in existence at the beginning of period t, as given by the period t distributional strategy τ_t, must coincide with the measure mapped from period $t-1$: in any equilibrium, a strategy must be consistent with itself, in this sense. Note that given a distributional strategy τ, and aggregate shock history θ^t, the distribution over characteristics at time t, "implied" by τ is given by $\int_Y \mathbf{P}_{\xi_t}(B, \tau_{t-1}, \theta^{t-1}, y)\tau_{t-1}(dy; \theta^{t-1})$, for all Borel sets $B \in \mathcal{B}_\Lambda$. Any distribution on $\Lambda \times A$ whose marginal distribution agrees with this distribution is consistent with τ at time t, and there are a continuum of such distributions. Thus, given τ, t and θ^t, there are a continuum of distributions $\tilde{\tau}_t(\bullet, \bullet; \theta^t)$ on $\Lambda \times A$, such that the marginal of $\tilde{\tau}_t(\bullet, \bullet; \theta^t)$ on Λ agrees with that implied by τ. The collection of such distributions (as t and θ^t vary) is the set of distributions consistent with τ.

The intemporal consistency conditions are formally defined:

Definition 1 *Let* $\tilde{\tau} = \{\tilde{\tau}_t\}_{t=1}^{\infty}$ *and* $\tau = \{\tau_t\}_{t=1}^{\infty}$, *with* $\tau_1(\bullet, A) = \mu_1(\bullet)$. *Say that* $\tilde{\tau}$ *is consistent with* τ *if:*

$$\int_\Theta \tilde{\tau}_1(f, A; \theta)g(\theta)\nu_1(d\theta) = \int_\Theta \mu_1(f)g(\theta)\nu_1(d\theta), \forall f \in \mathcal{C}(\Lambda), g \in \mathbf{L}_1(\Theta, \mathcal{R}, \nu_1),$$

$$\int_{\Theta^2} \tilde{\tau}_2(f, A; \theta^2)g(\theta^2)\nu_2(d\theta^2) = \int_{\Theta^2}\int_Y \mathbf{P}_{\xi_2}(f, \tau_1, \theta^1, y)\tau_1(dy; \theta^1)g(\theta^2)\nu_2(d\theta^2),$$

$$\forall f \in \mathcal{C}(\Lambda), g \in \mathbf{L}_1(\Theta^2, \mathcal{R}, \nu_2),$$

and for period t,

$$\int_{\Theta^t} \tilde{\tau}_t(f, A; \theta^t)g(\theta^t)\nu_t(d\theta^t) = \int_{\Theta^t}\int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)\tau_{t-1}(dy; \theta^{t-1})g(\theta^t)\nu_t(d\theta^t),$$

$$\forall f \in \mathcal{C}(\Lambda), g \in \mathbf{L}_1(\Theta^t, \mathcal{R}, \nu_t).$$

These conditions imply that $\tilde{\tau}_t(f, A; \theta^t) = \int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)\tau_{t-1}(dy; \theta^{t-1})$, almost everywhere θ^t (relative to ν_t). Recall, if $\mu \in \mathcal{M}(\Lambda)$ and f is a measurable function on Λ, $\mu(f)$ denotes $\int f d\mu$. Thus, the condition imposed is that the distribution over characteristics space Λ, determined by the distributional strategy $\tilde{\tau}$ at time t, $\tilde{\tau}_t(\bullet, A; \theta^t)$, is consistent, given the distributional strategy τ, with the characteristics distribution implied by the characteristics transition function, $\mathbf{P}_{\xi_t}(\bullet, \tau_{t-1}, \theta^{t-1}, y)$, and the distribution over previous state variables determined by τ, $\tau_{t-1}(\bullet, \bullet; \theta^{t-1})$.

Denote the collection of strategies which are consistent with τ by $\mathbf{C}(\tau)$. A strategy consistent with itself is then a fixed point of $\mathbf{C}$. Norm continuity of $\mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)$ (viewing $\mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)$ as an element of $\mathbf{L}_1(\Theta^{t-1}, \mathcal{C}(Y), \nu_{t-1})$, for fixed f) ensures that these equalities are continuous in τ sequences. This implies that $\mathbf{C}$ is an upperhemicontinous correspondence and so has a fixed point. Formally, define a collection of consistency mappings:

$$\mathbf{C}_1(\tau) = \{\tilde{\tau}_1 \mid \int_\Theta \tilde{\tau}_1(f, A; \theta)g(\theta)\nu_1(d\theta) = \int_\Theta \mu_1(f)g(\theta)\nu_1(d\theta), \forall f \in \mathcal{C}(\Lambda), g \in \mathbf{L}_1(\Theta, \mathcal{R}, \nu_1)\},$$

and for $t \geq 2$,

$$\mathbf{C}_t(\tau) = \{\tilde{\tau}_t \mid \int_{\Theta^t} \tilde{\tau}_t(f, A; \theta^t)g(\theta^t)\nu_t(d\theta^t)$$

$= \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)\tau_{t-1}(dy; \theta^{t-1})g(\theta^t)\nu_t(d\theta^t), \forall f \in \mathcal{C}(\Lambda), g \in \mathbf{L}_1(\Theta^t, \mathcal{R}, \nu_t)\}.$

Then the following result holds.

Theorem 2 *The correspondence* $\mathbf{C}(\tau) \equiv \times_{t=1}^{\infty}\mathbf{C}_t(\tau)$ *is non-empty, upper-hemicontinuous and convex-valued.*

We now consider those distributional strategy sequences in which almost all agents are maximizing for almost all aggregate shock histories, θ^t. Consider the time t valuation function $V_t(\alpha, a, \tau, \theta^t)$. This gives the payoff to agent α if the distributional strategy is given by τ, the aggregate shock history to time t is θ^t and α chooses a. Given τ, $\mathbf{C}_t(\tau)$ gives the set of period t distributional strategies whose marginal distributions on characteristics space agrees with the distribution over characteristics space implied by τ and the transistion functions. For a strategy to be an equilibrium, we require that it be consistent (with itself) and that at every time period, at almost all histories (θ shocks), almost all agents are optimizing. If, for the moment, we fix a "representative" θ^t, then τ and the transition functions imply some distribution, say $\lambda_t(\bullet; \theta^t)$ on Λ. With agents selecting actions optimally, the payoff to α is $max_a V_t(\alpha, a, \tau, \theta^t) = W_t(\alpha, \tau, \theta^t)$. Let $h(\alpha, \theta^t)$, be an optimal choice for α (at θ^t), with h a measurable function on $\Lambda \times \Theta^t$. Then h and λ_t determine a joint distribution $\hat{\tau}_t$ on $\Lambda \times A$, for each θ^t: $\hat{\tau}_t(\bullet, \bullet; \theta^t)$. By construction, if $\tilde{\tau}_t \in \mathbf{C}_t(\tau)$, for almost all θ^t, $\int_Y V_t(\alpha, a, \tau, \theta^t)\tilde{\tau}_t(dy; \theta^t) \le \int_Y V_t(\alpha, a, \tau, \theta^t)\hat{\tau}_t(dy; \theta^t)$. On the other hand, if $\hat{\tau}_t \in \mathbf{C}_t(\tau)$ and for all $\tilde{\tau}_t \in \mathbf{C}(\tau)$, $\int_Y V_t(\alpha, a, \tau, \theta^t)\tilde{\tau}_t(dy; \theta^t) \le \int_Y V_t(\alpha, a, \tau, \theta^t)\hat{\tau}_t(dy; \theta^t)$, for almost all θ^t, then at almost all θ^t, almost all agents are optimizing. If τ_t and $\hat{\tau}_t$ coincide in this definition, for each t and θ^t, then under the distributional strategy τ, every period almost all agents are optimizing at almost all aggregate shocks. In this case as τ is consistent with itself, it is an equilibrium. Formally,

Definition 2 *Let τ be a distributional strategy consistent with itself. Then τ is an equilibrium if for each t,*

$$sup_{\tilde{\tau} \in \mathbf{C}_t(\tau)} \int_{\Theta^t} \int_Y V_t(\alpha, a, \tau, \theta^t)\tilde{\tau}_t(dy; \theta^t)\nu_t(d\theta^t) \le \int_{\Theta^t} \int_Y V_t(\alpha, a, \tau, \theta^t)\tau_t(dy; \theta^t)\nu_t(d\theta^t).$$

Define a best response mapping, $\mathbf{B}(\tau)$:

$$\mathbf{B}(\tau) = \{\hat{\tau} \in \mathbf{C}(\tau) \mid \forall t, sup_{\tilde{\tau} \in \mathbf{C}_t(\tau)} \int_{\Theta^t} \int_Y V_t(\alpha, a, \tau, \theta^t)\tilde{\tau}_t(dy; \theta^t)\nu_t(d\theta^t)$$

$$\le \int_{\Theta^t} \int_Y V_t(\alpha, a, \tau, \theta^t)\hat{\tau}_t(dy; \theta^t)\nu_t(d\theta^t)\},$$

so a fixed point of $\mathbf{B}$ is an equilibrium. By construction, if $\bar{\tau}$ in $\mathbf{B}(\tau)$, then for all t, for almost all θ^t, almost all agents are maximizing at state θ^t. The next theorem shows that $\mathbf{B}$ satisfies the conditions of the Glicksberg Fan Theorem. That is, $\mathbf{B}$ is convex-valued, non-empty and upper-hemicontinuous.

Theorem 3 *The correspondence* **B** *satisfies the conditions of the Glicksberg Fan theorem and hence has a fixed point, which is an equilibrium of the game.*

4 Markov Equilibria

We now show that when the model is stationary and the θ process Markov, there exists a *Markov equilibrium.* More precisely, we show constructively that for every equilibrium, there is an (expected payoff) equivalent Markov equilibrium. This result uses the conditional no aggregate uncertainty formulation involving the θ process. The Markov equilibrium is on an enlarged state space which includes payoffs. This is similar to DGMM who also use an enlarged state space which includes payoffs. However, we impose relatively weak assumptions on the transition function of the process. We conclude this section by dropping the conditional no aggregate uncertainty hypothesis and returning to a general model of aggregate uncertainty as a random measure μ^{η}. In that environment we provide a result on the existence of Markov equilibrium where the state space is just the aggregate distribution over characteristics. This illustrates an alternative approach to modelling aggregate uncertainty, and does not require that expectations enter the state space.

For the model with conditional no aggregate uncertainty we first impose the following stationarity assumptions (with a slight abuse of notation):

1. $u_t(\alpha, a, \tau_t, \theta^t) = u(\alpha, a, \tau_t, \theta_t)$: utility is time independent and depends only on the current value of θ.
2. $\mathbf{P}_{\xi_t}(\bullet, \tau_{t-1}, \theta^{t-1}, y) = \mathbf{P}_{\xi}(\bullet, \tau_{t-1}, \theta_{t-1}, y)$: the transition function is Markov.
3. $\nu(\bullet \mid \theta^t) = \nu(\bullet \mid \theta_t)$: the aggregate shock process is Markov.

In addition, we assume that

1. $u(\alpha, a, \tau_t, \theta_t)$ is continuous in all variables.
2. $\mathbf{P}_{\xi}$ is *weak** continuous on $\mathcal{M}(\Lambda \times A) \times \Theta \times Y$.
3. $\nu(\bullet \mid \theta_t)$ is *weak** continuous on Θ.
4. Θ is a compact metric space.

These additional assumptions imply that the value functions $V_t(\alpha, a, \tau, \theta^t)$ and $W_t(\alpha, \tau, \theta^t)$ (given in theorem 1) are continuous in θ^t. We now introduce a state space, **S**, and define equilibrium Markov strategies relative to this state space.

Given an initial distribution μ over the characteristics space and an initial aggregate shock θ, we denote the associated set of equilibrium distributional strategies as:

$\mathbf{E}(\mu,\theta) = \{\tau \in \mathcal{M}_\infty \mid \tau$ *is an equilibrium of the game with initial characteristics distribution* μ *and initial aggregate shock* $\theta\}$, where $\mathcal{M}_\infty = \times_{t=0}^{\infty}\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$ and $\mathcal{F}(\Theta^0, \mathcal{M}(\Lambda \times A)) = \mathcal{M}(\Lambda \times A))$. Define the state space $\mathbf{S}$:

$$\mathbf{S} = \{(\mu, v, \theta) \in \mathcal{M}(\Lambda) \times \mathcal{C}(\Lambda) \times \Theta \mid \exists \tau \in \mathbf{E}(\mu,\theta) \textit{ and } v(\alpha) = W_1(\alpha, \tau, \theta), \forall \alpha \in \Lambda\}.$$

Thus, $(\mu, v, \theta) \in \mathbf{S}$, means that given initial conditions (μ, θ) there is an equilibrium strategy τ, such that the expected payoff to agent α in this equilibrium is $v(\alpha)$. In addition, define a correspondence $\varphi : \mathbf{S} \to \mathcal{M}_\infty$ according to:

$$\varphi(\mu, v, \theta) = \{\tau \in \mathcal{M}_\infty \mid \tau \in \mathbf{E}(\mu,\theta), v(\alpha) = W_1(\alpha, \tau, \theta), \forall \alpha \in \Lambda\}.$$

The correspondence φ associates to any point $(\mu, v, \theta) \in \mathbf{S}$ an equilibrium strategy τ (in the game with initial characteristics distribution μ and initial aggregate shock θ), with the property that the payoff to α is $v(\alpha)$. Under the additional assumption of continuity in θ, the correspondence φ is an upper-hemicontinuous correspondence. Define a Markov equilibrium:

Definition 3 *An equilibrium distributional strategy* $\bar{\tau}$ *is a Markov Equilibrium if for almost all* $\theta^t, \theta^{t'}$ *such that*

$$(i)\ \mu(\bullet \mid \theta^{t-1}) = \mu(\bullet \mid \theta^{t'-1}),\ (ii)\ W_t(\alpha, \bar{\tau}, \theta^t) = W_{t'}(\alpha, \bar{\tau}, \theta^{t'}) \textit{ and } (iii)\ \theta_t = \theta_{t'},$$

the strategy $\bar{\tau}$ *satisfies* $\bar{\tau}(\bullet, \bullet \mid \theta^t) = \bar{\tau}(\bullet, \bullet \mid \theta^{t'})$.

Thus, an equilibrium distributional strategy is a Markov equilibrium if the behavior at two different histories is the "same", when the distributions on characteristics, the expected payoffs to all agents, and the aggregate shocks agree.

Theorem 4 *Given an equilibrium* τ *of the game with initial characteristics distribution* μ *and initial state* θ, *there is a Markov equilibrium,* $\bar{\tau}$, *such that the first period payoff to each agent is unchanged: the expected payoff to* α *is the same under* $\bar{\tau}$ *as* τ.

That is, every equilibrium payoff in the game arises as the payoff to some Markov equilibrium. For the proof we take a pointwise measurable selection, τ^*, $\tau^*(\mu, v, \theta) \in \varphi(\mu, v, \theta)$, for all $(\mu, v, \theta) \in \mathbf{S}$ which we use to construct the Markov equilibrium $\bar{\tau}$. Future payoffs are supported by reapplying the first component of $\tau^*(\mu, v, \theta), \tau_1^*(\mu, v, \theta)$, in succeding periods $2, 3, \ldots$, thus introducing Markov stationarity.

For the final result, we return to a basic formulation of aggregate uncertainty. In the model with an aggregate uncertainty parameter (θ) identified explicitly, aggregate uncertainty is modelled with the aggregate distribution conditionally nonstochastic, given the current aggregate shock. Typically, however, "Markov - type" results require a degree of continuity in the transition process governing the state variable (in the sense of absolute continuity relative to a fixed measure or relative to the transition measure at all states, say). Conditional no aggregate uncertainty runs counter to this type of assumption. Reverting from the conditional no aggregate uncertainty assumption to the general specification permitting aggregate uncertainty allows us to address the issue of Markov structure with standard (although strong) assumptions on the transition process.

We return to the process $\xi(\eta, \alpha, a, \tau)$ governing the evolution of individual characteristics, modified slightly to include the distribution on characteristics, μ: $\xi(\eta, \alpha, a, \tau, \mu)$. In this case, for a given (η, τ, μ), next period's aggregate distribution is given by

$$\mu^{\eta}(B) = \tau(\{(\alpha, a) \mid \xi(\eta, \alpha, a, \tau, \mu) \in B\}), \forall B \in \mathcal{B}_{\Lambda}.$$

The corresponding distributions on the space of measures over characteristics are:

$$\psi_{\mu}(Q) \equiv \mathbf{p}(\{\eta \mid \mu^{\eta} \in Q\}), Q \in \mathcal{B}_{\mathcal{M}(\Lambda)},$$

and

$$\psi(Q) \equiv \mathbf{p}(\{\eta \mid (\mu^{\eta}, \xi(\eta, \alpha, a, \tau)) \in Q\}), Q \in \mathcal{B}_{\mathcal{M}(\Lambda)} \otimes \Lambda.$$

Since θ is no longer separate from η, it no longer enters the utility function explicitly: utility is given by a (time independent) function, $u(\alpha, a, \tau, \mu)$. The transition function now is a distribution on $\mathcal{M}(\Lambda) \times \Lambda$, where $\mathbf{P}_{\mathcal{M}\xi}(\bullet, \bullet, \alpha, a, \tau, \mu)$ gives a distribution over $\mathcal{M}(\Lambda) \times \Lambda$, given (α, a, τ, μ). A time t distributional strategy is a function from $\mathcal{M}(\Lambda)$ to $\mathcal{M}(\Lambda \times A)$. For a fixed measure ψ on $\mathcal{M}(\Lambda)$, the natural topology on the space of these functions is given by the following criterion of convergence: $\tau^k \to \tau$ if $\forall f \in \mathcal{C}(Y), g \in \mathbf{L}_1(\mathcal{M}(\Lambda), \mathcal{M}(\Lambda \times A), \psi)$,

$$\int_Y f(y)\tau^k(dy; \mu)g(\mu)\psi(d\mu) \to \int_Y f(y)\tau(dy; \mu)g(\mu)\psi(d\mu).$$

We make the following assumptions. There is a fixed measure ψ on $\mathcal{M}(\Lambda)$ such that

1. u is continuous in (α, a) and norm continuous in τ (relative to ψ):

$$\int_{\mathcal{M}(\Lambda)} sup_y \mid u(y, \tau^k, \tilde{\mu}) - u(y, \tau^k, \tilde{\mu}) \mid \psi(d\tilde{\mu}) \overset{\tau_t^k \to \tau_t}{\longrightarrow} 0.$$

2. $\mathbf{P}_{\mathcal{M}\xi}$ is continuous in (α, a) and norm continuous in τ (relative to ψ) on *measurable functions*, f, on $\mathcal{M}(\Lambda) \times \Lambda$:

$$\int_{\mathcal{M}(\Lambda)} sup_y \mid \mathbf{P}_{\mathcal{M}\xi}(f, y, \tau^k, \tilde{\mu}) - \mathbf{P}_{\mathcal{M}\xi}(f, y, \tau, \tilde{\mu}) \mid \psi(d\tilde{\mu}) \overset{\tau_t^k \to \tau_t}{\longrightarrow} 0.$$

3. The μ component of the transition functions is dominated by ψ uniformly, $\exists b < \infty$ such that for any $(y, \tau, \tilde{\mu})$, $\exists f$ measurable, $f : \mathcal{M}(\Lambda) \to \mathcal{R}$, $0 \leq f \leq b$, such that

$$\mathbf{P}_{\mathcal{M}\xi}(X, \Lambda, y, \tau, \tilde{\mu}) = \int_X f(\mu)\psi(d\mu).$$

In this model, the state space for the Markov formulation is $\mathcal{M}(\Lambda)$. An equilibrium is Markov if the current distributional strategy τ_t depends only on the current state, μ_t. The intertemporal consistency conditions on the distributions have the form: $\forall f \in \mathcal{C}(\mathcal{M}(\Lambda)), \forall g \in \mathbf{L}_1(\mathcal{M}(\Lambda), \mathcal{R}, \psi)$

$$\int_{\mathcal{M}(\Lambda)} \hat{\tau}_{t+1}(f, \Lambda; \mu) g(\mu) \psi(d\mu) = \int \mathbf{P}_{\mathcal{M}\xi}(f, \Lambda; y, \tau_t, \mu) g(\mu) \psi(d\mu), \forall y \in Y, t \geq 1,$$

and,

$$\int_{\mathcal{M}(\Lambda)} \hat{\tau}_1(f, \Lambda; \mu) g(\mu) \psi(d\mu) = \int_{\mathcal{M}} \mu(f) g(\mu) \psi(d\mu).$$

In this case, say that $\hat{\tau}$ is consistent with τ.

The proof proceeds along the lines of theorems 1 through 3. This again entails establishing the existence of valuation functions $\{V_t(\alpha, a, \tau, \mu)\}_{t\geq 1}$ and $\{W_t(\alpha, \tau, \mu)\}_{t\geq 1}$, where $\tau = (\tau_1, \tau_2, \ldots,)$, $\tau_t : \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda \times A)$. As before we do so by looking at a truncated n-period version of the game, establishing the existence of $\{V_t^n(\alpha, a, \tau, \mu)\}_{t\geq 1}$ and $\{W_t^n(\alpha, \tau, \mu)\}_{t\geq 1}$, and then take limits to obtain $\{V_t(\alpha, a, \tau, \mu)\}_{t\geq 1}$ and $\{W_t(\alpha, \tau, \mu)\}_{t\geq 1}$. Note that even if the function $W_t(\alpha, \tau, \mu)$ were continuous in μ (for fixed τ), since τ_t (for example) is an endogenously determined function of μ, τ will depend on μ as a measurable function which is generally not continuous. When the dependence of τ on μ is taken into account, $W_t(\alpha, \tau, \mu)$ depends measurably but not continuously on μ. As a result, the convergence of expressions such as $\int_{\mathcal{M}(\Lambda)\times\Lambda} W_t(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}_{\mathcal{M}\xi}(d\tilde{\mu} \times d\tilde{\alpha}; y, \tau^k, \mu) \psi(d\mu)$ to $\int_{\mathcal{M}(\Lambda)\times\Lambda} W_t(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}_{\mathcal{M}\xi}(d\tilde{\mu} \times d\tilde{\alpha}; y, \tau, \mu) \psi(d\mu)$ depends on the assumption of norm continuity on measurable functions. Assumption 3 of uniform boundedness of the Radon-Nikodym derivative is made for similar reasons. In this framework, the appropriate definition of equilibrium is:

Definition 4 *A strategy τ is a Markov equilibrium if τ is consistent with itself and for each t,*

$$\int_Y V_t(\alpha, a, \tau, \mu) \tau_t(dy; \mu) \psi(d\mu) \geq \int_Y V_t(\alpha, a, \tau, \mu) \tilde{\tau}_t(dy; \mu) \psi(d\mu),$$

for all $\tilde{\tau}$ consistent with τ.

We then argue that the consistency and best response mappings satisfy the conditions of the Glicksberg Fan Theorem so that an equilibrium exists.

Theorem 5 *There exists a Markov equilibrium.*

5 Appendix

It is useful to prove first the following lemmas which are used in the proof of theorem 1 below.

Lemma 1 *Let X, Y and Z be compact metric spaces. Let $\pi(x,y,z)$ be continuous in $(x,y,z) \in X \times Y \times Z$, and $\mathbf{P}(\bullet; y,z) : Y \times Z \to \mathcal{M}(X)$ be continuous in (y,z), so that $(y_k, z_k) \to (y,z)$ implies that the sequence of measures $\mathbf{P}(\bullet; y_k, z_k)$ converges (weak*) to $\mathbf{P}(\bullet; y,z)$. Finally, let $\mathbf{Q}(\bullet; z) : Z \to \mathcal{M}(Y)$ be (weak*) continuous in z. Then $z_k \to z$ implies that*

$$\int_Y \int_X \pi(x,y,z_k)\mathbf{P}(dx; y, z_k)\mathbf{Q}(dy; z_k) \to \int_Y \int_X \pi(x,y,z)\mathbf{P}(dz; y,z)\mathbf{Q}(dy; z).$$

Proof: Let $\gamma(y,z) = \int_X \pi(x,y,z)\mathbf{P}(dx; y,z)$ and note that $\gamma(y,z)$ is continuous. To see this, let $w = (y,z)$ and consider a sequence $w_k \to w$. Then

$$\mid \gamma(w_k) - \gamma(w) \mid = \mid \int \pi(x,w_k)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w) \mid \leq$$

$$\mid \int \pi(x,w_k)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w_k) \mid + \mid \int \pi(x,w)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w) \mid .$$

Since $\mid \int \pi(x,w_k)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w_k) \mid \leq \int \mid \pi(x,w_k) - \int \pi(x,w) \mid \mathbf{P}(dx; w_k)$ and π is uniformly continuous on $X \times W$ ($X \times W$ is a compact metric space), then given $\varepsilon > 0$, $\exists \bar{k}$ such that $k \geq \bar{k}$ implies that $\mid \pi(x,w_k) - \pi(x,w) \mid \leq \varepsilon$, for all x. Thus,

$$\mid \int \pi(x,w_k)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w_k) \mid \to 0.$$

Since π is continuous on (x,w) and $\mathbf{P}(\bullet; w_k)$ converges weakly to $\mathbf{P}(\bullet; w)$,

$$\mid \int \pi(x,w)\mathbf{P}(dx; w_k) - \int \pi(x,w)\mathbf{P}(dx; w) \mid \to 0.$$

Thus, γ is continuous on $W = Y \times Z$. Since W is a compact metric space, γ is uniformly continuous on W.

Now, $\int_Y \int_X \pi(x,y,z_k)\mathbf{P}(dx; y, z_k)\mathbf{Q}(dy; z_k) = \int_Y \gamma(y, z_k)\mathbf{Q}(dy; z_k)$, so that using the uniform continuity of γ and *weak** convergence of $\mathbf{Q}(\bullet; z_k)$ to $\mathbf{Q}(\bullet; z)$, the same argument as above gives

$$\int_Y \gamma(y,z_k)\mathbf{Q}(dy; z_k) \to \int_Y \gamma(y,z)\mathbf{Q}(dy; z) = \int_Y \int_X \pi(x,y,z)\mathbf{P}(dz; y,z)\mathbf{Q}(dy; z),$$

which completes the proof.

$\square$

For the next lemma, we need the following notation. Let $(\Omega, \mathcal{D}, \mu)$ be a given probability space and $\mathcal{M}(Y)$ the set of measures on $Y = \Lambda \times A$. Let $\mathcal{F}(\Omega, \mathcal{M}(Y))$ denote the set of measurable functions from Ω to $\mathcal{M}(Y)$. A sequence of measures, $\{\tau^k\}$ in $\mathcal{F}(\Omega, \mathcal{M}(Y))$ converges to a measure τ if and only if

$$\int_\Omega \int_Y f(y)\tau^k(dy;\omega)g(\omega)\mu(d\omega) \to \int_\Omega \int_Y f(y)\tau(dy;\omega)g(\omega)\mu(d\omega), \forall f \in \mathcal{C}(Y), g \in \mathbf{L}_1(\Omega, \mathcal{R}, \mu).$$

Lemma 2 *Let $r : Y \times \mathcal{M}(Y) \times \Omega \to \mathcal{R}$ (so that given (y, τ, ω), r has the value $r(y, \tau(\omega), \omega)$ or $r(y, \tau, \omega)$ for brevity) be continuous on Y and norm continuous with respect to τ: as $\tau^k \to \tau$, $\int_\Omega sup_y \mid r(y, \tau^k, \omega) - r(y, \tau, \omega) \mid \mu(d\omega) \to 0$. Then $s(\alpha, \tau, \omega) = max_a r(\alpha, a, \tau, \omega)$ is continuous in α and norm continuous in τ: $\tau^k \to \tau$ implies $\int_\Omega sup_\alpha \mid s(\alpha, \tau^k, \omega) - s(\alpha, \tau, \omega) \mid \mu(d\omega) \to 0$.*

Proof : Continuity of s in α is clear. To consider norm continuity of s in τ, let $\tau^k \to \tau$. Since r is norm continuous, given $\epsilon > 0$, $\exists \bar{k}$, such that $k \geq \bar{k}$ implies

$$\int_\Omega sup_y \mid r(\alpha, a, \tau^k, \omega) - r(\alpha, a, \tau, \omega) \mid \mu(d\omega) \leq \epsilon.$$

Let

$$\Omega_k(\beta\epsilon) = \{\omega \mid \mid r(\alpha, a, \tau^k, \omega) - r(\alpha, a, \tau, \omega) \mid \geq \beta\epsilon\}.$$

Then

$$\epsilon \geq \int_\Omega sup_y \mid r(\alpha, a, \tau^k, \omega) - r(\alpha, a, \tau, \omega) \mid \mu(d\omega)$$

$$\geq \int_{\Omega_k(\beta\epsilon)} sup_y \mid r(\alpha, a, \tau^k, \omega) - r(\alpha, a, \tau, \omega) \mid \mu(d\omega) \geq \beta\epsilon\mu(\Omega_k(\beta\epsilon)).$$

Thus, $1 \geq \beta\mu(\Omega_k(\beta\epsilon))$, and setting $\beta = 1/\sqrt{\epsilon}$ gives $\sqrt{\epsilon} \geq \mu(\Omega_k(\sqrt{\epsilon}))$. Let $a^k(\alpha, \omega)$ maximize $r(\alpha, a, \tau^k, \omega)$ and $a(\alpha, \omega)$ maximize $r(\alpha, a, \tau, \omega)$. On $\Omega_k(\sqrt{\epsilon})^c, \forall\alpha$

$$r(\alpha, a^k(\alpha, \omega), \tau^k, \omega) \geq r(\alpha, a(\alpha, \omega), \tau^k, \omega) \geq r(\alpha, a(\alpha, \omega), \tau, \omega) - \sqrt{\epsilon}.$$

The first inequality follows since $a^k(\alpha, \omega)$ is a maximizer of $r(\alpha, a, \tau^k, \omega)$ and the second inequality follows since $\omega \in \Omega_k(\sqrt{\epsilon})^c$. Similarly,

$$r(\alpha, a^k(\alpha, \omega), \tau^k, \omega) \leq r(\alpha, a(\alpha, \omega), \tau, \omega) + \sqrt{\epsilon} \leq r(\alpha, a(\alpha, \omega), \tau, \omega) + \sqrt{\epsilon}.$$

The first inequality follows since $\omega \in \Omega_k(\sqrt{\epsilon})^c$ and the second follows since $a(\alpha, \omega)$ is a maximizer of $r(\alpha, a, \tau, \omega)$. Consequently, $\forall\alpha, \mid s(\alpha, \tau^k, \omega) - s(\alpha, \tau, \omega) \mid \leq \sqrt{\epsilon}, \omega \in \Omega_k(\sqrt{\epsilon})^c$. Thus,

$$sup_\alpha \mid s(\alpha, \tau^k, \omega) - s(\alpha, \tau, \omega) \mid \leq \sqrt{\epsilon}, \omega \in \Omega_k(\sqrt{\epsilon})^c.$$

Therefore,

$$\int_\Omega \mid sup_\alpha \mid s(\alpha, \tau^k, \omega) - s(\alpha, \tau, \omega) \mid \mu(d\omega) =$$

$\int_{\Omega_k(\sqrt{\epsilon})} sup_\alpha \mid s(\alpha,\tau^k,\omega) - s(\alpha,\tau,\omega) \mid \mu(d\omega) + \int_{\Omega_k(\sqrt{\epsilon})^c} sup_\alpha \mid s(\alpha,\tau^k,\omega) - s(\alpha,\tau,\omega) \mid \mu(d\omega)$

The latter expression is bounded above by $2K\mu(\Omega_k(\sqrt{\epsilon})) + \sqrt{\epsilon}\mu(\Omega_k(\sqrt{\epsilon})^c) \le 2K\sqrt{\epsilon} + \sqrt{\epsilon} = [2K+1]\sqrt{\epsilon}$.

□

Proof of Theorem 1: We now use these results to demonstrate that there exist value functions $V_t(\alpha, a, \tau, \theta^t)$ and $W_t(\alpha, \tau, \theta^t)$ which are continuous in (α, a) and α respectively, and norm continuous in τ. First consider an n-period truncation of the game. Trivially in period n, $V_n^n(\alpha, a, \tau, \theta^n) = u_n(\alpha, a, \tau_n, \theta_n)$, which, by assumption, is continuous in (α, a) and norm continuous, as is $W_n^n(\alpha, \tau, \theta^n) = max_a u_n(\alpha, a, \tau_n, \theta_n)$, by lemma 2. Now define

$$V_{n-1}^n(\alpha, a, \tau, \theta^{n-1}) = u_{n-1}(\alpha, a, \tau_{n-1}, \theta_{n-1}) +$$
$$\delta_n \int_\Theta \int_\Lambda W_n^n(\xi, \tau, \theta^n) \mathbf{P}_{\xi_n}(d\xi, \tau_{n-1}, \theta^{n-1}, \alpha, a)\nu_n(d\theta^n \mid \theta^{n-1}).$$

$V_{n-1}^n(\alpha, a, \tau, \theta^{n-1})$ satisfies: (1) $V_{n-1}^n(\alpha, a, \tau, \theta^n)$ is continuous in (α, a) and (2) if $\tau^k \to \tau$, then $\int_\Theta sup_{\alpha,a} \mid V_{n-1}^n(\alpha, a, \tau^k, \theta^{n-1}) - V_{n-1}^n(\alpha, a, \tau, \theta^{n-1}) \mid \nu_n(d\theta^{n-1}) \to 0$. Continuity in (α, a) follows from lemma 1, treating $(\tau_{n-1}, \theta^{n-1})$ as parameters of u_{n-1} and $\mathbf{P}_{\xi_n}$, respectively. To simplify notation we now write $\mathbf{P}_{\xi_n}^k$ for $\mathbf{P}_{\xi_n}^k(d\xi, \tau_{n-1}^k, \theta^{n-1}, \alpha, a)$, and so forth. Norm continuity in τ can be seen by separating current and future components of expected payoffs, so that, using abbreviated notation,

$$\int_\Theta sup_{(\alpha,a)} \mid V_{n-1}^n(\alpha, a, \tau^k, \theta^{n-1}) - V_{n-1}^n(\alpha, a, \tau, \theta^{n-1}) \mid \nu_n(d\theta^{n-1})$$

becomes

$$\int_{\Theta^{n-1}} sup_{(\alpha,a)} \mid u_{n-1}^k - u_{n-1} + \delta_n \int_\Theta \int_\Lambda W_n^{nk} \mathbf{P}_{\xi_n}^k \nu_n(d\theta^n \mid \theta^{n-1}) -$$
$$\delta_n \int_\Theta \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \nu_n(d\theta^n \mid \theta^{n-1}) \mid \nu_{n-1}(d\theta^{n-1})$$

$$\le \int_{\Theta^{n-1}} sup_{(\alpha,a)} \mid u_{n-1}^k - u_{n-1} \mid \nu(d\theta^{n-1}) +$$
$$\delta_n \int_{\Theta^{n-1}} sup_{(\alpha,a)} \mid \int_\Theta \int_\Lambda W_n^{nk} \mathbf{P}_{\xi_n}^k \nu_n(d\theta^n \mid \theta^{n-1}) - \int_\Theta \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \nu_n(d\theta^n \mid \theta^{n-1}) \mid$$
$$\nu_{n-1}(d\theta^{n-1}).$$

The first term on the right hand side converges to 0 as $\tau^k \to \tau$, because u_{n-1} is norm continuous. For the second term consider

$$\int_{\Theta^{n-1}} sup_{(\alpha,a)} \mid \int_\Theta \int_\Lambda W_n^{nk} \mathbf{P}_{\xi_n}^k \nu_n(d\theta^n \mid \theta^{n-1}) - \int_\Theta \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \nu_n(d\theta^n \mid \theta^{n-1}) \mid \nu_{n-1}(d\theta^{n-1})$$

$$\le \int_{\Theta^{n-1}} \int_\Theta sup_{(\alpha,a)} \mid \int_\Lambda W_n^{nk} \mathbf{P}_{\xi_n}^k - \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \mid \nu_n(d\theta^n \mid \theta^{n-1}) \nu_{n-1}(d\theta^{n-1})$$

$$= \int_{\Theta^n} sup_{(\alpha,a)} \mid \int_\Lambda W_n^{nk}\mathbf{P}^k_{\xi_n} - \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \mid \nu_n(d\theta^n).$$

The last expression is no greater than:

$\int_{\Theta^n} sup_{(\alpha,a)} \mid \int_\Lambda W_n^{nk}\mathbf{P}^k_{\xi_n} - \int_\Lambda W_n^n \mathbf{P}^k_{\xi_n} \mid \nu_n(d\theta^n)+$

$\int_{\Theta^n} sup_{(\alpha,a)} \mid \int_\Lambda W_n^{n}\mathbf{P}^k_{\xi_n} - \int_\Lambda W_n^n \mathbf{P}_{\xi_n} \mid \nu_n(d\theta^n).$

The first term is bounded above by $\int_{\Theta^n} sup_\alpha \mid \int_\Lambda W_n^{nk} - W_n^n \mid \nu_n(d\theta^n)$, which converges to 0, from norm continuity of W_n^n. The second term is converges to 0, by norm continuity of $\mathbf{P}^k_{\xi_n}(d\xi, \tau_{n-1}, \theta^{n-1}, \alpha, a)$ in τ. $W^n_{n-1}(\alpha, \tau, \theta^{n-1})$ is defined from $V^n_{n-1}(\alpha, a, \tau, \theta^{n-1})$ and, as before, is continuous in α. Norm continuity of $W^n_{n-1}(\alpha, \tau, \theta^{n-1})$ in τ follows from lemma 2.

Proceed inductively in this way to define $V_t^n(\alpha, a, \tau, \theta^t)$ and $W_t^n(\alpha, \tau, \theta^t)$ for $1 \leq t \leq n$. The discussion above defines the recursion for fixed n and shows that for any t, $1 \leq t \leq n$, that both $V_t^n(\alpha, a, \tau, \theta^t)$ ($\equiv V^n_{n-(n-t)}(\alpha, a, \tau, \theta^{n-(n-t)})$) and $W_t^n(\alpha, \tau, \theta^t)$ are continuous functions of (α, a) and α respectively, and that both are norm continuous in τ. To conclude we show that the following limits exist for each j and are continuous functions of (α, a) and α respectively:

$$lim_{n\to\infty} V_j^n(\alpha, a, \tau, \theta^j) = V_j(\alpha, a, \tau, \theta^j) \; and \; lim_{n\to\infty} W_j^n(\alpha, \tau, \theta^j) = W_j(\alpha, \tau, \theta^j).$$

Taking $n > j$, observe that each of the functions $V_j^n(\alpha, a, \tau, \theta^j)$ and $W_j^n(\alpha, \tau, \theta^j)$ is increasing in n, and that

$$0 \leq V_j^{n+s}(\alpha, a, \tau, \theta^j) - V_j^n(\alpha, a, \tau, \theta^j) \leq \sum_{r=1}^{s} (\times_{h=j}^{n+r-1} \delta_{j+h})K \leq [\delta^{n-j+1}/(1-\delta)]K,$$

$$0 \leq W_j^{n+s}(\alpha, \tau, \theta^j) - W_j^n(\alpha, \tau, \theta^j) \leq \sum_{r=1}^{s} (\times_{h=j}^{n+r-1} \delta_{j+h})K \leq [\delta^{n-j+1}/(1-\delta)]K.$$

Therefore $V_j^n(\alpha, a, \tau, \theta^j)$ and $W_j^n(\alpha, \tau, \theta^j)$ are Cauchy sequences in n. Since $V_j^n(\alpha, a, \tau, \theta^j)$ and $W_j^n(\alpha, \tau, \theta^j)$ are continuous in (α, a) and α respectively and are both norm continuous in τ, the limits $V_t(\alpha, a, \tau, \theta^t) = lim_n V_t^n(\alpha, a, \tau, \theta^t)$ and $W_t(\alpha, \tau, \theta^t) = lim_n W_t^n(\alpha, \tau, \theta^t)$ inherit these properties also.

□

Proof of Theorem 2: To show upper-hemicontinuity of $C(\tau)$ let $\tilde{\tau}^n = (\tilde{\tau}_1^n, \ldots, \tilde{\tau}_t^n, \ldots)$ and $\tau^n = (\tau_1^n, \ldots, \tau_t^n, \ldots)$ where $\tilde{\tau}_t^n$, $\tau_t^n \in \mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$ with $\tilde{\tau}^n \in C(\tau^n)$, $\tilde{\tau}_t^n \to \tilde{\tau}_t$, $\tau_t^n \to \tau_t$, and

$$\int_{\Theta^t} \tilde{\tau}_t^n(f, A; \theta^t) g(\theta^t) \nu_t(d\theta^t) = \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}^n, \theta^{t-1}, y) \tau_{t-1}^n(dy; \theta^{t-1}) g(\theta^t) \nu_t(d\theta^t).$$

Then,

$$\int_{\Theta^t} \tilde{\tau}_t(f, A; \theta^t) g(\theta^t) \nu_t(d\theta^t) = \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1}) g(\theta^t) \nu_t(d\theta^t).$$

To see this, note directly from the topology on τ that

$$\int_{\Theta^t} \tilde{\tau}_t^n(f, A; \theta^t) g(\theta^t) \nu_t(d\theta^t) \to \int_{\Theta^t} \tilde{\tau}_t(f, A; \theta^t) g(\theta^t) \nu_t(d\theta^t).$$

Now consider the right hand side. Abbreviate $\mathbf{P}_{\xi_t}(f, \tau_{t-1}^n, \theta^{t-1}, y)$ by $\mathbf{P}_{\xi_t}^n(\theta^{t-1}, y)$ and $\mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y)$ by $\mathbf{P}_{\xi_t}(\theta^{t-1}, y)$. Then

$$\begin{aligned}
& \left| \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}^n(\theta^{t-1}, y) \tau_{t-1}^n(dy; \theta^{t-1}) g d\nu_t - \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1}) g d\nu_t \right| \\
\leq & \left| \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}^n(\theta^{t-1}, y) \tau_{t-1}^n(dy; \theta^{t-1}) g d\nu_t - \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \tau_{t-1}^n(dy; \theta^{t-1}) g d\nu_t \right| \\
+ & \left| \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \tau_{t-1}^n(dy; \theta^{t-1}) g d\nu_t - \int_{\Theta^t} \int_Y \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1}) g d\nu_t \right|.
\end{aligned}$$

The first term on the right of the inequality is less than or equal to

$$\int_{\Theta^t} \int_Y \| \mathbf{P}_{\xi_t}^n(\theta^{t-1}, y) - \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \|_y \, \tau_{t-1}^n(dy; \theta^{t-1}) g d\nu_t$$

$$= \int_{\Theta^t} \| \mathbf{P}_{\xi_t}^n(\theta^{t-1}, y) - \mathbf{P}_{\xi_t}(\theta^{t-1}, y) \|_y \, g d\nu_t,$$

and the norm continuity condition on $\mathbf{P}_{\xi_t}$ implies that this goes to zero. The second term converges to zero from the topology on τ. Convexity follows since the restrictions are linear. It remains to show that non-emptyness is also satisfied. To see this, given $\mu_1(\bullet)$, the initial measure on Λ, and given the measure ν_1 on Θ, let h be a measurable function from $\Lambda \times \Theta$ to A. Define a measure φ on $\Lambda \times A \times \Theta$ according to the property that $\varphi(X \times Z) = \mu_1 \otimes \nu_1(h^{-1}(X) \cap Z)$ for any measurable sets X and Z in A and $\Lambda \times \Theta$ respectively. (Interpret φ as the unique extension from such measurable rectangles). Let $\hat{\tau}_1(\bullet, \bullet; \theta_1) = \varphi(\bullet, \bullet; \theta_1)$, where $\varphi(\bullet, \bullet; \theta_1)$ is the conditional distribution of φ on $\Lambda \times A$, given θ_1. Note that $\varphi(A \times Z) = \mu_1 \otimes \nu_1(Z)$ so that $\hat{\tau}_1(\bullet, A; \theta_1) = \mu_1 \otimes \nu_1(\bullet; \theta_1) = \mu_1(\bullet)$. For $t \geq 2$, a similar discussion applies. View $\int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1})$ as a conditional distribution on Λ given θ^{t-1}. Let $\mathbf{Q}$ be the joint distribution on $\Lambda \times \Theta^t$ determined by $\int_Y \mathbf{P}_{\xi_t}(f, \tau_{t-1}, \theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1})$ and ν_t. As before, let h be a measurable function from $\Lambda \times \Theta^t$ to A. Define a measure on $\Lambda \times A \times \Theta^t$, φ, determined on rectangles $X \times Z$, where X and Z are measurable subsets of A and $\Lambda \times \Theta^t$ respectively, by $\varphi(X \times Z) = \mathbf{Q}(h^{-1}(X) \cap Z)$. Let $\tilde{\tau}_t(\bullet, \bullet; \theta^t) = \varphi(\bullet, \bullet; \theta^t)$. Lastly, note that $\tilde{\tau}_t(\bullet, A; \theta^t) = \varphi(\bullet, A; \theta^t) = \mathbf{Q}(h^{-1}(A) \cap \bullet; \theta^t)$ $= \mathbf{Q}(\bullet; \theta^t) = \int_Y \mathbf{P}_{\xi t}(\bullet, \tau_{t-1}, \theta^{t-1}, y) \tau_{t-1}(dy; \theta^{t-1})$.

□

Proof of Theorem 3: To see that $\mathbf{B}$ is convex-valued, observe that $\mathbf{C}$ is convex-valued and the additional constraints on $\hat{\tau}$ in the definition of $\mathbf{B}$ are defined by linear inequalities, so that $\mathbf{B}$ is convex-valued. Note that $\mathbf{B}$ is non-empty since for any t, $\mathbf{C}_t(\tau)$ is closed and non-empty and $\int_{\Theta^t}\int_Y V_t(\alpha,a,\tau,\theta^t)\tau_t(dy;\theta^t)\nu_t(d\theta^t)$ is continuous in τ.

Recall that $W_t(\alpha,\tau,\theta^t) \equiv max_a V_t(\alpha,a,\tau,\theta^t)$ and that the set of distributions in $\mathbf{C}_t(\tau)$ are required to have the same marginal distribution on Λ: if $\hat{\tau}_t$, $\bar{\tau}_t \in \mathbf{C}_t(\tau)$, then $\tau_t(\bullet,A;\theta^t) = \tau_t(\bullet,A;\theta^t)$, ν_t almost everywhere θ^t. Let this distribution on Λ be denoted $\lambda_t(\bullet;\theta^t)$. Then $\hat{\tau} \in \mathbf{B}(\tau)$ if and only if,

$$\int_{\Theta^t}\int_Y V_t(\alpha,a,\tau,\theta^t)\hat{\tau}_t(dy;\theta^t)\nu_t(d\theta^t) = \int_{\Theta^t}\int_\Lambda W_t(\alpha,\tau,\theta^t)\lambda_t(d\alpha;\theta^t)\nu_t(d\theta^t).$$

We use this result below to prove upper-hemicontinuity of $\mathbf{B}(\bullet)$. To see this, consider the correspondence

$$\psi(\alpha,\theta^t) \equiv \{a \mid max_a V_t(\alpha,a,\tau,\theta^t) \leq V_t(\alpha,a,\tau,\theta^t)\} \equiv \{a \mid W_t(\alpha,\tau,\theta^t) \leq V_t(\alpha,a,\tau,\theta^t)\}.$$

Denote the graph of ψ by $\mathcal{G}_\psi$ and observe that $\mathcal{G}_\psi \in \mathcal{B}_\Lambda \times \mathcal{B}_A \times \mathcal{B}^t_\Theta$ (ψ has a measurable graph) since

$$\mathcal{G}_\psi = \{(\alpha,a,\theta^t) \mid a \in \psi(\alpha,\theta^t)\} = \{(\alpha,a,\theta^t) \mid W_t(\alpha,\tau,\theta^t) \leq V_t(\alpha,a,\tau,\theta^t)\}.$$

Denote by $\lambda \otimes \nu_t$ the measure on $\Lambda \times \Theta^t$ determined by λ and ν_t. Viewing ψ as a correspondence from $(\Lambda \times \Theta^t, \mathcal{B}_\Lambda \times \mathcal{B}^t, \lambda \otimes \nu_t)$, there is a measurable selection $h : \Lambda \times \Theta^t \to A$, with $h(\alpha,\theta^t) \in \psi(\alpha,\theta^t)$ almost everywhere $\lambda \otimes \nu_t$ since $\mathcal{G}_\psi \in \mathcal{B}_\Lambda \times \mathcal{B}_A \times \mathcal{B}^t$ (using the measurable selection theorem). Thus $V_t(\alpha,h(\alpha,\theta^t),\tau,\theta^t) = W_t(\alpha,\tau,\theta^t)$, almost everywhere $\lambda \otimes \nu_t$. Now, define a distribution on $\Lambda \times A \times \Theta^t$ by $\varphi(W \times Z) = \lambda \otimes \nu_t(h^{-1}(W) \cap Z), \forall W \in \mathcal{B}_A$ and $Z \in \mathcal{B}_\Lambda \times \mathcal{B}^t$. Observe that $\varphi(A \times Z) = (\lambda \otimes \nu_t)(Z)$ so that ν_t almost everywhere θ^t, $\varphi(A,\bullet;\theta^t) = \lambda(\bullet;\theta^t)$. Define $\tau^*_t : \tau^*_t(\bullet,\bullet;\theta^t) = \varphi(\bullet,\bullet;\theta^t)$. Then $\tau^*_t \in \mathbf{C}_t(\tau)$ and,

$$\begin{aligned}\int_{\Theta^t}\int_Y V_t(\alpha,a,\tau,\theta^t)\tau^*_t(dy;\theta^t)\nu_t(d\theta^t) &= \\ \int_{\Theta^t}\int_\Lambda V_t(\alpha,h(\alpha,\theta^t),\theta^t)\lambda_t(d\alpha;\theta^t)\nu_t(d\theta^t) &\geq \int_{\Theta^t}\int_\Lambda W_t(\alpha,\tau,\theta^t)\Lambda_t(d\alpha;\theta^t)\nu_t(d\theta^t).\end{aligned}$$

However, since $W_t(\alpha,\tau,\theta^t) \geq V_t(\alpha,a,\tau,\theta^t), \forall(\alpha,a) \in \Lambda \times A$, we have

$$\int_{\Theta^t}\int_Y V_t(\alpha,a,\tau,\theta^t)\tau^*_t(dy;\theta^t)\nu_t(d\theta^t) = \int_{\Theta^t}\int_\Lambda W_t(\alpha,\tau,\theta^t)\lambda_t(d\alpha;\theta^t)\nu_t(d\theta^t).$$

It remains to show that $\mathbf{B}(\tau)$ is upper-hemicontinuous. Let $\tau^n \to \tau$ and suppose that $\tilde{\tau}^n \in \mathbf{B}(\tau^n)$, with $\tilde{\tau}^n \to \tilde{\tau}$. It is necessary to show that $\tilde{\tau} \in \mathbf{B}(\tau)$. Recall $W_t(\alpha,\tau,\theta^t)$ is norm continuous:

$$\int_{\Theta^t} sup_\alpha \mid W_t(\alpha,\tau^n,\theta^t) - W_t(\alpha,\tau,\theta^t) \mid \nu(d\theta^t) \to 0.$$

Let $\lambda^n_t(\bullet;\theta^t)$ be the distribution on Λ determined by $\mathbf{C}_t(\tau^n)$, so that if $\hat{\tau}^n_t \in \mathbf{C}(\tau^n)$, then $\hat{\tau}^n_t(\bullet,A;\theta^t) = \lambda^n_t(\bullet;\theta^t)$, ν_t almost everywhere. Norm continuity of $W_t(\alpha,\tau,\theta^t)$ implies that

$$\int_{\Theta^t}\int_\Lambda \mid W_t(\alpha,\tau^n,\theta^t) - W_t(\alpha,\tau,\theta^t) \mid \lambda^n_t(d\alpha;\theta^t)\nu(d\theta^t) \to 0.$$

Hence,

$$| \int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau^n, \theta^t) \lambda_t^n(d\alpha; \theta^t) \nu(d\theta^t) - \int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau, \theta^t) \lambda_t^n(d\alpha; \theta^t) \nu(d\theta^t) | \rightarrow 0.$$

Now in the topology given on measures, let λ_t be the limit of λ_t^n and note that $\hat{\tau}_t \in \mathbf{C}(\tau)$, so $\hat{\tau}_t(A, \bullet; \theta^t) = \lambda_t(\bullet; \theta^t)$, ν_t almost everywhere. Observe also that $W_t(\alpha, \tau, \theta^t)$ is continuous in α (since $W_t(\alpha, \tau, \theta^t) = max_a V_t(\alpha, a, \tau, \theta^t)$ and $V_t(\alpha, a, \tau, \theta^t)$ is continuous in (α, a), then $W_t(\alpha, \tau, \theta^t)$ is continuous in α (for each θ^t)). Thus

$$\int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau, \theta^t) \lambda_t^n(d\alpha; \theta^t) \nu(d\theta^t) \rightarrow \int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau, \theta^t) \lambda_t(d\alpha; \theta^t) \nu(d\theta^t).$$

Now recall that since $\tilde{\tau}^n \in \mathbf{B}(\tau^n)$,

$$\int_{\Theta^t} \int_{\Lambda} V_t(\alpha, a, \tau^n, \theta^t) \tilde{\tau}_t^n(d\alpha \times da; \theta^t) \nu(d\theta^t) = \int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau^n, \theta^t) \lambda_t^n(d\alpha; \theta^t) \nu(d\theta^t).$$

Since $\tilde{\tau}^n \rightarrow \tilde{\tau}$ as $\tau^n \rightarrow \tau$, using the norm continuity of $V_t(\alpha, a, \tau^n, \theta^t)$ we find that

$$\int_{\Theta^t} \int_{\Lambda} V_t(\alpha, a, \tau^n, \theta^t) \tilde{\tau}_t^n(d\alpha \times da; \theta^t) \nu(d\theta^t) \rightarrow \int_{\Theta^t} \int_{\Lambda} V_t(\alpha, a, \tau, \theta^t) \tilde{\tau}_t(d\alpha \times da; \theta^t) \nu(d\theta^t).$$

Therefore,

$$\int_{\Theta^t} \int_{\Lambda} V_t(\alpha, a, \tau, \theta^t) \tilde{\tau}_t(d\alpha \times da; \theta^t) \nu(d\theta^t) = \int_{\Theta^t} \int_{\Lambda} W_t(\alpha, \tau, \theta^t) \lambda_t(d\alpha; \theta^t) \nu(d\theta^t).$$

Thus, $\tilde{\tau} \in \mathbf{B}(\tau)$, so that $\mathbf{B}$ is upper-hemicontinuous. Therefore $\mathbf{B}$ is convex-valued, non-empty and upper-hemicontinuous and so has a fixed point.

□

Proof of Theorem 4: The following discussion describes the construction of the Markov equilibrium. In view of the following facts:

1. $\mathbf{E}$ is upper-hemicontinuous,
2. W_1 is norm continuous in τ, and continuous in (α, θ),
3. $\mathcal{M}_\infty$ is metrizable and compact,

it follows that φ is an upper-hemicontinuous correspondence into a complete separable space. Hence there is a pointwise measurable selection, τ^*, $\tau^*(\mu, v, \theta) \in \varphi(\mu, v, \theta)$, for all $(\mu, v, \theta) \in \mathbf{S}$. We use τ^* to construct the Markov equilibrium $\bar{\tau}$. Consider the first component of $\tau^*(\mu, v, \theta)$, $\tau_1^*(\mu, v, \theta)$. This is a measure on $\Lambda \times A$ which is optimal in the sense that at (μ, v, θ):

$$\tau_1^*(\mu, v, \theta)\{(\alpha, a) \mid V_1(\alpha, a, \tau^*, \theta) \geq W_1(\alpha, \tau^*, \theta)\} = 1.$$

To implement the strategy in period one, knowledge of (μ, v, θ) is required. Now given θ, let

$$\mu_2(\bullet \mid \theta) = \int_Y \mathbf{P}_\xi(\bullet, \tau_1^*, \theta, y)\tau_1^*(dy).$$

The measure $\mu_2(\bullet \mid \theta)$ is the second period distribution on characteristics. Given a realization of the aggregate shock in the second period, say θ', the expected payoff to agent α over the remainder of the game is: $W_2(\alpha, \tau^*, (\theta, \theta'))$ $(= v\mid_{(\theta,\theta')} (\alpha)$. Considering τ^* and (θ, θ') fixed, $W(\alpha, \tau^*, (\theta, \theta'))$ is an element of $\mathcal{C}(\Lambda))$, which we can write as $v_2(\alpha)$. Now observe that τ^* induces an equilibrium from period 2 on, for all "histories" except possibly a set of ν measure 0. Thus, except for a set θ''s of ν measure 0,

$$(\mu_2(\bullet \mid \theta),\ W_2(\alpha, a, \tau^*, (\theta, \theta')\,),\ \theta') \in \mathbf{S}.$$

Denote this "state" by (μ_2, v_2, θ'). Viewed as a subgame, the expected payoff to agent α is $v_2(\alpha)$. Note that this payoff is generated at this subgame by τ^*: $\tau_2^*(\bullet, \bullet \mid (\theta, \theta')\,)$. However, note that exactly the same payoff is obtained on this subgame if $(\tau_2^*, \tau_3^*, \ldots)$ is replaced by $\tau^*(\mu_2, \nu_2, \theta')$. For this reason, τ_1^* remains optimal and the strategy obtained in this way is an equilibrium. Denote this strategy by $\tau^*(2) \in \mathcal{M}_\infty$ (given the initial θ) as

$$\tau^*(2) = (\tau_1^*(\mu, v, \theta),\ \tau^*[\mu_2(\bullet \mid \theta),\ W_2(\alpha, \tau^*, (\theta, \theta')),\ \theta']_{\theta' \in \Theta}).$$

Thus, $\tau^*(2)$ is composed of $\tau_1^*(\mu, v, \theta)$ in the first period, and then τ^* is "restarted" in period 2: at the subgame reached by history θ', the "state" is $s_2 = (\mu_2(\bullet \mid \theta),\ W_2(\alpha, \tau^*, (\theta, \theta')),\ \theta')$ and this state is sustained by the strategy $\tau^*(s_2)$, at that subgame.

The important point about this construction is that τ_1^* is being applied at *period two* and this is the way in which Markov stationarity is introduced. Note that $\tau^*(2)$ induces an equilibrium on almost all histories, θ', and gives the same continuation payoffs from the second period at each history as did τ^*. This ensures that τ_1^* is optimal at almost all θ in period one. Thus, the strategy $\tau^*(2)$ is also an equilibrium which gives the same first period payoff (v) as τ^*. First period "strategies", $\tau_1^*(\mu, v, \theta)$ are unchanged while second period strategies under $\tau^*(2)$ generate the same expected payoff there as did τ^*. The result of this construction is that the Markov property holds for the first and second period.

Now, replace the equilibrium strategy τ^* by the equilibrium strategy $\tau^*(2)$. This alters the evolution of the characteristics distribution and the valuation functions. In particular,

$$\mu_3(\bullet \mid \theta, \theta') = \int_Y \mathbf{P}_\xi(\bullet, \tau_2^*(2)[\mu_2(\bullet \mid \theta),\ W_2(\alpha, \tau_2^*(2), (\theta, \theta')), \theta']\, \theta', y) \times$$

$$\tau_2^*(2)[\mu_2(\bullet \mid \theta),\ W_2(\alpha, \tau^*, (\theta, \theta'),\ \theta'](dy).$$

Similarly, there is a valuation function for period 3, $W_3(\alpha, \tau^*(2), \theta, \theta', \tilde{\theta})$. Here again, for a fixed history, $(\theta, \theta', \tilde{\theta})$, $W_3(\alpha, \tau^*(2), \theta, \theta', \tilde{\theta}) \in \mathcal{C}(\Lambda)$. Now, define $\tau^*(3)$

$$\tau^*(3) = (\tau_1^*(2), \tau_2^*(2), \tau^*[(\mu_3(\bullet \mid \theta, \theta'), W_2(\alpha, \tau^*, (\theta, \theta', \tilde{\theta})), \tilde{\theta}\,]_{(\theta,\theta')\in\Theta^2}).$$

As with $\tau^*(2)$, $\tau^*(3)$ is an equilibrium. Proceed in this way to define iteratively a sequence of equilibria $\tau^*(n)$ from $\tau^*(n-1)$ and observe that the sequence $\{\tau^*(n)\}_n$ converges, say to $\bar{\tau}$. Under $\bar{\tau}$ and the Markov distribution on Θ, the state variable $s = (\mu, v, \theta)$ evolves stochastically as a Markov chain. Schematically,

$$s_1 = (\mu, v \mid_\theta, \theta) \xrightarrow{\theta'} (\mu \mid_\theta, v \mid_{(\theta,\theta')}, \theta') = s_2 \xrightarrow{\tilde{\theta}}$$

$$(\mu \mid_{\theta,\theta'}, v \mid_{(\theta,\theta',\tilde{\theta})}, \tilde{\theta}) = s_3 \xrightarrow{\theta^0} (\mu \mid_{(\theta,\theta',\tilde{\theta})}, v \mid_{(\theta,\theta',\tilde{\theta},\theta^0)}, \theta^0).$$

or alternatively,

$$s_1 = (\mu, v \mid_\theta, \theta) \xrightarrow{\theta'} (\mu \mid_\theta, v \mid_{(\theta,\theta')}, \theta') = s_2 = (\mu', v', \theta') \xrightarrow{\tilde{\theta}}$$

$$(\mu' \mid_{\theta'}, v' \mid_{\tilde{\theta}}, \tilde{\theta}) = (\hat{\mu}, \hat{v}, \tilde{\theta}) = s_3 \xrightarrow{\theta^0} (\hat{\mu} \mid_{\theta^0}, \hat{v} \mid_{\theta^0}, \theta^0).$$

The evolution of the states may be described as follows. With $\bar{\tau}$, given s_1, the distributional strategy at time 1 is $\tau_1^*(s_1)$. (Note that the first components of τ^* and $\bar{\tau}$ are related: $\tau_1^*(\mu, v, \theta) = \bar{\tau}_1(\mu, v, \theta), \forall(\mu, v, \theta)$. At $t = 2$, the distributional strategy is $\tau_1^*(s_2)$, and at time t, $\tau_1^*(s_t)$. The influence of the θ sequence on strategies is only through the s variables, since given τ, s_t depends on $\theta^t = (\theta_1, \theta_2, \cdots \theta_t)$. The behaviour of $\bar{\tau}$ throughout the remainder of the game (from period t on) depends only on θ^t through s_t, so we can write the value function $W_t(\alpha, \tau, \theta^t)$ as $\overline{W}_t(\alpha, \tau, \theta_t, s_t(\theta^{t-1}))$. Note also that, since the environment is stationary, if $s_l(\theta^{l-1}) = s_t(\theta^{t-1})$, then $(\tau_1^*, s_l(\theta^{l-1}))$ and $(\tau_1^*, s_t(\theta^{t-1}))$ induce the same distribution over the state space in subsequent periods so that $\overline{W}_t(\alpha, \bar{\tau}, \theta, s_t(\theta^{t-1})) = \overline{W}_l(\alpha, \bar{\tau}, \theta, s_l(\theta^{l-1}))$. Consequently, we may write $\overline{W}(\alpha, \bar{\tau}, \theta, s_t(\theta^{t-1}))$ to denote the time t value function (without the time subscript).

A play of the game in this formulation may be described as follows. Fix an initial state $s = (\mu, \nu, \theta)$. At time $t = 1$ the distributional strategy $\tau_1^*(s)$ is played. Depending on the realization of the second period aggregate uncertainty variable, θ' a new state $s_2 = (\mu \mid_\theta, v \mid_{(\theta,\theta')}, \theta')$ is reached. The first component of $s_2 = (\mu \mid_\theta, v \mid_{(\theta,\theta')}, \theta')$ is equal to $\mu_2(\bullet \mid \theta) = \int_Y \mathbf{P}_\xi(\bullet, \tau_1^*, \theta, y)\tau_1^*(dy)$ and the second component is equal to $\overline{W}_2(\alpha, \bar{\tau}, \theta, s_2) = \overline{W}(\alpha, \theta, \bar{\tau}, s_2)$. For fixed s_2 and given $\bar{\tau}$, $\overline{W}(\alpha, a, \theta, \bar{\tau}, s_2) \in \mathbf{L}_1(\Theta, \mathcal{C}(\Lambda), \nu_1)$ and for fixed θ, $\mu_2(\bullet \mid \theta) \in \mathcal{M}(\Lambda)$. This completes the description of the Markov equilibrium.

□

Proof of Theorem 5: The proof follows essentially the same plan as the proof of theorem 3. This requires showing first that the consistency mapping is an upper- hemicontinuous correspondence and that there exist value functions for this case, analogous to those given in theorem 1. To define the value functions, follow theorem 1 and consider a game truncated to n periods. Given $\tau = \{\tau_t\}_{t=1}^{\infty}$, define $V_n^n(\alpha, a, \tau, \mu) = u(\alpha, a, \tau_n, \mu)$. Continuity in (α, a) and norm continuity in τ of $V_n^n(\alpha, a, \tau, \mu)$ follow directly since $u(\alpha, a, \tau_n, \mu)$ has these properties. Let $W_n^n(\alpha, \tau, \mu) = max_a V_n^n(\alpha, a, \tau, \mu)$, $W_n^n(\alpha, \tau, \mu)$ is continuous in α and norm continuous in τ by lemma 2. Next, define

$$V_{n-1}^n(\alpha, a, \tau, \mu) = u(\alpha, a, \tau_{n-1}, \mu) + \delta \int_{\Lambda \times \mathcal{M}(\Lambda)} W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}, \mu).$$

To see that $V_{n-1}^n(\alpha, a, \tau, \mu)$ is continuous in (α, a) and norm continuous in τ, observe that

$$\int sup_y \mid V_{n-1}^n(y, \tau^k, \mu) - V_{n-1}^n(y, \tau, \mu) \mid \psi(d\mu)$$

$$\leq \int sup_y \mid u(\alpha, a, \tau_{n-1}^k, \mu) - u(\alpha, a, \tau_{n-1}, \mu) \mid \psi(d\mu) +$$

$$\delta \int sup_\alpha \mid \int W_n^n(\tilde{\alpha}, \tau^k, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}^k, \mu) -$$

$$\int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}, \mu) \mid \psi(d\mu).$$

The first term on the right goes to 0, by norm continuity of u. The second term is bounded from above by

$$\int sup_y \mid \int W_n^n(\tilde{\alpha}, \tau^k, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}^k, \mu) -$$

$$\int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}^k, \mu) \mid \psi(d\mu)$$

$$+ \int sup_y \mid \int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}^k, \mu) -$$

$$\int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}, \mu) \mid \psi(d\mu).$$

The second of these terms converges to 0, since $\mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid \alpha, a, \tau_{n-1}, \mu)$ is norm continuous on measurable functions. The first of these two terms is bounded from above by

$$\int sup_y \int sup_{\tilde{\alpha}} \mid W_n^n(\tilde{\alpha}, \tau^k, \tilde{\mu}) - \int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mid \mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid y, \tau_{n-1}^k, \mu) \psi(d\mu)$$

$$\leq \int_\mu \int_{\tilde{\mu}} sup_{\tilde{\alpha}} \mid W_n^n(\tilde{\alpha}, \tau^k, \tilde{\mu}) - \int W_n^n(\tilde{\alpha}, \tau, \tilde{\mu}) \mid b\psi(d\mu)\psi(d\tilde{\mu}).$$

The latter term converges to 0, hence $V_{n-1}^n(\alpha, a, \tau, \mu)$ is norm continuous in τ. Finally, continuity of $V_{n-1}^n(\alpha, a, \tau, \mu)$ in $y = (\alpha, a)$ follows directly from continuity of $u(\alpha, a, \tau_{n-1}, \mu)$ in y and since $\mathbf{P}(d(\tilde{\alpha}, \tilde{\mu}) \mid y, \tau_{n-1}^k, \mu)$ is assumed continuous in y on measurable functions.

Let $W_{n-1}^n(\alpha, \tau, \mu) = max_a V_{n-1}^n(\alpha, a, \tau, \mu)$, and proceed inductively to define sequences of functions, $\{V_t^n(\alpha, a, \tau, \mu)\}_{t=1}^n$ and $\{W_t^n(\alpha, \tau, \mu)\}_{t=1}^n$ As in the proof of theorem 1, the limits

$\lim_n V_t^n(\alpha, a, \tau, \mu)$ and $\lim_n W_t^n(\alpha, a, \tau, \mu)$ exist and are norm continuous in τ and continuous in (α, a) and α respectively.

Next, observe that the intertemporal consistency conditions satisfy upper-hemicontinuity. To see this let $\tau^n \to \tau$ and let $\tilde{\tau}^n$ be a consistent sequence in the range of the correspondence with $\tilde{\tau}^n \to \tilde{\tau}$. Thus, considering period t, $\forall f \in \mathcal{C}(\mathcal{M}(\Lambda)), \forall g \in \mathbf{L}_1(\mathcal{M}(\Lambda), \mathcal{R}, \psi)$

$$\int_{\mathcal{M}(\Lambda)} \tilde{\tau}_{t+1}(f, \Lambda; \mu) g(\mu) \psi(d\mu) = \int \mathbf{P}_{\mathcal{M}\xi}(f, \Lambda; y, \tau_t, \mu) g(\mu) \psi(d\mu), \forall y \in Y.$$

$\int_{\mathcal{M}(\Lambda)} \tilde{\tau}_{t+1}(f, \Lambda; \mu) g(\mu) \psi(d\mu)$ converges to $\int_{\mathcal{M}(\Lambda)} \tau_{t+1}(f, \Lambda; \mu) g(\mu) \psi(d\mu)$, in view of the topology on τ. Comparing $\int \mathbf{P}_{\mathcal{M}\xi}(f, \Lambda; y, \tau_t^n, \mu) g(\mu) \psi(d\mu)$ and $\int \mathbf{P}_{\mathcal{M}\xi}(f, \Lambda; y, \tau_t, \mu) g(\mu) \psi(d\mu)$, the difference (in absolute value) converges to 0, by norm continuity of $\mathbf{P}_{\mathcal{M}\xi}$.

Finally, we construct the best response mapping in exactly the same way as was done in theorem 3. A consistent strategy $\hat{\tau}$ is a best response (with consistent marginal λ on $\mathcal{M}(\Lambda)$: $\hat{\tau}(\bullet, \Lambda : \mu) = \lambda_t(\bullet : \mu), \psi$ *a.e.* μ),

$$\int_{\mathcal{M}(\Lambda)} \int_Y V_t(\alpha, a, \tau, \mu) \hat{\tau}_t(dy; \mu) \psi_t(d\mu) = \int_{\mathcal{M}(\Lambda)} \int_\Lambda W_t(\alpha, \tau, \mu) \lambda_t(d\alpha; \mu) \psi(d\mu).$$

The reasoning given in the proof of theorem 3, establishes existence here also.

$\square$

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6 Table of Notation

- Λ: Agents' characteristics space ($\alpha \in \Lambda$).
- A: Action space of each agent ($a \in A$).
- Y: $Y = \Lambda \times A$.
- μ: Aggregate distribution on agents' characteristics.
- $\mathcal{M}(X)$: Space of probability measures on X.
- $\mathcal{C}(X)$: Space of of continuous functions on X.
- $\mathcal{B}_X$: Family of Borel sets of X.
- Θ: State space of aggregate uncertainty ($\theta \in \Theta$).
- Θ^∞: $\Theta^\infty \equiv \times_{t=1}^\infty \Theta$ aggregate uncertainty for the infinite game.
- θ^∞: $\theta^\infty \cdots = (\theta_1, \theta_2, \cdots \theta_t, \cdots) \in \Theta^\infty$.
- θ^t: $\theta^t \equiv (\theta_1, \theta_2, \cdots \theta_t)$.
- $\mathbf{L}_1(\Theta^t, \mathcal{C}(\Lambda \times A), \nu_t)$: Normed space of measurable functions from Θ^t to $\mathcal{C}(\Lambda \times A)$.
- $\mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$: Space of measurable functions from Θ^t to $\mathcal{M}(\Lambda \times A)$.
- X^t: $X^t = \times_{s=1}^t X$.
- $\mathcal{B}_X^t$: Borel field on X^t.
- ν: Distribution on Θ^∞.
- ν_t: Marginal distribution of ν on Θ^t.
- $\nu(\bullet \mid \theta^t)$: Conditional distribution on Θ^∞ given θ^t.
- $\nu_t(\bullet \mid \theta^s)$: Conditional distribution on Θ^t given θ^s (where $s < t$).
- τ_t: "Period t" distributional strategy.
- τ: Distributional strategy for all periods $\tau = (\tau_1, \tau_2, \cdots, \tau_t, \cdots)$.
- ξ_t: Transition process for agents' types.
- $\mathbf{P}_{\xi_{t+1}}(\bullet, \tau_t, \theta^t, y)$: Transition function associated with ξ_t.
- u_t: Utility function.
- $V_t(\alpha, a, \tau, \theta^t)$: Value function for each collection $(\alpha, a, \tau, \theta^t)$.

- $W_t(\alpha, \tau, \theta^t)$: Value function given optimal action a.
- $\mathbf{C}(\tau)$: Consistency correspondence. Distributions consistent with τ and characteristics transition functions, $\mathbf{P}_{\xi_t}$.
- $\mathbf{B}(\tau)$: Best response correspondence (which also satisfy consistency).
- $\mathbf{E}(\mu)$: Set of equilibrium distributional strategies.
- $\mathcal{M}_\infty$: $\times_{t=1}^{\infty} \mathcal{F}(\Theta^t, \mathcal{M}(\Lambda \times A))$.
- $\mathbf{S}$: Expanded state space for Markov construction.
- $v(\alpha, a, \theta)$: Value function for Markov construction.
- $\mathbf{P}(\bullet, \tau_t^*, \theta_t, y)$: Invariant charateristics transition function for Markov game.