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Strategic Behavior and Information Revelation in Dynamic Auctions

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STRATEGIC BEHAVIOR AND INFORMATION REVELATION IN DYNAMIC AUCTIONS

by

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Abstract

In this paper, we study a two-period common-value auction model in which the seller possesses some private information about the value of the object being sold. Even though the probability of revealing his information in a later period could be different, the seller's equilibrium strategy is similar and information revealing: sell late if and only if the information is favorable enough. We also show that it is the seller's best policy to always reveal his private information in some equilibria but not in others. One implication of the model is that the owner's ability to sell on more occasions generally reduces his expected revenue.

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1. Introduction

There is a large literature on auctions[1]. Most papers deal with the private information of the bidders. Some deal with the private information of the seller. Milgrom and Weber (1982), for example, prove that the owner of an auctioned object should reveal truthfully any private information he possesses, if he can commit to the policy he chooses. In the model, the seller has strong incentive to cheat after he receives his private information. Kyle (1985), as another example, examines the revelation of information in dynamic trading in which an informed trader' private information is completely incorporated into prices by the end of the trading process. It is assumed that other agents in the market are noise traders.

Under circumstances that all traders are rational and the seller cannot commit to any policy that is not ex post revenue maximizing, we may ask the following questions: What is our prediction about the seller's behavior? How is the private information of the seller revealed from his behavior? Can information revealing increase the seller's revenue? Can the seller be benefited from having the option of selling in one more period if he cannot sell it successfully in previous periods?

Our paper answers these questions. Specifically, a two-period model is studied. In the model, the owner has an indivisible common-valued object for sale. The owner offers to sell it to the highest bid in an auction, if that bid is above his reservation price. If the highest bid is too low, he can sell it in a second-period auction. Participants in those auctions are assumed to be the same for notational simplicity. In the model, whether or not the owner can sell the object successfully in the first period reveals some of the owner's private information. Potential buyers compete with each other in the auctions. Their bids for the asset, of course, depend on what information about the asset they can infer from the actions of the owner and other private information they themselves have. We allow for the possibility that the seller's information is revealed before the next auction starts if the seller does not sell the object successfully in the first auction. More importantly, the potential buyers are assumed to get more private information from exogenous sources as time passes by. When the buyers get additional private information, they tend to receive more of the surplus, which means that the owner loses more on average. Therefore, the owner of the asset faces an interesting tradeoff as a result of his private information: if he sells it low in the current auction, the bidders infer that the owner's private information is likely to be unfavorable; if he sets a high reservation price and cannot sell it until next auction, they get more of the surplus.

We find that no matter what the exogenous probability the seller's information will be revealed later, in equilibrium the owner sets a low reservation price and sells successfully in the first period when he receives unfavorable information about the asset and sets a high reservation price and sells late when he receives favorable information. There could be many such Nash equilibria in the two-period game. The best equilibrium for the owner is always to sell the asset in the first period auction; but this may not be revenue maximizing ex post and may not survive if we impose certain intuitive restrictions on the bidders' beliefs. We define a sequential Nash equilibrium for the two-period game and argue that sequential equilibrium is more appropriate than Nash equilibrium for analyzing this model. If we consider only the profit maximizing equilibrium among those sequential equilibria, we find that the revenue to the seller is an increasing function of the probability of publicly revealing of the seller's private information. This is not true, however, if we consider other sequential equilibria in the game.

Our results suggest that if the seller can somehow commit himself to always sell in one auction, then he will get more out of the sale. This can prevent buyers from getting more private information. Thus by committing himself to a quick sale, he at the same time benefits himself.[2] Other evidence that favors a quick sale includes "must sell"s or "out-of-business-sale" in advertising. This also provides a justification for auctions: quick sale--it prevents bidders from getting more private information.

We have made some simplifying assumptions to make the analysis tractable. Although these assumptions limit the direct applicability of the model, it is suggestive of certain principles concerning the sale of a house, a used car, a share of common stock, etc., when the owner possesses some private information that the potential buyers do not have at the time of sale, and when the potential buyers can acquire more private information as time passes.

The paper is organized as follows. In Section 2, we examine the properties of the equilibria in the two-period auction game. In Section 3, we investigate the effect of information revelation on the seller's revenue. Section 4 contains a summary.

2. The Model

Consider the following two-period auctioning model: There are n bidders, named 1, 2, ..., n, and the owner of an asset is named 0. The asset has a common value V (which is random) to every bidder. The owner can sell the asset in either of the two periods by means of a first-price, sealed-bid auction in each period.[3] Bidders have identical information before the first auction starts; the common prior among these bidders is given by a cumulative distribution function H(v). In the first period, bidders receive no private information[4]. The owner, however, receives a private random signal $I_0 \in \Sigma$ and determines a reservation price for the auction in the first period. If he sells successfully, the game ends. If the asset is not sold in the first period, then it must be sold in the second period. In the second period, each bidder, say i, receives some private information denoted by the random variable $I_i \in \Omega$, where $i \in N \equiv \{1,2,\ldots,n\}$ is the set of bidders. At the end of the second period, the value of the asset is realized. For simplicity, assume that there is no discounting.

We make the following additional assumption on the information bidders have about the value of the auctioned asset in the second period: the owner's private information I_0 is revealed to the bidders with probability α before the second-period auction starts. Bidders are Bayesian and update their prior beliefs using whatever information is available to them.

The following assumptions in no way affect the qualitative results of our paper but simplifying the analysis greatly: the support of V is $\{0,v\}$ (where $v>0$); the support of I_i is $\Omega=\{0,1\}$, $\forall i \in N$; the support of I_0 is $\Sigma=[\underline{s},\overline{s}]$; the distribution of I_0 given V is atomless; $I_1,\ldots,I_n$ are mutually independent and identically distributed conditional on V; and I_0, I_1, ... , I_n are affiliated with V according to the definition below:

Two random variables X and Y are *affiliated* if $\forall$ $x>x'$, $y>y'$, where x,x' are in the support of X, y,y' are in the support of Y,

$$\Pr\{x/y\}\Pr\{x'/y'\} \geq \Pr\{x'/y\}\Pr\{x/y'\},$$

where the respective conditional probabilities are properly defined.[5]

The concept of affiliation has been introduced into auction theory by Milgrom and Weber (1982). We use a simple version of their definition. The following lemma will be needed; its proof and a more detailed treatment of affiliation can be found in their paper.

Lemma 1

Assume that X and Y are affiliated and that all the following conditional probabilities are properly defined. Then $\forall\ y>y'$,

i) $E(X/y) \geq E(X/y')$;

ii) $\forall x$, $\Pr\{X \geq x/y) \geq \Pr\{X \geq x/y'\}$.

Let $q_0=\Pr\{I_1=0/V=0\}$, $q_1=\Pr\{I_1=1/V=v\}$. Since V and I_1 are affiliated, from Lemma 1, we have $\Pr\{I_1 \geq 1/V=v\} \geq \Pr\{I_1 \geq 1/V=0\}$; i.e., $q_0+q_1 \geq 1$.

Since the bidders have no private information in the first period, in any equilibrium, because of competition the bidders will pay the expected value of the asset conditional on the event that the asset is sold successfully in the first period. Given this form of strategy of the bidders, the seller's choice reduces to choosing whether or not to sell in the first period, given I_0. A reservation price in the first-period auction will not convey more information to the bidders than simply a choice of whether or not selling in the first period. In the case of anonymous reservation price, bidders will simply pay a single price in the first period. Thus, the seller will set a reservation price higher than that price if he prefers to wait, and set the reservation price lower than that if he prefers to sell it right away. The situation is more complicated in the case when the seller announces his reservation price publicly in the first period. Bids from bidders may depend on the announced reservation price. But if the seller would like to sell it in the first period, then he would announce the reservation price that induces the highest payment in the first period. If he would like to sell in the second period, then he would like to announce the price that induces the highest expected payment in the second period. Thus, the effective decision space is virtually a set of {1,2}, where 1 indicates that he sells in the first period, and 2 indicates that he sells in the second period.

A pure strategy for the owner in this two-period game is thus given by a function that maps his set of possible private information signals Σ into $\{1,2\}$.

A mixed strategy for the owner is given by a function ψ: $\Sigma\times[0,1] \to \{1,2\}$. Let U_0 serves as a randomizing device for the mixed strategy (following the notation of Aumann (1964)), where U_0 is uniformly distributed on [0,1] and independent of $I_0, I_1, \ldots, I_n$, and V. Then the owner's decision is characterized by the function $\psi(I_0,U_0)$. If $\psi(I_0,U_0)=1$, he sells in period 1. If $\psi(I_0,U_0)=2$, he sells in period 2.

Suppose that $U_1,\ldots,U_n$, are the randomizing devices for the bidders, where U_i, $i\in N$, are mutually independent and independent of $I_0, I_1, \ldots, I_n$, V and U_0. In the case that I_0 is not revealed, bidder i's strategy is given by a pair of functions b_{i1}, b_{i2}, such that if the owner sells the asset in the first period, he will bid $b_{i1}=b_{i1}(U_i)$, and if the owner sells it in the second period, he will bid $b_{i2}(I_i,U_i)$ or $\tilde{b}_{i2}(I_0,I_i,U_i)$ depending on whether I_0 is revealed or not. The payoff to any bidder is the expected value of the asset conditional on all information available to him multiplied by his winning probability minus his expected payment.

Suppose that the owner's equilibrium strategy is to sell in the first period if and only if $(I_0,U_0)\in S_1$, i.e., $\psi(\sigma,u_0)=1$ if and only if $(\sigma,u_0)\in S_1$. Suppose also that this occurs with positive probability. Because of competition between bidders who possess identical information, the equilibrium price in the first period is equal to the expected value of the auctioned object conditional on S, or $E(V/(I_0,U_0)\in S_1)$.

With probability α, the private information of the seller (I_0=s) is revealed publicly at the beginning of the second period; with probability $1-\alpha$, that information is not revealed. Let $S_2=\Sigma\times[0,1]\backslash S_1$ in the case when the private information of the seller is not revealed and $S_2=\{s\}\times[0,1]$ in the case when that information is revealed. Thus, bidders update their prior according to $(I_0,U_0)\in S_2$, the common information in the second period. In addition to this, each bidder receives some private information (a private signal of either 0 or 1) about the value of the asset. As the signal space is discrete, there is no pure-strategy equilibrium in the game. We outline below the proof for the non-existence of pure-strategy equilibria in this second-period game. Suppose that there were a pure-strategy equilibrium. Consider one of the players, say 1. When player 1 receives signal 0, he must make non-negative expected profits, since he can always do so by bidding 0. When he receives signal 1, however, he must make strictly positive profits, since he can certainly do so by making the same bid he makes at signal 0. But then there must be another player, say 2, bidding the same amount conditional upon some private signal as player 1 does upon seeing signal 1, since otherwise 1 could lower his bid a bit and improve his payoff. Player 2, however, can improve his payoff in this event by bidding a bit higher, since he could earn extra profits by taking over player 1's winning opportunities. Thus, the hypothesized strategy combination cannot be an equilibrium. (A rigorous proof can be found in Wang (1990).) We have the following lemma[6] for the second-period game:

<u>Lemma 2</u>

The second-period subgame has a mixed-strategy equilibrium in which each bidder adopts the following strategy: bid $\underline{x}=E(V/I_1=\ldots=I_n=0,(I_0,U_0)\in S_2)$ if 0 is observed; bid according to a continuous c.d.f. $F(x)$ with an interval support $[\underline{x}, \bar{x}]$ if 1 is observed, where $F(x)$ and $\bar{x}$ are given in the proof.

<u>Proof</u>

Given the strategy of the owner is $\psi(I_0,U_0)$, bidders update their common prior in the second period using $\psi(I_0,U_0)=2$.

We first construct a mixed strategy and then prove that if each player uses this strategy, the resulting symmetric strategy combination constitutes an equilibrium: upon observing 0, a player bids

$$\underline{x} = E(V/I_1=0,\ldots,I_n=0,(I_0,U_0)\in S_2) = \frac{(1-q_1)^n d\cdot v}{(1-q_1)^n d + q_0^n(1-d)}$$

where $d = \Pr\{V=v/(I_0,U_0)\in S_2\}$, and upon observing 1 a player bids x according to the continuous c.d.f. F(x) with support $[\underline{x},\bar{x}]$, where F is constructed as follows: The equation that describes the indifference of a player, say 1, over the support of his mixture is

$$\Pi_1 = \Pr\{1 \text{ wins } /I_1=1,V=v, \text{ and 1 bids } x\}(v-x)\Pr\{V=v/I_1=1,(I_0,U_0)\in S_2\}$$
$$+ \Pr\{1 \text{ wins } /I_1=1,V=0, \text{ and 1 bids } x\}(-x)\Pr\{V=0/I_1=1,(I_0,U_0)\in S_2\}$$
$$= [q_1F(x) + (1-q_1)]^{n-1}(v-x)\frac{dq_1}{dq_1+(1-d)(1-q_0)}$$
$$+ [(1-q_0)F(x) + q_0]^{n-1}(-x)\frac{(1-d)(1-q_0)}{(1-d)(1-q_0)+dq_1}.$$

Since $F(\underline{x})=0$, this equation with $x=\underline{x}$ pins down the value of Π_1 at

$$\frac{(1-q_1)^{n-1}q_0^{n-1}d(1-d)}{(1-q_1)^n d + q_0^n(1-d)}\cdot\frac{q_0+q_1-1}{(1-d)(1-q_0)+d\,q_1}\cdot v,$$

and since $F(\bar{x})=1$, the equation similarly pins down the value of $\bar{x}$.

For $x\in(\underline{x},\bar{x})$, rewriting the equation for Π_1 generates the inverse function:

$$x = \frac{vd[q_1F(x) + (1-q_1)]^{n-1}q_1 - \Pi_1[(1-d)(1-q_0)+dq_1]}{d[q_1F(x) + (1-q_1)]^{n-1}q_1 + (1-d)(1-q_0)[(1-q_0)F(x)+q_0]^{n-1}}$$

It is easy to prove that the numerator is always positive for $0 \le F(x) \le 1$; and since $q_1F(x) + (1-q_1)$ is monotone strictly increasing and continuous as a function of $F(x)$ and

$$\frac{(1-q_0)F(x) + q_0}{q_1F(x) + (1-q_1)}$$

is strictly decreasing and continuous in $F(x)$ (since $q_0 + q_1 > 1$), x is strictly increasing and continuous in $F(x)$. Therefore $F(x)$ must be strictly increasing and continuous in x on $(\underline{x},\overline{x})$; hence F is a c.d.f.

To see that the symmetric strategy combination is an equilibrium, notice that upon observing 1 a player receives constant profit $\Pi_1>0$ by bidding anywhere on $[\underline{x},\overline{x}]$, less by bidding above $\overline{x}$ (since $\overline{x}$ wins with probability 1) and zero by bidding below $\underline{x}$. Upon observing 0, bidding above $\overline{x}$ is never optimal for the bidder for similar reasons and bidding below $\underline{x}$ earns zero expected profits, while at $x\in[\underline{x},\overline{x}]$, his profits are

$$\Pi_0(x)=\Pr\{1 \text{ wins}/I_1=0, V=v, \text{ and } 1 \text{ bids } x\}(v-x)\Pr\{V=v/I_1=0, (I_0,U_0)\in S_2\}$$
$$+\Pr\{1 \text{ wins}/I_1=0, V=0, \text{ and } 1 \text{ bids } x\}(-x)\Pr\{V=0/I_1=0, (I_0,U_0)\in S_2\}$$

$$= [q_1F(x) + (1-q_1)]^{n-1} (v-x) \frac{d\,(1-q_1)}{d\,(1-q_1) + (1-d)q_0}$$

$$+ \quad [(1-q_0)F(x) + q_0]^{n-1}(-x) \frac{(1-d)q_0}{d\,(1-q_1) + (1-d)q_0}$$

$$= \frac{[d\,q_1 + (1-d)(1-q_0)](1-q_1)}{[d\,(1-q_1) + (1-d)q_0]\,q}\,\Pi_1$$

$$- x[(1-q_0)F(x) + q_0]^{n-1} \frac{(1-d)[q_0 + q_1 - 1]}{[d\,(1-q_1) + (1-d)q_0]q_1}$$

which is a decreasing function of x. So, bidding $\underline{x}$ maximizes $\Pi_0(x)$ on $[\underline{x},\overline{x}]$ and $\Pi_0(\underline{x}) = 0$. Thus, each player using the constructed strategy constitutes an equilibrium. ∎

A bidders makes zero expected profit when 0 is received. When 1 is received, however, a bidder makes positive profit by simply bidding $\underline{x}$. Given that other bidders are bidding randomly over $[\underline{x},\overline{x}]$ when 1 is received, a bidder bidding higher will have a higher probability of winning but at the same time will pay more upon winning. Thus, by constructing a suitable randomization, a bidder is indifferent to bidding any point in the support. This payoff of the bidder is proved positive, which means that the seller does not receive the full amount of the expected value of the object from the auction.

Let $\Gamma(s;\Sigma')$ denote the owner's revenue from the second-period auction when bidders' common prior is $\{I_0 \in \Sigma'\}$, whereas in fact $I_0=s$. From Lemma 2, each bidder bids $\underline{x}$ with probability $Pr\{I_1=0/I_0=s\}$, and bids $x \in [\underline{x},\overline{x}]$ according to a c.d.f. $F(x)$ with probability $Pr\{I_1=1/I_0=s\}$. Because of affiliation between I_i and I_0, from Lemma 1 $Pr\{I_1=0/I_0=s\}$ is decreasing in s while $Pr\{I_1=1/I_0=s\}$ is increasing in s. Let $B_s(x)$ denote the c.d.f. of the bid by any bidder. Then for $x \geq \underline{x}$,

$$B_s(x) = Pr\{I_1=0/I_0=s\}+F(x)Pr\{I_1=1/I_0=s\} = (1-F(x))\ Pr\{I_1=0/I_0=s\} + F(x)$$

$B_s(x)$ is decreasing in s, so that $[B_s(x)]^n$, the c.d.f. of the highest bid must also be decreasing in s. Thus the distribution of the highest bid is first order stochastically monotone increasing in s. Therefore we have:

<u>Lemma 3</u>

$\Gamma(s;\Sigma')$ is increasing in s.

For $\Sigma' = [s', s'']$, we have the following additional property:

Lemma 4

$\Gamma(s; [s', s''])$ is increasing in s' and s''.

Proof

We index $\underline{x}$, $\overline{x}$, and $F(x)$ by s' and s'' for $\Sigma' = [s', s'']$. With probability $Pr\{I_1=0/I_0=s\}$, a bidder bids $\underline{x}(s', s'')$; with probability $Pr\{I_1=0/I_0=s\}$, a bidder bids $x \in [\underline{x}(s', s''), \overline{x}(s', s'')]$ according to $F(x; s', s'')$. Since these probabilities do not change as s' and s'' change, we only need to prove that $F(x; s', s'')$ is first-order stochastically increasing in s' and s''. Because of the affiliation between I_0 and V, $\underline{x}(s', s'')$ must be increasing in s' and s''. $F(x; s', s'')$, however, is determined by the following equation:

$$\Pi_1 = Pr\{1 \text{ wins}/I_1=1,\ V=v, \text{ and } 1 \text{ bids } x\}\ (v-x)\ Pr\{V=v/I_1=1, I_0 \in [s', s'']\}$$

$$+ Pr\{1 \text{ wins}/I_1=1,\ V=0, \text{ and } 1 \text{ bids } x\}\ (-x)\ Pr\{V=0/I_1=1, I_0 \in [s', s'']\}$$

$$= \left[\ q_1 F(x; s', s'') + (1-q_1)\ \right]^{n-1}\ (v-x)\ h(s', s'')$$

$$+ \left[\ (1-q_0) F(x; s', s'') + q_0\ \right]^{n-1}\ (-x)\ (1-h(s', s''))$$

$$\equiv\ (1-q_1)^{n-1}\ (v-\underline{x}(s', s''))\ h(s', s'')\ + q_0^{n-1}\ (-\underline{x}(s', s''))\ (1-h(s', s''))$$

where $h(s', s'') = Pr\{V=v/I_1=1,\ I_0 \in [s', s'']\}$. It follows from the affiliation properties that h is increasing in s' and s''.

Let

$$\Delta = \left\{\left[\ q_1 F(x; s', s'') + (1-q_1)\ \right]^{n-1}\ (v-x)\ -\ (1-q_1)^{n-1}(v-\underline{x}(s', s''))\right\}\ h(s', s'')$$

$$+ \left\{\left[\ (1-q_0) F(x; s', s'') + q_0\ \right]^{n-1}\ (-x)\ -\ q_0^{n-1}(-\underline{x}(s', s''))\right\}\ (1-h(s', s''))$$

$$\equiv 0$$

We have

$$\frac{\partial\Delta}{\partial h} = \Big\{[\, q_1F(x;s',s'') + (1-q_1)\,]^{n-1}\ (v-x) - (1-q_1)^{n-1}(v-\underline{x}(s',s''))\Big\}$$

$$- \Big\{[\,(1-q_0)F(x;s',s'') + q_0\,]^{n-1}\ (-x) - q_0^{n-1}(-\underline{x}(s',s''))\Big\}$$

$$= \frac{1}{h(s',s'')}\Big[[\,(1-q_0)F(x;s',s'') + q_0\,]^{n-1}\ x - q_0^{n-1}\underline{x}(s',s'')\Big] \quad > 0$$

$$\frac{\partial\Delta}{\partial F} = h(n-1)\ q_1[q_1F(x) + (1-q_1)]^{n-2}\ (v-x)$$

$$+ (1-h)(n-1)(1-q_0)[(1-q_0)F(x) + q_0]^{n-2}\ (-x)$$

$$= \frac{(1-q_0)(n-1)}{(1-q_0)F(x) + q_0}\left\{\frac{q_1[(1-q_0)F(x)+q_0]}{(1-q_0)[q_1F(x)+(1-q_1)]}\cdot h\ [q_1F(x)+(1-q_1)]^{n-1}\ (v-x)\right.$$

$$\left. + (1-h)[(1-q_0)F(x)+q_0]^{n-1}\ (-x)\right\}$$

$$> \frac{(1-q_0)(n-1)}{(1-q_0)F(x) + q_0}\,\Pi_1 \quad > 0$$

$$\frac{\partial\Delta}{\partial\underline{x}} = h(1-q_1)^{n-1} + (1-h)\ q_0^{n-1} > 0$$

From

$$0 = \frac{d\Delta}{ds'} = \frac{\partial\Delta}{\partial h}\ \frac{dh}{ds'} + \frac{\partial\Delta}{\partial F}\ \left(\frac{dF}{ds'}\right) + \frac{\partial\Delta}{\partial\underline{x}}\ \frac{d\underline{x}}{ds'}$$

and $\frac{\partial\Delta}{\partial h} > 0$, $\frac{dh}{ds'} > 0$, $\frac{\partial\Delta}{\partial F} > 0$, $\frac{\partial\Delta}{\partial\underline{x}} > 0$, $\frac{d\underline{x}}{ds'} > 0$, we can conclude that $\dfrac{dF(x;s',s'')}{ds'} < 0 \quad \forall\ x\in[\underline{s},\bar{s}]$.

Similarly,

$$\frac{dF(x;s',s'')}{ds''} < 0 \quad \forall\ x\in[\underline{s},\bar{s}].$$

Thus, $F(\bullet;s',s'')$ is first-order stochastically increasing in s' and s'. Therefore, $\Gamma(s;[s',s''])$ is increasing in both s' and s''. ∎

Recall that $\Gamma(s;[s',s''])$ is the expected revenue to the owner when the bidders' common prior is $I_0\in[s',s'']$, and $I_0=s$. An increase in s' or s'' represents a higher expectation in V, since I_0 and V are affiliated. Thus it is intuitive that $\Gamma(s;[s',s''])$ is increasing in both s' and s''. Together with Lemma 3, $\Gamma(s;[s',s''])$ is increasing in all of its arguments.

Let $I_0\in\Sigma'$ be the common belief of the bidders in the second period when $I_0=s$ is not revealed (with probability $1-\alpha$). $I_0=s$ is known with probability α. Let $S^*=\{s^*: (1-\alpha)\Gamma(s^*;[s^*,\bar{s}])+\alpha\Gamma(s^*;[s^*,s^*]))=E(V/I_0\in[\underline{s},s^*])\} \cup \{\underline{s},\bar{s}\}$ be the collection of s at the intersections of $y_2 = (1-\alpha)\Gamma(s,[s,\bar{s}])+\alpha\Gamma(s;[s,s])$ and $y_1 = E(V/I_0\in[\underline{s},s])$, together with the two extreme points of Σ. The following theorem characterize the equilibria in the two-period auction game:

<u>Theorem 1</u>

In every Nash equilibrium, the seller uses a pure strategy characterized by some $s^* \in S^*$ having the form: sell the asset in the first period if $I_0 \in [\underline{s},s^*]$, otherwise sell in the second period. In each such equilibrium, in the first period the bidders bid $E(V/I_0\in[\underline{s},s^*])$ if $s^* \neq \underline{s}$, and bid 0 if $s^* = \underline{s}$; in the second period the bidders use the symmetric mixed strategy detailed in Lemma 2 if $s^* \neq \bar{s}$, and bid 0 if $s^* = \bar{s}$.

Proof

Suppose that the owner uses the following mixed strategy: sell the asset in the first period if and only if $(I_0,U_0)\in S_1\subset\Sigma\times[0,1]$. The equilibrium price must be $E(V/(I_0,U_0)\in S_1)$ in the first period because of competition and the fact that they all have the same information. Suppose that $(s_1,u_0)\in S_1$; then it must be true that $E(V/(I_0,U_0)\in S_1) \geq \alpha\Gamma(s_1,[s_1,s_1]) + (1-\alpha)\Gamma(s_1;\Sigma')$, otherwise the owner would wait and do better. From Lemmas 3 and 4, the right hand side is increasing in s_1. So $E(V/(I_0,U_0)\in S_1) > \alpha\Gamma(s_1,[s_1,s_1]) + (1-\alpha)\Gamma(s_1;\Sigma')$ $\forall s<s_1$, which means that there are no s_1, s_2, and u_0, u_0', such that $s_1>s_2$, $(s_1,u_0)\in S_1$ but $(s_2,u_0')\notin S_1$. So the owner's strategy must be (essentially) a pure one and have the form: sell the asset in the first period if $I_0 \in [\underline{s}, s^*]$, otherwise sell in the second period. Notice that the owner is indifferent between selling in the first period or second period if $I_0=s^*\in(\underline{s},\overline{s})$, since

$$E(V/I_0\in [\underline{s}, s^*]) = \alpha\Gamma(s^*,[s^*,s^*]) + (1-\alpha)\Gamma(s^*;[s^*,\overline{s}]).$$

He could equally well randomize in this event, but the event has probability zero of occurring. It is easy to check that any $s^* \in \Sigma$ satisfying the above equation can be used to construct such an equilibrium.

We have not yet considered, however, the cases $s^* = \underline{s}$ and $s^* = \overline{s}$. If $s^* = \underline{s}$, then $\Pr\{I_0\in[\underline{s},s^*]\}=0$. Competition in the first period does not necessarily lead the bidders to all bid $E(V/I_0\in[\underline{s},s^*])$. The bidders bidding zero in the first period supports the Nash equilibrium characterized by $s^*=\underline{s}$.

Similarly, $s^*=\overline{s}$, the owner always selling in the first period and all bidders bidding 0 in the second period is another Nash equilibrium. ▮

The intuition for the structure of the seller's strategy is that given any bidder's belief, the owner's revenue in the second period is an increasing function of his private signal. Since bidders pay a single price in the first period, it must generate more revenue to the owner to sell in the second period when he receives a more favorable signal than a less favorable one. There are two Nash equilibria which might not satisfy sequential rationality: the owner never sells in either of the periods; bidders pay zero in that period. We shall discuss this in more detail later in the paper.

In the equilibrium characterized by $s^*\in S^*$, the owner's expected total revenue is

$$\Pi(s^*,\alpha) = E(V/I_0\in [\underline{s},s^*])\, \Pr\{I_0\in [\underline{s},s^*]\} + \int_{s\in[s^*,\overline{s}]} \left\{(1-\alpha)\Gamma(s;[s^*,\overline{s}])+\alpha\Gamma(s;[s,s])\right\} dG_0(s)\ , \tag{1}$$

where $G_0(s)$ is the c.d.f. of I_0.

Since the distribution of I_0 conditional on V is atomless, any interior $s^*\in S^*$ must satisfy the following equation:

$$E(V/I_0\in[\underline{s},s^*]) = (1-\alpha)\Gamma(s^*;[s^*,\overline{s}\ ]) + \alpha\Gamma(s^*;[s^*,s^*]) \tag{2}$$

The left-hand side is increasing in s^*; from Lemmas 3 and 4 the right-hand side is also increasing in s^*.

Suggestive sketches of $y_1 = E(V/I_0\in[\underline{s},s])$ and $y_2 = (1-\alpha)\Gamma(s;[s,\overline{s}]) + \alpha\Gamma(s;[s,s])$ are drawn in (s, y) space in Figure 1. Both curves are increasing. Given that bidders presume that the owner sells in the first period if and only if $I_0\in[\underline{s},s]$ and the owner receives a private signal $I_0=s$, the first curve represents the owner's revenue from selling in the first period, while the second one represents his revenue from selling in the second period.

(Insert Figure 1 about here)

As we argued before, allowing the owner to set a reservation price (publicly or secretly) in the first-period auction neither adds equilibria to nor eliminates equilibria from those characterized by Theorem 1. In the first period, bidders always bid the expected value of V conditional on the event that the owner sells his asset successfully in the first period, taking into account the owner's reservation price. This event can be characterized by an equation of the form $\psi(I_0,U_0)=1$, where the owner sells successfully in the first period if and only if (I_0,U_0) satisfies this equation. Thus, there is no difference in the conditions that characterize the equilibria with or without reservation prices.

In Hendricks, Porter, and Wilson (1990), it is assumed that the reservation prices of the government in the offshore oil lease auctions are random and affiliated with the bids of the informed bidders. Our analysis shows that this may be the result of the strategic considerations of the government to maximize the sale revenue. Since the government can offer the unsold tracks for sale at a later date, the reservation prices in an auction is affiliated with the private information of the government. Even if the government does not have any private information before the auction starts, the bids in the auction are informative and the government can base her estimate on those bids. Thus, from the point of any particular bidder, the reservation prices are seen to be affiliated with his private information. Since bids are usually increasing functions of the private information in equilibrium, the reservation prices must be affiliated with the highest bid in an auction.

There are many Nash equilibria in the game, but some of them are unreasonable. By imposing a criterion of "rational" beliefs along the lines of sequential equilibrium concept in Kreps and Wilson (1982), some of the equilibria may be eliminated. We define a *rational belief* about I_0 of a bidder in either period of the two-period game given the owner's strategy as: i) the Bayesian updated distribution whenever the probability of the event that the owner sells in that period is positive; ii) any distribution on Σ if that probability is zero.

Definition

A Nash equilibrium is called a *sequential equilibrium* in this game if given the owner's strategy, there exist rational beliefs about I_0 for each bidder in both periods, and given these beliefs and other players' strategies, each player is acting optimally.

Notice that sequential equilibrium is a weak refinement of Nash equilibrium for this game. For any Nash equilibrium characterized by an interior $s^* \in S^*$, $\underline{s}<s^*<\overline{s}$, it is easy to see that the conventional Bayesian beliefs are rational. For the Nash equilibria characterized by the two extreme points, however, the probability of selling in one of the periods is zero. Taking the equilibrium characterized by $s^*=\overline{s}$, for example, the most unfavorable rational belief for I_0 is $I_0=\underline{s}$ with probability one. With this belief of the bidders, the owner's revenue from selling in the second period when $I_0=\overline{s}$ is $\Gamma(\overline{s};[\underline{s},\underline{s}])$. So if $\Gamma(\overline{s};[\underline{s},\underline{s}])$ is greater than $E(V)$, which is the selling price in the first period in this case, then this Nash equilibrium cannot be a sequential equilibrium. This is quite intuitive: the expected selling price is at least $\Gamma(\overline{s};[\underline{s},\underline{s}])$ in the second period when $I_0=\overline{s}$ since bidders will believe that I_0 is at least $\underline{s}$ even though the probability of selling in the second period is zero. Identical arguments apply for the case when the Nash equilibrium is characterized by $s^*=\underline{s}$.

Let $\tilde{S}^*$ be S^* but excluding $\{\underline{s}\}$ if $E(V/I_0=\underline{s}) > \alpha\Gamma(\underline{s};[\underline{s},\underline{s}]) + (1-\alpha)\Gamma(\underline{s};[\underline{s},\bar{s}])$, and excluding $\{\bar{s}\}$ if $\alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}]) > E(V)$. We characterize all sequential equilibria in the two-period game in the following proposition:

Proposition

Every sequential equilibrium is characterized as in Theorem 1 by some $s^* \in \tilde{S}^*$.

Proof

It is easy to see that any sequential equilibrium must be characterized by some $s^* \in S^*$. For an interior $s^* \in \tilde{s}^*$, all bidders bidding $E(V/I_0 \leq s^*)$ in the first period and using the mixed-strategy in Lemma 2 with the updated distribution of V conditional on $I_0 \geq s^*$ combined with the conventional Bayesian updated beliefs constitutes a sequential equilibrium.

For $s^* = \bar{s}$, however, the owner sells in the second period with probability zero. Given the bidders' most unfavorable belief about I_0 is $I_0=\underline{s}$ with probability one if the seller sells in the second period, the revenue for the owner when $I_0=\bar{s}$ is $\alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}])$, which must not exceed $E(V)$, the revenue of selling in the first period, in a sequential equilibrium. So $s^* = \bar{s}$ is not a part of any sequential equilibrium if $\alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}]) > E(V)$.

For similar reasons, $s^* = \underline{s}$ is not a part of any sequential equilibrium if $E(V/I_0=\underline{s}) > \alpha\Gamma(\underline{s};[\underline{s},\underline{s}])+(1-\alpha)\Gamma(\underline{s};[\underline{s},\bar{s}])$. ∎

Notice that $E(V/I_0\in[\underline{s},s^*])$ is increasing in s^*, and $(1-\alpha)\Gamma(s;[s^*,\bar{s}]) + \alpha\Gamma(s;[s,s])$ is increasing in s^* and is always less than $E(V/I_0\in[\underline{s},s^*])$ for $s<s^*$. The owner's revenue in the equilibrium characterized by $s^* \in S^*$ must then be increasing in s^*. Therefore, the Nash equilibrium characterized by $s^*=\bar{s}$

generates the most revenue (which is equal to E(V)) to the owner. In any other equilibrium, the owner's expected revenue is less than E(V), the selling price in a single auction. This leads to the conclusion that the flexibility of the owner of being able to sell in more possible auctions is generally not good to him.

An important implication can be deduced from this result. A seller's power of selling is bounded by his ability to commit. If the seller can commit himself to always sell in the first period, then he can obtain the full expected value of the object. Auctions can be used to bound oneself from trading further. Those given rules of auctions are easy to monitor. The sale is so quick that it prevents bidders from obtaining further private information which is harmful to the benefit of the seller.

One may question the existence of equilibrium in the game. It is easy to see that $\tilde{S}^*$ is never empty. This is because $\underline{s} \notin \tilde{S}^*$ if and only if $E(V/I_0=\underline{s}) > \alpha\Gamma(\underline{s};[\underline{s},\underline{s}])+(1-\alpha)\Gamma(\underline{s};[\underline{s},\bar{s}])$. $\bar{s} \notin \tilde{S}^*$ if and only if $\alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}]) > E(V)$. From this, we have $y_1 > y_2$ at $s=\underline{s}$, and $y_1 < y_2$ at $s=\bar{s}$. Because both curves are continuous in s, if $\underline{s} \notin \tilde{S}^*$ and $\bar{s} \notin \tilde{S}^*$, they must intercept at least once (cf. Figure 1).

We define the equilibrium that is characterized by the largest element in $\tilde{S}^*$ as the *optimal equilibrium* in the game. This equilibrium generates the most revenue to the seller among all sequential equilibria. From the continuity of y_1 and y_2 and the compactness of Σ, $\tilde{S}^*$ must have a largest element. Thus, an optimal equilibrium always exists in the game.

There may not be an s* satisfying (2) in general, and hence in some examples the only sequential equilibrium has all trade occurring in either period 1 or 2. This could occur since the private information of the owner might not be explicitly revealed in the second period. One may wonder whether there can ever be an interior equilibrium for some α, especially when

$\alpha=0$. The example below shows that there exists an interior equilibrium in some cases even when $\alpha=0$.

Example

Let $\alpha = 0$, let the support of I_0 be $[0,1]$, let the support of V be $\{0,1\}$, let $f(s/0)=1+\varepsilon-2s\varepsilon$, $\forall s\in[0,1]$ be the density function of I_0 conditional on $V=0$, and let

$$g(s/1) = \begin{cases} \dfrac{1-\varepsilon}{1+\varepsilon} + \dfrac{2s\varepsilon}{(1+\varepsilon)(1-\varepsilon^2)}, & \text{if } 1-\varepsilon^2 \ge s \ge 0; \\[2ex] 1 + \dfrac{2(1-\varepsilon)}{\varepsilon^3}[s - (1-\varepsilon^2)], & \text{if } 1 \ge s > 1-\varepsilon^2, \end{cases}$$

be the (continuous) density function of I_0 conditional on $V=1$, where $0<\varepsilon<1$.

It is easy to check that these are genuine conditional densities satisfying the affiliation assumption plus the following conditions:

(i) $f(0/0)/g(0/1) \to 1$ as $\varepsilon \to 0$;

$f(1/0)/g(1/1) \to 0$ as $\varepsilon \to 0$;

(ii) $\Pr\{V=1/I_0=0\} \to \Pr\{V=1\}$ as $\varepsilon \to 0$;

$\Pr\{V=1/I_0=1\} \to 1$ as $\varepsilon \to 0$.

That is, $I_0 = 0$ provides little information about V, while $I_0 = 1$ almost ensures that $V = 1$ as $\varepsilon \to 0$. Thus, $\Gamma(0,[0,1]) \to \Gamma([0,1];[0,1])$ (i.e., the revenue for the owner when bidders believe that $I_0\in[0,1]$ and it is really so), and $E(V/I_0\in[0,0]) \to E(V)$. Since $E(V) > \Gamma([0,1];[0,1])$, as $\varepsilon \to 0$, $\Gamma(0;[0,1]) < E(V/I_0\in[0,0])$. On the other hand, $\Gamma(1;[1,1]) \to 1$, while $E(V/I_0\in[0,1]) = E(V) = \Pr\{V=1\} < 1$. So as $\varepsilon \to 0$, $\Gamma(1;[1,1]) > E(V/I_0\in[0,1])$.

Since both sides in (2) are continuous, there must be at least one $s^* \in (0,1)$ satisfying (2) for ε small enough.

3. The Effect of Information Revelation

In this section, we consider the effect of a change in the probability of revelation of the seller's private information. Even though this probability does not affect the form of the seller's strategy in equilibrium, it may affect the revenue to the seller from the sale. Similar to the well-established intuition that it is to the seller's benefit to truthfully reveal whatever private information he possesses, this is also true for the optimal equilibrium in our dynamic game:

Theorem 2

The seller's expected revenue is an increasing function of α in the optimal equilibrium.

Proof

If $E(V/I_0 \in [\underline{s},\bar{s}]) \geq \alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}])$, then the optimal equilibrium is characterized by $s^*=\bar{s}$; that is, the seller always sells in the first period.

If $E(V/I_0 \in [\underline{s},\bar{s}]) < \alpha\Gamma(\bar{s};[\bar{s},\bar{s}]) + (1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}])$, however, the optimal equilibrium is characterized by the s^* at the last interception between y_1 and y_2. In this case, $y_2 > y_1$ at $s=\bar{s}$. Thus, at $s=s^*$, y_1 must intercept y_2 from above; that is,

$$\frac{\partial y_1(s^*)}{\partial s} \leq \frac{\partial y_2(s^*)}{\partial s} .$$

But $\dfrac{\partial y_1}{\partial \alpha} = 0$, $\dfrac{\partial y_2}{\partial \alpha} = \Gamma(s;[s,s]) - \Gamma(s;[s,\bar{s}]) < 0$. We have

$$0 = \frac{d(y_1-y_2)}{d\alpha} = \left(\frac{\partial y_1(s^*)}{\partial s} - \frac{\partial y_2(s^*)}{\partial s} \right) \frac{ds^*(\alpha)}{d\alpha} - \frac{\partial y_2}{\partial \alpha}$$

for $s^*=s^*(\alpha)$ which is the last interception of y_1 and y_2 for α. Thus,

$ds^*(\alpha)/ds$ must be positive.

The seller's revenue is given by (1). Notice that $\forall\ s<s^*$,

$$(1-\alpha)\Gamma(s;[s^*,\bar{s}])+\alpha\Gamma(s;[s,s]) < (1-\alpha)\Gamma(s^*;[s^*,\bar{s}])+\alpha\Gamma(s^*;[s^*,s^*])=E(V/I_0\in[\underline{s},s^*])$$

Moreover,
$$\int_{s\in[s^*,\bar{s}]} \Gamma(s;[s^*,\bar{s}]\ dG_0(s) \le \int_{s\in[s^*,\bar{s}]} \Gamma(s;[s,s])\ dG_0(s)\ .$$

This is because the left hand side is the revenue to the seller given $s\in[s^*,\bar{s}]$, and the right hand side is the same revenue except that I_0 is revealed each time. Since I_0 is affiliated with V, from Theorem 17 in Milgrom and Weber (1982), we conclude that the inequality holds. Together with $ds^*(\alpha)/d\alpha > 0$, we conclude that $\Pi(s^*(\alpha),\alpha)$ is increasing in α.∎

Notice that an optimal equilibrium is characterized by s^* which is either $\bar{s}$ or the last interception of y_1 and y_2. As α increases, y_2, the revenue for the s at margin, is decreased. This is because $\Gamma(s;[s,s])<\Gamma(s;[s,\bar{s}])$. Since y_1 must cross y_2 from above at s^*, the interception moves towards a larger s^*. That is, there will be more occasions that the seller sells in the first period, which is good for the seller. The result is consistent with the well-establish result in auction theory: truthful revelation of seller's information increases the expected selling price (c.f. Milgrom and Weber (1982)). In this model α is the probability of truthful revelation of seller's information. An increase in α lowers the payoff of selling in the second period at the margin, and thus increases the probability of selling in the first period. Since bidders receive no private information in the first period, the seller receives the expected value of the object conditional on successful sale in the first period.

Recall that the optimal equilibrium may not be the only equilibrium in our game. In other equilibria, the direction of the change in the selling revenue is not obvious. Circumstance may occur that an increase in the probability of releasing information in the second period hurts the seller, as is explained by the following example: Consider the case when y_1 intercepts y_2 at $s=\bar{s}$. $s^*=\bar{s}$ is an equilibrium. As α increases, y_2 decreases and intercepts y_1 at $s' < \bar{s}$. Since the seller receives full expected value of the object from the equilibrium characterized by $s^*=\bar{s}$, he is worse off at the equilibrium characterized by $s^*=s'$. Thus, an increase in α does not always increase the seller's revenue.

4. Concluding Remarks

As we have shown in the above sections, all equilibria have similar forms despite the different probability of the revelation of the owner's information: $\exists s^*\in[\underline{s},\bar{s}]$, the owner sells early if and only if $I_0 \leq s^*$. In a sequential equilibrium, the expected revenue to the owner is less when he has the option of delaying the sale after receiving his private information, except for the equilibrium characterized by always selling in the first period. Any Nash equilibrium that is not sequential is not a suitable equilibrium in our model, because given that bidders have rational beliefs, there are circumstances that it is profitable for the seller to deviate ex post.

Given that the owner will sell in the first period only if he receives unfavorable signal, potential buyers will not pay much for the asset in the first period. When the seller delays his sale, he loses his flexibility and has to sell the asset no matter what the price is. As bidders receive private information in this period, they are able to make a profit out of their private information, and the owner suffers a loss in revenue.

It is difficult to tell how the probability of information revelation affects the set of sequential equilibria. We do know, however, that if the owner's information is always publicly revealed in the second period, the equilibrium characterized by $s^*=\underline{s}$ is never a sequential equilibrium, whereas if the information is never revealed it can be. Similarly, the equilibrium characterized by $s^*=\bar{s}$ is a sequential equilibrium in fewer circumstances for a larger α, since $\Gamma(\bar{s};[\bar{s},\bar{s}]) > \Gamma(\bar{s};[\underline{s},\underline{s}])$. So, $\alpha\Gamma(\bar{s};[\bar{s},\bar{s}]+(1-\alpha)\Gamma(\bar{s};[\underline{s},\underline{s}]) \le E(V)$ is violated in more circumstances for larger α.

The analysis in previous sections can be extended to models in which the asset is not completely common-valued. As long as there is a common part in the valuation (say, for example, the value to a trader consists of a common-value term plus an independent private-value term), the private signals of the traders are affiliated. One can expect that the equilibrium strategies will be similar: the owner sells early when the owner's signal about the common term is unfavorable, otherwise he sells late.

This model might also be extended to the case when the owner can delay the sale for many periods and in each period, bidders receive some private information. As long as bidders get more private information as they wait, the equilibrium strategies of the owner in the multi-period game are conjectured to be characterized by a reservation price in each period; these reservation prices are increasing in the seller's private signal. Thus, they must also be affiliated with each bidder's private information. This provides a justification for the random-reservation-price assumption in Hendricks, Porter, and Wilson (1990).

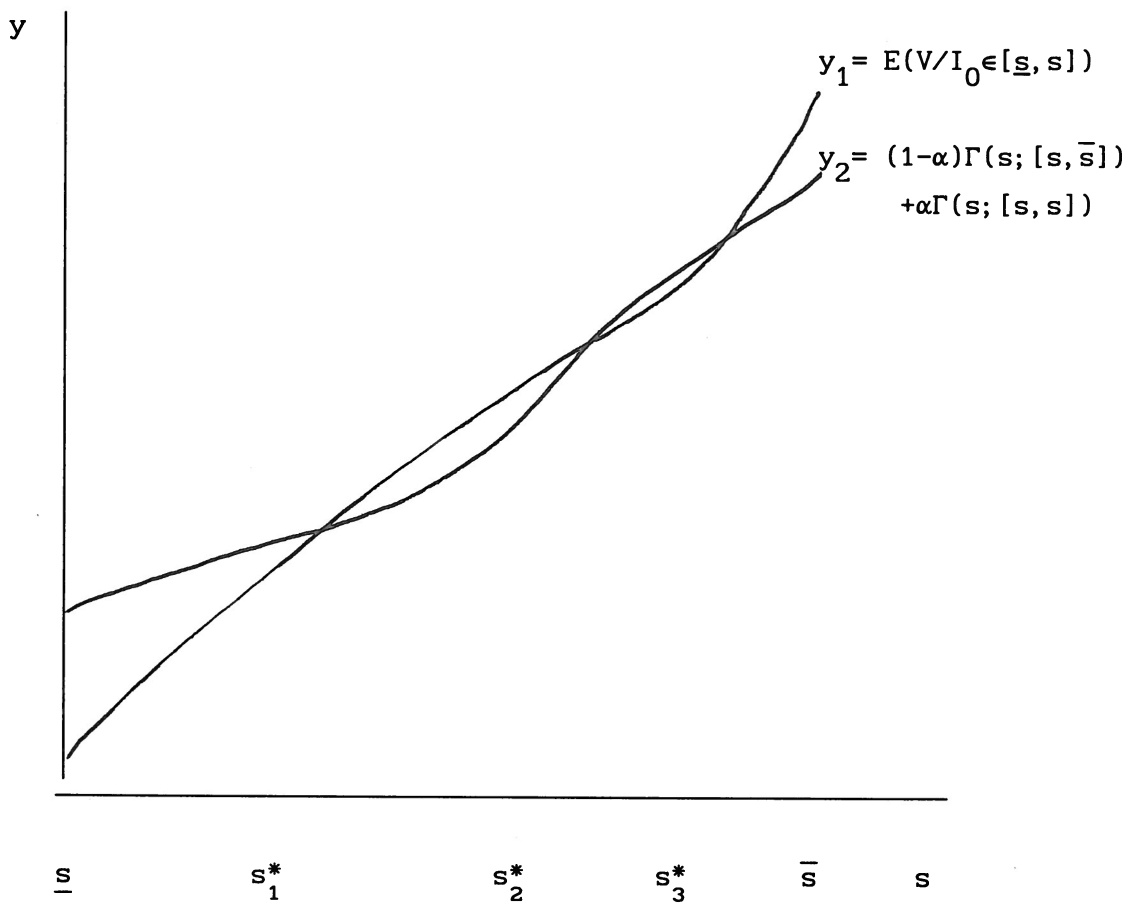

Figure 1 The Owner's Revenues in the First and the Second Period at the Marginal Signal

Endnotes

[1] See Milgrom (1989) or McAfee and McMillan (1987) for an excellent survey.

[2] Milgrom (1990) has found evidence that sellers may buy back the objects being sold themselves, but they do it quietly. This should be regarded as evidence to support, but not to oppose, this important feature of auctions.

[3] The results will not change if we change the selling mechanism to other types of auctions.

[4] This assumption makes the analysis simpler. The qualitative results of the model will be preserved, if we assume that bidders possess some private information on some aspects of the object but not others, on which they will get more information later.

[5] Refer to Wang (1990) or Shiryayev (1984) for a definition of conditional probability when $Pr\{Y=y\}$ or $Pr\{Y=y'\}$ is zero.

[6] Maskin and Riley (1985) have constructed the equilibrium strategy for a private-value auction game that is similar.

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