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Asset Pricing Puzzles and Incomplete Markets

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Comments Welcome

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ABSTRACT

An endowment economy with heterogeneous agents and incomplete asset markets is specified, parameterized and solved using a numerical solution algorithm. The model features two types of infinitely lived agents who are endowed with different sources of non-tradable income. Despite not being able to insure against endowment risk, individuals are able to *partially* diversify away idiosyncratic risk by trading in a limited set of competitive asset markets. Numerical results indicate that the model can account for substantially more of the variability in intertemporal marginal rates of substitution documented by Hansen and Jagannathan (1990) than can models based on a representative agent. In addition, the model can generate a mean risk-free rate of interest smaller than the rate of time preference and potentially account for the so called ‘risk-free rate puzzle’.

A large literature in financial economics has documented the inability of the representative agent, complete markets theory of portfolio choice (Breeden (1979), Lucas (1978), Merton (1973)) to account for various simple properties of the joint behaviour of aggregate consumption and asset returns. Mehra and Prescott (1985) show that, given the stochastic properties of U.S. consumption, the theory drastically underpredicts the average excess rate of return on stocks — the so-called 'equity premium puzzle'. Weil (1989) points out, as did Mehra and Prescott, that the magnitude of the average risk-free rate is far below that predicted by the theory: this has been dubbed the 'risk-free rate puzzle'. Numerous authors, including Grossman, Melino and Shiller (1987) and Backus, Gregory and Zin (1989), have shown that the theory cannot account for term premia inherent in the yield curve. Finally, there exists a vast literature that documents the theory's shortcomings in accounting for deviations from 'unbiasedness' in foreign exchange markets. Hodrick (1987) provides a comprehensive survey of theoretical and empirical applications of asset pricing theory to foreign exchange markets.

All of these empirical anomalies are associated with the same class of consumption-based asset pricing models. However, they also involve asset markets that may be very dissimilar. Hansen and Jagannathan (1990) and Shiller (1982) provide a method of summarizing information on asset returns from various different markets, thereby generating an 'asset pricing puzzle' that in a certain sense encompasses those mentioned above. These authors develop tools which one can use to infer statistical properties of an investor's *intertemporal marginal rate of substitution* (IMRS) using data on returns alone. They show, for example, that should two assets exhibit large differences in expected return, the implication for a broad class of asset pricing models is that the IMRS must be highly variable. Consequently, an alternative view of the equity premium puzzle is that of a 'variability puzzle'. The parametric family of models examined by Mehra and Prescott implies an IMRS for the representative agent that is much less variable than that implied by the large excess return on stocks observed in their data-set. Papers by Breen, Glosten and Jagannathan (1989), Hansen and Jagannathan (1990) and Heaton (1990) document evidence in this regard. Backus, Gregory and Telmer (1990) and Bekaert and Hodrick (1990) reach similar conclusions using foreign exchange market data. This line of research shows that a useful way to characterize the shortcomings of the representative agent model is in terms of its inability to generate sufficient variability in the IMRS.

One branch of subsequent research has explored the possibility that relaxing auxiliary assumptions, within the representative agent framework, can help bridge the gap between theory and data. Papers by Cecchetti, Lam and Mark (1990), Kandel and Stambaugh (1989, 1990) and Rietz (1988) examine alternative assumptions about the driving processes for aggregate consumption. Labadie (1989) investigates the effects of stochastic inflation on asset prices. Rouwenhorst (1989) takes into account production decisions and examines asset prices in a version of the neo-classical growth model. Benninga and Protopapadakis (1990) and Kocherlakota (1990) allow for firm leverage and unconventional values of the rate of time preference. Finally, the effects of alternative preference structures are examined by a number of authors, including Abel (1990), Burnside (1990), Constantinides (1990), Dunn and Singleton (1986), Epstein and Zin (1989), Heaton (1990), Nason (1988), and Weil (1989). While this line of research has offered valuable insights, such as the role played by habit formation in generating variability in the IMRS, many issues remain unresolved. From the point of view of a theory based on a representative agent, the stochastic properties of aggregate U.S. consumption appear to be inconsistent with those of asset returns.

This paper abandons the representative agent, complete markets paradigm and investigates the extent to which deviations from frictionless Arrow-Debreu markets can help to account for the above asset pricing anomalies. The Lucas (1978) model is generalized in that agents are endowed with idiosyncratic risk, in the form of different stochastic endowment processes, that cannot be perfectly hedged due to incomplete asset markets. Aggregation, or the construction of a representative agent, is invalid (Brennan and Kraus (1976), Milne (1979), Wilson (1968)), and the IMRS relevant for intertemporal asset pricing is based on disaggregate consumption data. The underlying idea is that, in equilibrium, individual risk and consumption variability may be larger than measures based on the per-capita aggregate if agents are not able to perfectly share their idiosyncratic income risk. This leads to one potential interpretation of the asset pricing anomalies. Asset prices in the model with imperfect risk sharing arrangements will reflect the fact that a certain amount of idiosyncratic risk is priced in equilibrium, while quantities based on aggregate data, such as the IMRS of a (misspecified) representative agent, will indicate that only systematic risk is priced. One might therefore expect moments such as the equity premium and the variability of the IMRS to be understated by a model based on aggregate data and perfect risk sharing. This paper investigates the magnitude of these effects in the context of a general equilibrium model with incomplete asset markets.

There are some other empirical observations which suggest that generalizing the perfect risk sharing model of Wilson (1968) may be useful from the point of view of intertemporal asset pricing theory. Eun and Resnick (1988) and Solnik (1974), for example, have shown that the potential gains from international portfolio diversification can be substantial. One interpretation of their result is that risk sharing across countries is non-optimal. Brennan and Solnik (1989) demonstrate that, in the context of a consumption-based asset pricing model, the economic significance of imperfect international risk sharing can be considerable. In the macro-economic literature on consumption dynamics a number of papers, including Flavin (1981), Hall and Mishkin (1982) and Zeldes (1989a,b), document evidence suggesting that consumers face binding liquidity constraints, thereby rendering them unable to pool risk associated with future income streams. Finally, in a recent study using panel data from the PSID, Mankiw and Zeldes (1990) find that a large proportion (75 percent) of households hold very little of their wealth in the form of stocks, and that the consumption of stockholders is substantially more volatile and more highly correlated with stock returns than the consumption of non-stockholders. A theory of asset prices based on optimal risk sharing is unlikely to be able to account for such evidence. Furthermore, given the important theoretical role of co-movements between consumption and returns, Mankiw and Zeldes's results suggest that aggregate data may be highly inappropriate for tests of intertemporal asset pricing theory.

Specific features of the model are as follows. There are two types of infinitely lived agents who are distinguished by different, exogenously determined endowment processes. Direct trading in claims to future endowment income, or labour income, is prohibited. One interpretation of this restriction is that, due to informational asymmetries, incentive compatible contracts written on individual specific outcomes do not exist. This aspect of the economy is not explicitly modeled. Individuals are able to trade in competitive asset markets in an attempt to *partially* diversify away idiosyncratic risk. The asset market structure consists solely of a riskless discount bond. A competitive equilibrium is characterized by stochastic processes for the distribution of wealth and the risk-free rate of interest. As closed form solutions are not available, a computational algorithm based on dynamic programming methods is developed. A series of quantitative experiments are conducted by calibrating the model to monthly U.S. per-capita consumption data and varying the amount of heterogeneity inherent in individual income processes.

Related literature on idiosyncratic risk and asset prices includes the following papers. Mankiw (1986), which helped to motivate this paper, uses a static model and finds that incomplete markets can potentially account for the equity premium puzzle. Aiyagari and Gertler (1990) and Hugget (1990) use dynamic models with a continuum of agents and no aggregate uncertainty. Kahn (1990) and Weil (1990) study overlapping generations economies. Finally, Lucas (1990) and Marcet and Singleton (1990), in independent work, use a two-agent economy similar to that in this paper to examine the equity premium puzzle and several other issues.

The remainder of the paper is organized as follows. Section 2 contains a derivation of a simple version of the Hansen-Jagannathan variance bounds technique. Empirical evidence on the implied statistical properties of the IMRS is documented in order to provide a metric with which to evaluate the model. The incomplete markets asset pricing model is outlined in Section 3. Since closed form solutions are not available, the model is solved by using a numerical solution technique. The algorithm, which is closely related to several other well known approaches, is briefly described in Section 4 and then in more detail in an Appendix. Section 5 presents numerical results from a sequence of heterogeneous agent economies and Section 6 conducts a brief sensitivity analysis. A summary of the results, some conclusions regarding the general approach and suggestions for future research are offered in Section 7.

2. Hansen-Jagannathan Bounds

This section makes use of tools developed by Hansen and Jagannathan (1990) and Shiller (1982) in order to develop a simple diagnostic that can be used to evaluate a broad class of asset pricing models. The procedure has the added advantage of being able to summarize the implications of data on returns from many different asset markets in a parsimonious manner. Empirical evidence from a number of different sources is reviewed in order to provide a metric with which the model of Section 3 can be evaluated.

The Hansen-Jagannathan procedure begins by making use of a tenet of financial economics: that securities markets do not admit any riskless arbitrage opportunities. The consequence (Harrison and Kreps (1979), Huang and Litzenberger (1988)) is that there exists a positive random variable, say m_t, that satisfies the following conditional moment restriction:

$$p_t = E_t(m_{t+1}X_{t+1}). \qquad [1]$$

In equation [1] p_t represents a vector of asset prices, p_t^i, and X_t represents a vector of payoffs, x_t^i, on a set of securities indexed by i. The conditional expectations operator, E_t, is defined in terms of an information set, $\mathcal{F}_t$, that represents the intersection of all individual information sets. Random variables subscripted with a t are assumed to be adapted to the information structure, $F \equiv \{\mathcal{F}_t; t=0,1,2,\ldots\}$. The variable m_t will be referred to as the *intertemporal marginal rate of substitution* (IMRS).

The Hansen-Jagannathan technique consists of using the moment condition [1] and data on asset returns to infer an unconditional mean-variance frontier for m_t. What follows is a simplified version of their procedure. Writing [1] in terms of returns, it is easily shown that, for any two assets, the following relation must hold:

$$E_t(m_{t+1}[R^i_{t+1} - R^j_{t+1}]) = 0 \qquad [2]$$

where $R^i_{t+1} \equiv x^i_t/p^i_t$ is the gross rate of return on asset i between time t and t+1. Define r_{t+1} as the return differential, $r_{t+1} \equiv R^i_{t+1} - R^j_{t+1}$. Since [2] holds in conditional mean, it holds unconditionally as well. The definition of covariance implies the following:

$$E(m)\ E(r) + Cov(r,m) = 0\ , \qquad [3]$$

where the absence of subscripts denotes an unconditional moment. Since m_t is a positive random variable, and correlations are less than unity in absolute value, [3] implies the inequality:

$$\sigma_m/E(m) \geq |E(r)|/\sigma_r\ , \qquad [4]$$

where σ denotes standard deviation. The expression [4] implies that the absolute value of the 'Sharpe ratio', $|E(r)|/\sigma_r$, provides a lower bound on the ratio of σ_m to its mean.

The inequality restriction [4] applies to any random variable that satisfies the no-arbitrage condition [1]. The essential link between [1] and the consumption based asset pricing model is that, in general, the latter identifies the random variable m_t with the ratio of an investors' marginal utilities of consumption at two points in time: her IMRS. The one exception to this statement involves constraints on asset holdings, or corner solutions. The theoretical random variable corresponding to m_t in this case is a non-standard quantity involving more than one individual. This is made clear in Section 5. The important point for now is that equation [1] and the resulting inequality [4] are applicable in a wide class of economies including those with heterogeneous agents and/or incomplete markets. There will always exist at least one random variable that satisfies the moment restriction [1]. Following the existing literature, this variable will be referred to as an IMRS. Should it fail to satisfy [4], given the Sharpe ratio from some portfolio, the model can be deemed inconsistent with the data in these dimensions.

This framework suggests a reinterpretation of Mehra and Prescott's equity premium puzzle. Equilibrium in their model is always associated with an interior solution to the representative agent's maximization problem. The random variable m_t from [1] is therefore unambiguously identified with the representative agent's IMRS. In addition, note that the mean IMRS is roughly equal to unity (since it is approximately equal to the inverse of 1 plus the mean risk-free rate of interest). The Sharpe ratio in [4] therefore provides a lower bound on the standard deviation of the IMRS alone. Suppose that the rate of return differential, r_t, is interpreted as the excess return on equity. Equation [4] indicates that large equity premiums imply large standard deviations for the IMRS. Herein lies the reinterpretation of the equity premium puzzle. The parametric family of models examined by Mehra and

Prescott imply an IMRS that is not nearly variable enough to be compatible with the data. This observation is documented by Hansen and Jagannathan (1990) and Heaton (1990).

The existence of predictable components in asset returns also presents a challenge to theories of asset pricing. To see this, consider the return, r_{t+1}, on a managed portfolio that is generated by a zero net investment trading strategy. Such a portfolio is constructed by going short in one asset in order to finance a long position in another. Since the price of this asset is zero, the asset pricing model implies that:

$$E(r\ m) = 0 \qquad [5]$$

The same steps as shown above imply that the absolute value of the Sharpe ratio for the balanced portfolio, $|E(r)|/\sigma_r$, constitutes a lower bound on σ_m. Now suppose that returns are predictable to a certain extent, and that the trading strategy alluded to above uses this information. Examples of simple trading strategies based on linear projections are documented below. Loosely speaking, one would expect the return generated by the trading strategy, and therefore the portfolio's Sharpe ratio, to be increasing in the amount of predictability. Highly variable IMRS's are therefore implied by predictable components in asset returns. Lower bounds obtained in this manner make use of more information than those based on simple return differentials, and are in general more restrictive (see Bekaert and Hodrick (1990) in particular).

A number of lower bounds on the variability of the IMRS, based on data from a number of different asset markets, are now reported. Breen, Glosten and Jagannathan (1989) use a trading strategy based on a continually updated projection of excess stock index returns onto t-bill returns. Their managed portfolio consists solely of the stock index fund whenever the fitted value from the projection is positive, and t-bills whenever the fitted value is negative. Sharpe ratios of 0.140 and 0.130 are reported for the NYSE value and equally weighted stock indexes, respectively (based on Table II, monthly data, 70:8-86:12). Backus, Gregory and Telmer (1990) use a similar trading strategy to construct a portfolio based on forward and spot market data for various currencies vis-a-vis the U.S. dollar (monthly data, 74:7-86:10). They report Sharpe ratios that range from 0.094 for the deutsche mark, to 0.358 for the yen.

Bekaert and Hodrick (1990), Hansen and Jagannathan (1990) and Heaton (1990) use a generalization of the above procedure to derive bounds based on large numbers of assets. Hansen and Jagannathan (1990) use monthly data on U.S. t-bills and the value-weighted NYSE index to construct a number of payoff series that correspond to X_t in [1] (see their Figure 5). They infer a lower bound of approximately 0.150. Heaton (1990) documents similar results using somewhat different techniques and the same data set. These authors characterize the entire admissible region for the IMRS in mean-standard deviation space. The value of 0.150 is conditional on a mean IMRS of slightly less than unity. Finally, Bekaert and Hodrick (1990) characterize the lower bound using two approaches. The first is based on a linear projection and uses a wide array of monthly data from international equity, bond and currency markets. They find lower bounds that vary from 0.100 to 0.300. Their second approach makes use of more conditioning information by scaling asset returns with elements of the assumed information set. They obtain substantially more restrictive lower bounds, the largest being 0.635. Bekaert and Hodrick's work illustrates the extent to which predictable components in asset prices can be made use of to infer strong restrictions on the IMRS.

These lower bounds are all estimates and are therefore subject to sampling variability. Hansen and Jagannathan (1990) and Bekaert and Hodrick (1990) report standard errors computed using the Newey-West (1987) estimator. Bounds that are calculated using a minimal amount of information on predictable components appear to be estimated imprecisely. This is not generally the case for tighter bounds that are obtained by using more conditioning information.

To summarize, using data on monthly asset returns, the Hansen-Jagannathan procedure allows one to infer that a reasonable model of monthly consumption and asset returns should feature an IMRS with a standard deviation of at least 0.150 and a mean between 0.995 and unity. The range for the mean implies an average risk-free return between 0 and 6 percent, at annual rates. This metric can now be used to highlight the shortcomings of the representative agent model. Figure 1 plots population mean-standard deviation pairs for the IMRS from a monthly calibration of the Mehra and Prescott (1985) representative agent economy. The discount factor is set to 0.9983, while the value of the risk aversion parameter, α, is varied from 1 to 10. The graph highlights two 'puzzles' associated with additively time separable preferences. The 'variability puzzle' refers to the fact that the largest standard deviation of the theoretical IMRS is a factor of 3 less than the lower bound of 0.150 (the α=10 case). The 'risk-free rate puzzle' refers to the fact that increased amounts of variability in the IMRS can only be achieved at the expense of unrealistically high mean risk-free rates. In the most extreme case, α=10, the expected risk-free rate is roughly 16 percent (annualized). The corresponding sample moment from the Mehra-Prescott data-set is 0.80 percent. For more reasonable attitudes towards risk, such as

$\alpha=2$ (which is used in this paper), the expected risk-free rate is more realistic, at roughly 5 percent, but the standard deviation of the IMRS is a factor of 10 less than the lower bound. A challenge for the incomplete markets model is to account for IMRS variability while at the same time generating a reasonable mean risk-free rate.

3. Asset Pricing Model

Consider an economy in which many rational investors trade in competitive securities markets and solve optimal portfolio allocation problems in order to determine their asset holdings and consumption sequences. It is assumed that individuals, indexed by k, derive utility from the consumption of a single good, in terms of which asset prices and payoffs are denominated. Denote $\tilde{m}_{k,t}$ as investor k's IMRS: the ratio of her time t marginal utility of consumption to that obtained at time t-1. As noted in Section 2, the following conditional moment restriction relates $\tilde{m}_{k,t+1}$ to asset prices and payoffs:

$$p_t = E_t(\tilde{m}_{k,t+1} X_{t+1}) \ . \qquad [5]$$

This relation is only valid insofar as individuals are at interior solutions to their portfolio allocation problems. In this case the variable $\tilde{m}_{k,t+1}$ is unambiguously identified with m_{t+1} in [1]. Equilibria featuring corner solutions, in which the usual Kuhn-Tucker conditions replace [5], are discussed in Section 5.

The asset pricing relation [5] is consistent with a wide class of models, the two of particular interest being the Mehra-Prescott representative agent model and the incomplete markets model derived below. In order to see the relationship between the two, consider the following additional structure to the model economy. Suppose there are two types of agents, k=1,2, and that agents of each type are identical in all respects. It is valid to construct two different representative agents, again indexed by k=1,2, and deal strictly in terms of this two agent economy. Next, assume that these two agents are each endowed with a claim to an exogenously determined income stream, hereafter called labour income, and that incomes are not perfectly correlated across individuals. For reasons outlined further below, claims to individual labour income are assumed to be non-tradable. Finally, assume that preference orderings over the single good are the same across agents 1 and 2: the determinant of agent type is solely related to endowments. At any point in time the following resource constraint must hold:

$$y_{1,t} + y_{2,t} = c_{1,t} + c_{2,t} \, , \qquad [6]$$

where $c_{k,t}$ and $y_{k,t}$ are, respectively, individual k's period t consumption and labour income. Assuming, for now, that agents are not at corner solutions, equations [5] and [6], along with the requirements that all assets are held and that both agents obey their intertemporal budget constraints, define a competitive equilibrium.

If we further assume that agents are able to trade in a complete set of Arrow-Debreu markets for contingent claims, then the Mehra-Prescott model results. As noted, the only characteristic that distinguishes agents 1 and 2 is their non-tradable labour income endowments. However, since markets are complete, individual IMRS's will be equated across states of the world, and no idiosyncratic risk will be priced in equilibrium. A unique equilibrium consumption sequence is easily identified in which each agent gets half of the aggregate endowment in every state. Supporting asset prices are then obtainable using [5]. The Mehra-Prescott exercise consists of placing further structure on preferences and the endowment process such that the pricing functional implicitly defined by [5] can be solved and exact numerical solutions can be obtained.

The remainder of this paper will be concerned with an environment in which a complete set of contingent claims is not available. The resulting problem becomes somewhat more complex. Since agent 1 and agent 2's IMRS need not be equated in equilibrium, standard welfare theorems from micro-economics cannot be relied upon to provide a tractable two-stage solution strategy. That is, equilibrium consumption sequences and asset prices must be solved for *simultaneously*: the invalidity of the welfare theorems renders centralized methods for finding equilibrium consumption allocations inapplicable. Many other properties of the Arrow-Debreu model, such as efficiency, also fail to hold in the incomplete markets environment. See Duffie (1990) and Geanakoplos (1990) for a survey of these and other important aspects of general equilibrium with incomplete markets.

Additional structure on preferences, technology and asset markets is now required in order to make [5] operational. For both expository and tractability reasons, an attempt is made to keep the model economy similar in spirit to that of Mehra and Prescott. The process that is observable at the aggregate level is outlined first, followed by a description of how the idiosyncratic component of individual labour income evolves.

3A. Aggregate and Individual Labour Income

The specification of the endowment processes for agents 1 and 2 follows in the spirit of Mankiw (1986). Aggregate labour income evolves according to some exogenous stochastic process while individual income is generated by a stochastic sharing rule. The idea is to build a model that accounts for both aggregate and disaggregate data. In doing so, links are maintained with the more easily measured aggregate time series while at the same time explaining why per-capita consumption is inappropriate for asset pricing.

Aggregate labour income, y_t, is assumed to grow at rate λ_t, where the stochastic process $\{\lambda_t\}$ is governed by a first order Markov chain with transition probabilities π_{ij}. Therefore, $y_{t+1}=\lambda_{t+1}y_t$. The random variable λ_t is restricted to take on one of two values, λ_1 or λ_2, which define two states of the world for aggregate labour income, high and low growth, respectively.

The characteristic that defines agents 1 and 2 is that in the low growth state of the world they are endowed with different proportions of the total amount of available labour income. Conditional on the growth rate of aggregate income being λ_2, agent 1 receives a fraction γ, $0<\gamma\leq 1/2$, of y_t with probability δ. With probability $(1-\delta)$ she receives $(1-\gamma)y_t$. Agent 2 receives the remainder in both of these idiosyncratic states of the world. When aggregate income growth is high (λ_1) both agents receive $1/2(y_t)$. It will be convenient to define a random variable, $Q_{1,t}$, as follows:

$$
\begin{aligned}
Q_{1,t} &= 1/2 && \text{with probability } \pi_{i1} \\
&= \gamma && \text{with probability } \pi_{i2}\delta \\
&= (1-\gamma) && \text{with probability } \pi_{i2}(1-\delta)
\end{aligned}
$$

Defining $Q_{2,t}$ as $(1-Q_{1,t})$, it is valid to write $y_{k,t} = Q_{k,t}y_t$. $Q_{k,t}$, the stochastic fraction of aggregate income that individual k receives, is referred to as the idiosyncratic shock process.

One interpretation of the individual endowment process is that in the low growth state agents receive an iid productivity shock. This shock renders them either more or less productive, relative to agents of the opposite type. The idea, due to Mankiw (1986), is that aggregate shocks to the economy may be distributed unevenly across the population.

The parameters γ and δ play an important role in the subsequent analysis. γ determines the degree of heterogeneity in the model. Should $\gamma = 1/2$ the model collapses to that of Mehra and Prescott (1985). As the absolute value of $(1/2-\gamma)$ grows, individuals become more heterogeneous in that the amount of idiosyncratic labour income risk grows. The parameter δ determines the amount of '*ex-ante* heterogeneity' present in the economy. Should $\delta=1/2$ individuals have the same expected labour incomes in the future. Note that only in the special case of a uniform distribution of wealth are agents truly the same in an *ex-ante* sense. This will be the case at t=0, by assumption, but generally not for t>0. δ also represents the extent to which one agent is wealthy relative to the other. For instance should δ be greater than 1/2, agent 2 would have a greater expected future labour income than agent 1.

3B. Asset Markets

A crucial assumption is that agents cannot trade claims to their individual specific labour income. This requirement, alongside a suitably restricted asset market structure, is sufficient to imply that markets are *dynamically incomplete* (see Huang and Litzenberger (1988) or Duffie and Huang (1985)). An obvious interpretation of such a restriction is that moral hazard type problems result in an inability to write incentive compatible contracts based upon idiosyncratic outcomes.

Imperfect insurance is available through trade in asset markets, even though perfect insurance for all states of the world is not. The only way in which agents are allowed to trade intertemporally is with a riskless bond that pays one unit of the consumption good with certainty one period hence. A more complex asset market structure is easily incorporated, including equity for instance, but the riskless bond should be sufficient for the question at hand: to what extent are agents able to diversify idiosyncratic risk in the presence of incomplete markets? Denote $b_{k,t}$ as the number of bonds held by individual k between time t and (t+1), and p_t as the time t price of one discount bond.

Because of the simplicity of the model, the formulation has been made intentionally non-technical. However a heuristic word on the underlying mathematics will be useful at this point. The economy can be thought of as being endowed with two information structures, an increasing set of σ-algebras, $F \equiv \{\mathcal{F}_t: \mathcal{F}_t \subseteq \mathcal{F}_{t+1}\}$, that correspond to the evolution of <u>both</u> the idiosyncratic and aggregate shocks, and another set, $G = \{\mathcal{G}_t: \mathcal{G}_t \subseteq \mathcal{G}_{t+1}\}$, that corresponds only to the evolution of the aggregate process. Note that $\mathcal{G}_t \subset \mathcal{F}_t$. Note also that $Q_{k,t}$ is $\mathcal{F}_t$ measurable but is not $\mathcal{G}_t$ measurable. It is understood that the probability measure associated with the operator $E_t(\cdot)$ is associated with the set $\mathcal{F}_t$. The assumption that drives the results is that while asset prices are $\mathcal{F}_t$ measurable, asset payouts are restricted to be $\mathcal{G}_t$ measurable. The requirement that contracts contingent upon the idiosyncratic processes cannot be written seems natural, given some sort of moral hazard motive. As long as this restriction is maintained, asset markets will be dynamically incomplete, regardless of the number of securities traded.

3C. Competitive Equilibrium

Individual k's intertemporal budget constraint can be written as follows:

$$y_{k,t} + b_{k,t-1} \geq c_{k,t} + p_t b_{k,t} \,. \qquad [7]$$

In order to formulate the model in terms of stationary processes, [7] is normalized by aggregate income, y_t.

$$Q_{k,t} + b^*_{k,t-1}/\lambda_t \geq c^*_{k,t} + p_t b^*_{k,t} \qquad [8]$$

Variables with an asterisk are expressed in terms of aggregate income. For example, $b^*_{k,t}$ is individual k's period t purchases of discount bonds, expressed as a fraction of y_t. For notational simplicity, the asterisks will be omitted hereafter.

Individual optimization problems are standard. Agent k chooses a consumption sequence $\{c_{k,t}\}$ supported by an admissible trading strategy, $\{b_{k,t}\}$, to maximize the following additively time separable expected utility function:

$$U(c_k) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_{k,t}) \ .$$

It is assumed that both agents begin their lives with no bond holdings. Finally, $u(\cdot)$ is assumed to take the form: $u(c) = c^{1-\alpha}/1-\alpha$ for both agents. The IMRS inherent in [5] is therefore equal $\tilde{m}_{k,t+1} \equiv (c_{k,t+1}/c_{k,t})^{-\alpha}$.

A competitive equilibrium with incomplete markets is defined as a sequence of prices $\{p_t\}$ and feasible trading strategies $\{b_{k,t}\}$, k=1,2, that result in the following conditions holding. Both agents agree on all aspects of the stochastic process describing prices. Each agent attains a maximum in terms of lifetime expected utility, taking prices as given. This implies that [5] holds for each time, t. In addition, markets for goods and assets must clear at all times. This condition is equivalent (by Walras Law) to requiring that the bond market clears, or that $b_{1,t} + b_{2,t} = 0$. All remaining quantities, such as equilibrium consumption sequences, can be calculated using budget identities.

4. Solving the Model

The equilibrium pricing functional and trading strategies are time invariant mappings from the state space, represented by Ω say, to the real line. The following random vector constitutes a set of state variables sufficient to describe the system at time t and predict its subsequent evolution:

$$S_t \equiv [Q_{1,t} \quad b_{1,t-1}] \ .$$

The tractability obtained by using a two agent model should now be apparent. Unlike standard asset pricing theory, asset prices in the current framework depend on the distribution of wealth. The two agent assumption allows the cross-sectional distribution of wealth to be fully characterized by the single variable, $b_{1,t-1}$.

Evaluating the relation [5] involves computing the conditional mean of a non-linear function of an endogenous state variable (the wealth variable). This is easily seen by substituting the budget constraint [7] into [5]. Closed-form solutions to such expressions do not exist in general. This paper follows much of the recent literature on dynamic stochastic general equilibrium theory by using a numerical solution technique to obtain approximate solutions for the endogenous policy functionals. Taylor and Uhlig (1990) present a summary of related methods that have been used to solve various specifications of the neo-classical growth model.

The algorithm developed here obtains approximations to the true pricing and trading functionals by iterating on a suitably defined contraction mapping to which the functionals represent a fixed point. An intuitive way of thinking about the solution method is that a dynamic programming problem is solved for an increasing (in time horizon) sequence of finite period economies. If the control rules to such a problem converge as the sequence grows, they represent a solution to the infinitely lived problem. The reason that the solution is an approximation to the true one is that the state space is discretized to a certain extent. The variable $b_{1,t-1}$ is restricted to take on only a finite number of values. Equilibrium trading rules will therefore take the form of a mapping from points on a two-dimensional state grid to points on a one dimensional grid. Asset prices however, are allowed to take on a continuum of different values, the important reason being that prices are not required to describe the state of the system. Consequently, the pricing functional will take the form of a mapping from the state grid onto the real line. A more detailed description of the algorithm is provided in the Appendix.

An important issue is the existence of borrowing constraints, which are a fundamental component of the solution algorithm. Since the state space for the variable $b_{1,t}$ is discretized, its distribution must have finite support. Furthermore, since the function $b_1(S_t)$ must be characterized at the boundary points, the solution will feature states of the world in which agents are constrained from taking a more extreme position than the boundaries allow for. These boundaries are symmetric around zero and are referred to as a borrowing constraint. Prices for states in which the constraint is binding are uniquely defined by the non-constrained agent's (the lender's) first order condition. The constrained agent's Euler equation [5] will not hold with equality in these states, but will satisfy the usual Kuhn-Tucker conditions. In addition, the random variable, m_t, that satisfies [1] and therefore the inequality [4], is not identified with the individual specific IMRS, $\tilde{m}_{k,t+1}$, but is instead a variable defined in the same way as the bond price. This is made clear in the next Section.

Constraints on asset holdings have played an important role in much of the related literature. Many authors have suggested that it may not be possible to study dynamic incomplete markets models without imposing short-selling constraints of one form or another. Duffie, Geanakoplos, Mas-Colell and McLennen (1989), for instance, establish that lower bounds on asset holdings are an element of a set of sufficient conditions required to prove the existence of a stationary Markovian equilibrium. The question of necessity remains an open one. In addition, Marcet and Ketterer (1989) and Marcet and Singleton (1990) argue that when agents are infinitely lived, constraints on asset holdings are necessary to rule out Ponzi schemes. This is an important issue for these authors, since their numerical solution method obtains approximate control rules for the infinitely lived problem directly. The method employed here obtains the infinitely lived solutions as the limit of a sequence of finite solutions in which agents are required to be solvent in the last period. Ponzi schemes are therefore directly ruled out by the algorithm.

The next section focuses on the effects of missing asset markets by specifying the borrowing constraint to be as unrestrictive as possible. The effects of more restrictive borrowing constraints, which constitute an interesting economic question in and of themselves, are examined subsequently.

5. Numerical Results

This section asks whether or not incomplete markets can provide an explanation for the observed variability in the IMRS. In addition, the extent to which the model can resolve the risk-free rate puzzle is examined. Since agents in the theoretical economy can have incomes that are highly variable (in growth rates) relative to the per-capita aggregate, one might imagine that the model would do quite well in terms of the second moment of the IMRS. However, as will become clear, this conclusion depends on the extent to which individuals are able to pool idiosyncratic risk by trading in the bond market.

The numerical exercise consists of choosing suitable values for the parameters describing tastes and the endowment processes, and then characterizing the effects of incomplete markets by varying the parameter γ from 0.50 (no heterogeneity) to 0.35 (extreme heterogeneity). Recall that the case without heterogeneity is equivalent to a complete markets model. Theoretical population moments are computed and compared to the sample moments reported in Section 2.

As in Mehra and Prescott (1985) the process for aggregate income (which equals aggregate consumption in this environment) is chosen to match the mean, standard deviation and coefficient of first-order autocorrelation from U.S. data. The difference is that in the current model the consumption data are monthly rather than annual. The source is CITIBASE, from which U.S. per-capita consumption of non-durables and services net of clothing and medical care was obtained. The mean, standard deviation and autocorrelation of the growth rate, for the period 1974-1986, are 1.0012, 0.0047, and -0.212, respectively. The transition matrix, Π, is restricted to be symmetric, which implies that it is completely described by one parameter defined by $\phi=\pi_{11}=\pi_{22}$. Trivially, $\pi_{12}=\pi_{21}=(1-\phi)$. The parameter values that satisfy the above criteria are $\lambda_1 = 1.0062$, $\lambda_2 = 0.9962$, $\phi = 0.39$.

The preference parameters that are unrelated to heterogeneity are chosen in accordance to acceptable norms in the literature. The objective is to focus on incomplete markets and not rely on extreme risk aversion and/or unconventional values for the discount factor. Section 6A examines the sensitivity of the results to alternative values for these parameters. The curvature parameter, α, and discount factor, β, are set equal to 2.0 and 0.9983 respectively. The discount factor is simply chosen to be consistent with an annualized discount rate of approximately 2 percent. Note that the

Hansen-Jagannathan statistic is independent of β. The value for α is chosen to capture the generally accepted view that agent's preferences exhibit low to moderate degrees of aversion toward risk.

The parameter δ is set equal to 1/2, implying that agents 1 and 2 have the same expected income streams at any point in time. Recall that this does not imply *ex-ante* homogeneity except in special cases, since the distribution of wealth evolves stochastically and need not be uniform.

In regard to the numerical algorithm, the number of points defining the partition for $b_{1,t}$ was set to 150 per interval of length 0.10. Experiments which used a finer partition were conducted and the resulting unconditional moments were not affected to the degree of reporting accuracy used. The lower bound on bond holdings, $b_{1,t}$ was chosen to be as unrestrictive as possible: $(0.01 - \gamma)$ for each reported value of γ. Lower values result in the existence of states in which an agent is insolvent. The computer program was written in FORTRAN and is available upon request. Computations were done on an 25 Mhz IBM compatible personal computer with an 80486 processor. The longest time required to compute an equilibrium for a single economy was roughly 30 minutes.

The only parameter as yet unspecified is γ, the measure of heterogeneity in the theoretical economy. Recall that the absolute value of $(0.50-\gamma)$ can be interpreted as a measure of the incompleteness of asset markets. The magnitude of γ also determines the variability of individual income growth rates. Figure 2 reports the population standard deviation of income growth for the range γ=0.50 to γ=0.35, in increments of 0.01. Note that the range is normalized to $(0.50-\gamma)$ so that higher values imply greater heterogeneity.

Mean income growth is 0.12 percent per month for all values. Due to the symmetric nature of the economy, all moments associated with agent 1 and 2 are identical. Only one set of numbers are therefore reported. One way in which to choose an appropriate value for γ is to match the variability of the theoretical income process from Figure 2 to that observed in some panel data set. The strategy here, however, is to take an agnostic position and report results for the wide range of γ given in the graph.

The exact definition of the IMRS that corresponds to m_t in equation [1] is now given. Recall that, in states in which an agent faces a borrowing constraint, her IMRS, $\tilde{m}_{k,t+1}$, will not satisfy [5] with equality, and therefore will not be bounded by the Hansen-Jagannathan statistic. The solution is to construct two random variables that are always equal to the unconstrained agent's IMRS. Define the indicator variable, $I_{1,t}$ to be unity whenever agent 1 is unconstrained in the bond market, and zero whenever she is at a corner solution. Then:

$$m_{1,t} \equiv I_{1,t}\, \tilde{m}_{1,t} + (1-I_{1,t})\, \tilde{m}_{2,t} \qquad [9]$$

where $\tilde{m}_{k,t} = \beta(c_{k,t+1}/c_{k,t})^{-\alpha}$, k=1,2. A second random variable denoted $m_{2,t}$ is defined analogously. Since bond prices are defined to satisfy the unconstrained agent's first order condition, the random variables $m_{1,t}$ and $m_{2,t}$ will satisfy [5] at all times. Their standard deviation is therefore bounded by [4], under the null hypothesis that the model is correct. For economies in which the borrowing constraint is irrelevant, $\tilde{m}_{1,t} = m_{1,t}$. Consequently, a measure of the net effect of the borrowing constraint is provided by the difference between the IMRS defined as $m_{k,t}$ and that defined as $\tilde{m}_{k,t}$. Finally, the symmetric nature of the economy results in the statistical properties of $m_{2,t}$ being virtually identical to those of $m_{1,t}$. This turns out to be the case with all individual specific quantities in the model. Subscripts referring to an individual will therefore be omitted and just one set of results will be reported for the remainder of the paper.

Population moments from the theoretical economy are now reported in Figures 3 through 5. Figure 3 graphs mean-standard deviation pairs for the theoretical IMRS, m_t, each point on the graph representing a different value of γ. Figures 4 and 5 graph the mean risk-free rate and standard deviation of consumption growth, respectively, against the value of $(0.5-\gamma)$. Mean consumption growth is equal to 0.12 percent for all values of γ. Although it is not difficult to construct the Markovian transition matrix for the economy, calculation of the corresponding unconditional distribution involves iterating on a rather large matrix. Since unconditional moments are obtainable by simulation with an arbitrary amount of precision, all reported moments represent sample averages from an extremely large simulation of the artificial data generating process.

The results in Figure 3 show that the model is successful to a certain extent in accounting for the variability of the IMRS. For the extreme case, γ=0.35, the standard deviation is approximately 7 times larger than that associated with the representative agent model, which is represented by the point γ=0.50. The standard deviation of income growth for γ=0.35 is roughly 31 percent per month. For more moderate income variability, say a standard deviation of 20 percent (γ=0.40), the incomplete markets model still generates an IMRS that is roughly 4 times more variable than the complete markets model.

Just as important in Figure 3 is the behaviour of the mean IMRS, which, with a minor correction for Jensen's inequality, is equal to the inverse of 1 plus the mean risk-free rate of interest. Recall that in the representative agent model (Figure 1), higher values for the standard deviation of the IMRS could only be achieved (by increasing risk aversion) at the expense of a drastically lower mean IMRS. Consequently, moderate variation in the IMRS was associated with unrealistically high risk-free rates. The incomplete markets model resolves this puzzle. Figure 3 shows that the mean IMRS is increasing in its standard deviation, implying that high variability in the IMRS is consistent with low values for the mean risk-free rate.

The results on interest rates are clarified further in Figure 4 where the annualized mean risk-free rate is plotted against the value of $(0.50-\gamma)$. The point $\gamma=0.50$ emphasizes the 'risk-free rate puzzle' associated with the representative agent model (with time separable preferences). The puzzle is that the theoretical mean is roughly 400 basis points greater than that observed in the data (0.80 percent in the Mehra-Prescott data-set on U.S. treasury bills). The impact of heterogeneity is to drastically reduce the mean risk-free rate. Figure 4 shows that the mean actually becomes negative for low values of γ. The case $\gamma=0.35$ results in a mean of -4.61 percent. A value for γ of 0.38 results in a mean risk-free rate of 0.80 percent, which matches the sample moment. The standard deviation of the IMRS for $\gamma=0.38$ is 0.045, a factor of 5 larger that the representative agent model.

While the results are encouraging relative to the representative agent model, the model cannot account for all of the variability in the IMRS implied by the Hansen-Jagannathan statistic. The largest standard deviation observed in the sequence of theoretical economies is roughly half that of the (liberal) lower bound reported in Section 2. Furthermore, taking into account sampling variability is not likely to resolve the puzzle. The Hansen-Jagannathan statistics are lower bounds, not point estimates, and are estimated with a fair amount of precision, the more restrictive estimates in particular.

The lack of success in terms of IMRS variability is interesting in and of itself. Agents are apparently able to pool a great deal of idiosyncratic risk in equilibrium, in spite of the very limited set of assets at their disposal. The standard deviation of equilibrium consumption growth, shown in Figure 5, is surprisingly small given the variability of income growth shown in Figure 2. The ratio of the standard deviation of consumption growth to that of income growth is shown in Figure 6. When $\gamma=0.35$ this ratio is 0.11. For less extreme values of γ the ratio falls to 0.08 and 0.06 for $\gamma=0.40$ and $\gamma=0.45$, respectively. Evidently, the asset market structure is not as 'incomplete' as one might have thought *ex ante*. Equilibrium trading strategies are such that agents who receive bad income realizations are able to cushion most of the blow through riskless borrowing.

As noted above, the difference between the IMRS, m_t (defined in [9]), and the individual specific IMRS, $\tilde{m}_t$, provides a measure of the net effect on the economy of the borrowing constraint. This effect turns out to be substantial as the amount of heterogeneity is increased. For the case of $\gamma=0.35$, the mean of $\tilde{m}_t$ is 0.9980, compared to a mean of 1.0043 for m_t. As a loose indicator of the magnitude of this difference, suppose that one mistakenly computed the mean risk-free rate based $\tilde{m}_t$. The implied interest rate is 2.40 percent: 7.01 percent higher than the mean of the true market clearing rate (-4.61 percent). In economies with less heterogeneity this effect is not as pronounced. For $\gamma=0.40$ the difference is roughly 2 percent and for values of γ between 0.45 and 0.50 the effect is virtually zero.

These results are indicative of two further points. First, it appears that the formulation of the model does not allow for the independent examination of the effects of market incompleteness and constraints on bond holdings, at least for economies with a substantial amount of heterogeneity. Secondly, the substantial reduction in the average risk-free rate appears to be driven to a large extent by the borrowing constraints. Figure 4 shows that the mean decreases only marginally for economies in which the borrowing constraint has no effect ($0.45 \leq \gamma < 0.50$). As the constraint becomes more important, at roughly $\gamma = 0.44$, the interest rate begins to drop at an increasing rate. The economic reasoning behind this result is straightforward. When one agent is constrained in equilibrium, the agent on the long side of the market must be persuaded to keep from accumulating larger credit balances. The reduction in the rate of return, relative to an equilibrium with no constraint, accomplishes this.

6. Other Properties of the Economy

This section briefly reports two further sets of numerical results: the sensitivity of the results to changes in the preference parameters and the effects of more restrictive borrowing constraints. Recall that in Section 5 the constraints were chosen to be as unrestrictive as possible in order to focus on the incomplete markets issue.

6A. Sensitivity to Changes in Preference Parameters

There are two preference parameters of interest for the sensitivity analysis: α, the curvature parameter and β the discount factor. The effects of changing β are straightforward. The standard deviation of the IMRS is independent of β, implying that the analysis based on Hansen-Jagannathan bounds is unaffected. However, the mean risk-free rate is monotonically decreasing in β. Herein lies the reason that several authors have had success in accounting for the risk-free rate puzzle using values for β in excess of unity. The important difference in this study is that for values of γ below 0.38, the mean risk-free rate is actually too small. Therefore, the flexibility to choose values for β less than the value used above (β=0.9983), implies that an arbitrarily low (positive) mean risk-free rate can be achieved for these economies. This goal can be achieved without resorting to negative rates of time preference (implied by $\beta > 1.0$).

Two sets of experiments were conducted using higher values for the curvature parameter: α=4 and α=6. Higher risk aversion increases the model's ability to generate more variable IMRS's. For γ=0.35 the standard deviation of the IMRS was 0.187 and 0.251 for α=4 and α=6, respectively. As was the case for α=2, the standard deviation is monotonically increasing as the amount of heterogeneity increases. Moderate amounts of risk aversion are apparently sufficient to reconcile the model with the (liberal) Hansen-Jagannathan bounds from Section 2. Whether or not these values for α imply unreasonable attitudes towards risk is debatable. Mankiw and Zeldes (1990) report anecdotal evidence, in terms of the certainty equivalent value of a lottery, that suggests that α=6 is probably unrealistic.

In terms of the mean risk-free rate of interest, the effects of increased risk aversion are to increase the mean for values of γ near 0.50, and to decrease the mean for values near 0.35, relative to the α=2 case. For moderate values of γ (near 0.50), this effect reflects the well known fact that the mean risk-free rate is increasing in the coefficient of relative risk aversion for economies with a representative agent. As the value of γ falls however, the effect of the borrowing constraint becomes more dominant, to the point where the risk-free rate actually falls, relative to lower risk aversion cases. The increased consumption smoothing motive generated by the higher value of α results in a relatively low market clearing interest rate in states in which an agent is constrained (or close to it). In other words, the reduction in the risk-free rate required to induce the agent on the long side of the market to hold the maximum number of bonds is increasing in the risk aversion parameter.

6B. Incomplete Markets and Borrowing Constraints

In Section 5 the borrowing constraint was set as unrestrictively as possible at $(0.01 - \gamma)$. In this section, two further sets of experiments are conducted in which the borrowing constraint is tightened, first to -0.250 and then to -0.125. These values imply that an individual is prohibited from borrowing any more than 50 percent and 25 percent, respectively, of his average monthly income. All other parameter values are as in Section 5. The effects are consistent with what was observed previously. As the borrowing constraint become more restrictive, the mean risk-free rate goes down sharply. For example, in the case in which the constraint is -0.25, the mean, in percent, is 0.97 and -13.03 for γ=0.40 and γ=0.35, respectively. These values can be compared to those from Section 5 which were 2.76 percent and -4.61 percent, respectively. In the more extreme case, where the constraint is set to -0.125, the mean interest rate becomes substantially counterfactual. It is, for γ=0.40 and γ=0.35, -15.73 percent and -67.41 percent.

One might expect the effects of tighter borrowing constraints on the standard deviation of the IMRS to be quite noticeable. In the limit, as the constraint approaches zero and the economy becomes autarkic, the amount of idiosyncratic risk inherent in the IMRS (which, recall, is m_t as defined in [9]) should increase substantially. However, for the cases considered here the effect is surprisingly small. For the less restrictive constraint, -0.25, the standard deviation is 0.040 and 0.079 for $\gamma=0.40$ and $\gamma=0.35$, respectively. When the constraint is -0.125 the corresponding values are 0.072 and 0.145. These numbers are all smaller than the Sharpe ratio reported in Section 2. Evidently, agents are still able to pool a substantial amount of idiosyncratic risk, despite being faced with what appear to be (judging by the effects on prices) fairly restrictive constraints on asset holdings. The most extreme case, with a constraint of -0.125 and $\gamma=0.35$, implies a ratio of the standard deviation of consumption growth to income growth of less than 0.20.

The implications of this section are that tighter borrowing constraints increase the standard deviation of the IMRS only marginally. Moreover, this is achieved at the expense of counterfactual properties for expected real interest rates. Consequently, the formulation of the borrowing constraint used here cannot account for the variability of the IMRS implied by securities markets.

7. Conclusions

This paper was motivated by the thought that departures from frictionless Arrow-Debreu markets may be helpful in accounting for the properties of consumption and asset returns that are anomalous within the representative agent framework. A generalization of the representative agent model was formulated which featured two types of frictions: incomplete asset markets and constraints on asset holdings. These frictions were shown to be helpful in accounting for several important properties of securities markets data. Specifically, the model was able to account for up to 7 times more of the implied variability in the IMRS than was the representative agent model with additively time separable preferences. This is important since, as was shown using Hansen-Jagannathan bounds, many anomalies that arise in the context of dynamic asset pricing theory are indicative of inadequate variability in the theoretical IMRS. Another encouraging result was that increases in the standard deviation of the IMRS were not associated with unrealistically high risk-free rates of interest: a pervasive property of many representative agent models. The combination of incomplete asset markets and borrowing constraints resulted in a mean risk-free rate that was monotonically decreasing in the standard deviation of the IMRS. In this sense the model offers an explanation for the 'risk-free rate puzzle'.

Progress has also been made in understanding the limitations of the type of approach used here. Conditional on low values for the risk aversion parameter, the model can account for at most half of the standard deviation in the IMRS implied by the Hansen-Jagannathan lower bound. This result was unexpected at the outset. It was thought that, given variable enough individual income processes along with incomplete markets, the

Hansen-Jagannathan puzzle could be resolved, and that research efforts would focus on the predictive properties of IMRS's in economies with several assets. The results indicate otherwise. In spite of a very limited asset market structure, a substantial amount of idiosyncratic risk turns out to be diversifiable and is therefore not priced in equilibrium. Agents are able to pool a large portion of their individual endowment risk. In this sense, the single asset assumption sharpens the results: additional assets would lead to increased consumption smoothing opportunities and therefore a lower standard deviation of the IMRS.

The results are also suggestive of possible limitations of models based on short-sales, or borrowing constraints. For example, Deaton (1989) examines (in a partial equilibrium context) the consumption-savings decision of an investor facing a constant interest rate and a lower bound on the net value of her assets. He finds that if individual endowment processes are non-stationary in levels, which they are here, very little consumption smoothing occurs. The ratio of consumption variability to income variability in Deaton's model is much higher than in any case documented above, including those with restrictive borrowing constraints. This study points out that when interest rates are allowed to vary in response to changing IMRS's, the conclusions in terms of the dynamic properties of consumption can be drastically different. Furthermore, should constraints on asset holdings have a substantial impact on consumption allocations, as above in Section 6B, the model suggests that the stochastic properties of market clearing prices may be strongly counterfactual. The general equilibrium approach emphasizes that merely endowing individuals with highly variable incomes and not allowing them to trade intertemporally may be insufficient for explaining the behaviour of consumption and asset returns simultaneously.

There are a number of possibilities for future research. Most obvious is the issue of a more complex asset market structure including, for instance, claims to a dividend process (equity). On-going research, developed independently, is being conducted in this area by Lucas (1990), Marcet and Singleton (1990) and Telmer (1990). As noted, allowing agents to trade additional assets in the above framework is not likely to be helpful in terms of the Hansen-Jagannathan puzzle. However, the effect of incomplete markets on the joint behaviour of returns and consumption is another question. Lucas (1990) reports that the excess return on equity in her model is far below the sample average reported by Mehra and Prescott (1985). Insofar as small excess returns on equity are indicative of low variability in the IMRS, and visa-versa, the numerical results in this paper are consistent with Lucas's.

Other possibilities include altering the present model in a number of different ways. Asymmetries in terms of the distribution of wealth, attitudes towards risk, and time preference are easily incorporated. In addition, a whole host of issues that are degenerate in the representative agent framework, such as the behaviour of trade flows and the possibility of default, are important issues in an incomplete markets model. While the model in this paper is probably too simplistic to address these types of problems, the basic approach opens up a number of research areas that could not be addressed by the frictionless model. Duffie (1990), Geanakoplos (1990), Magill and Shafer (1990) and references therein provide a discussion of the many questions that can be asked using the general equilibrium model with incomplete markets.

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FIGURE 1

Representative Agent IMRS Moments

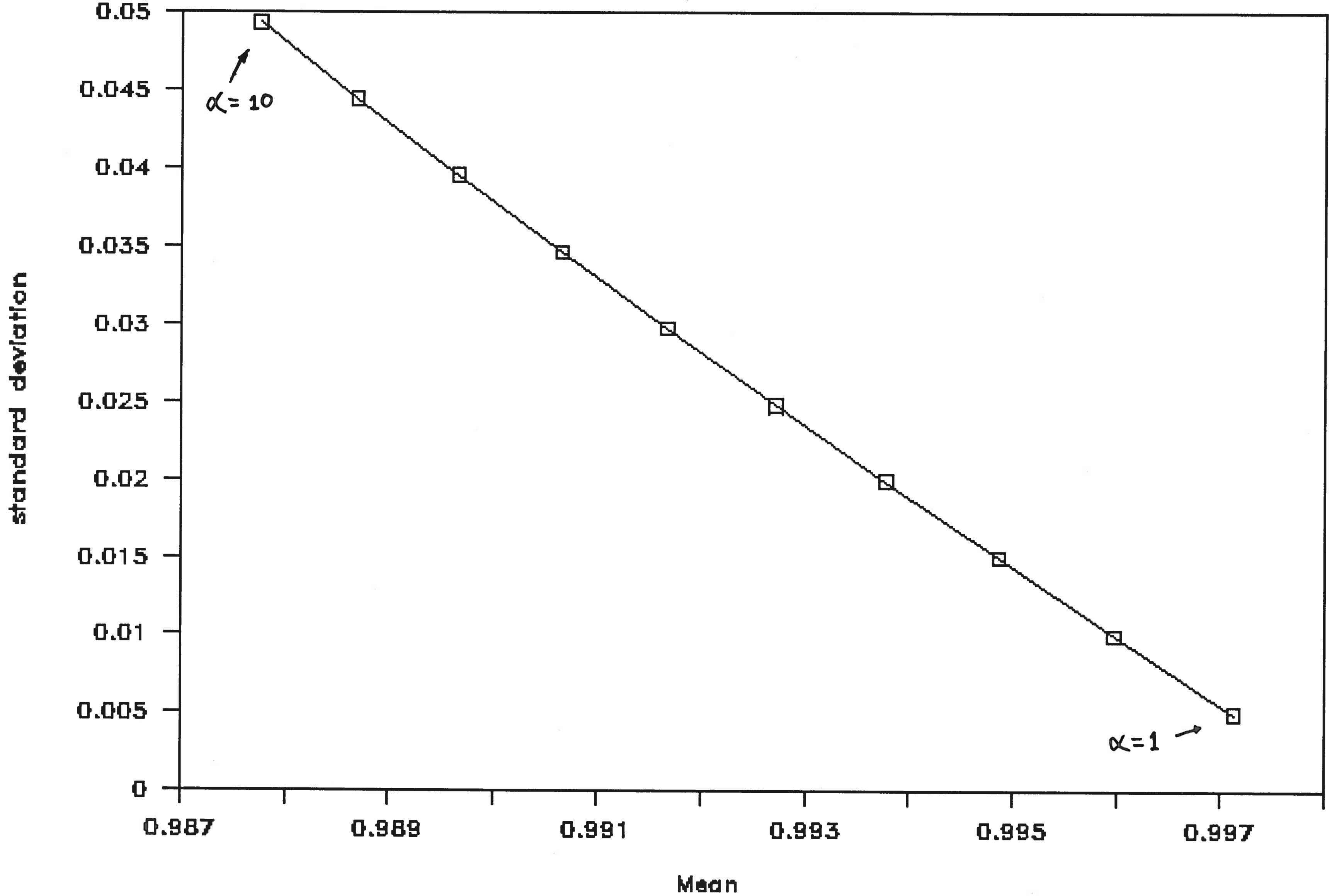


FIGURE 2

Standard Deviation of Income Growth

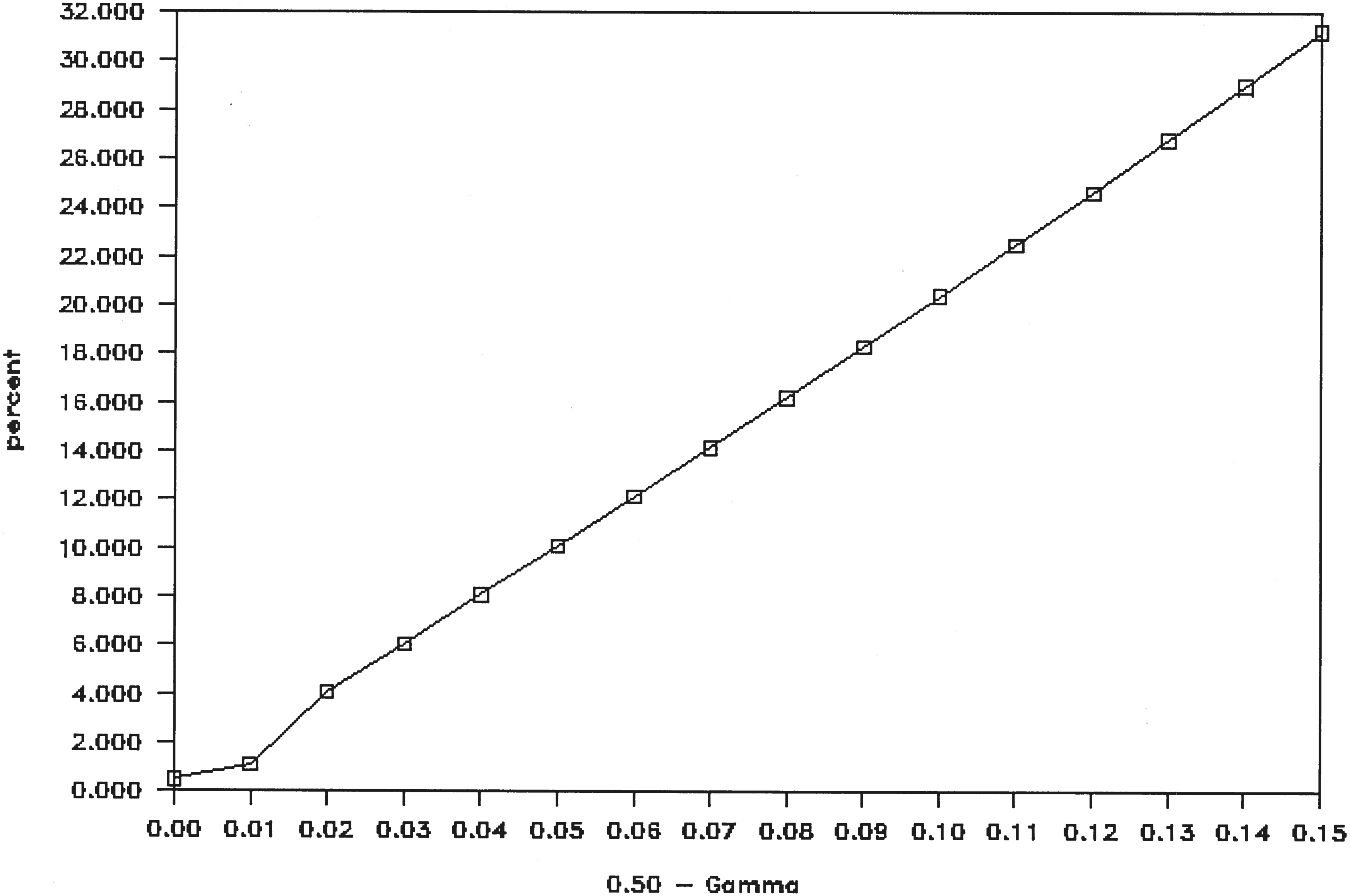


FIGURE 3

IMRS: Mean – Standard Deviation Pairs

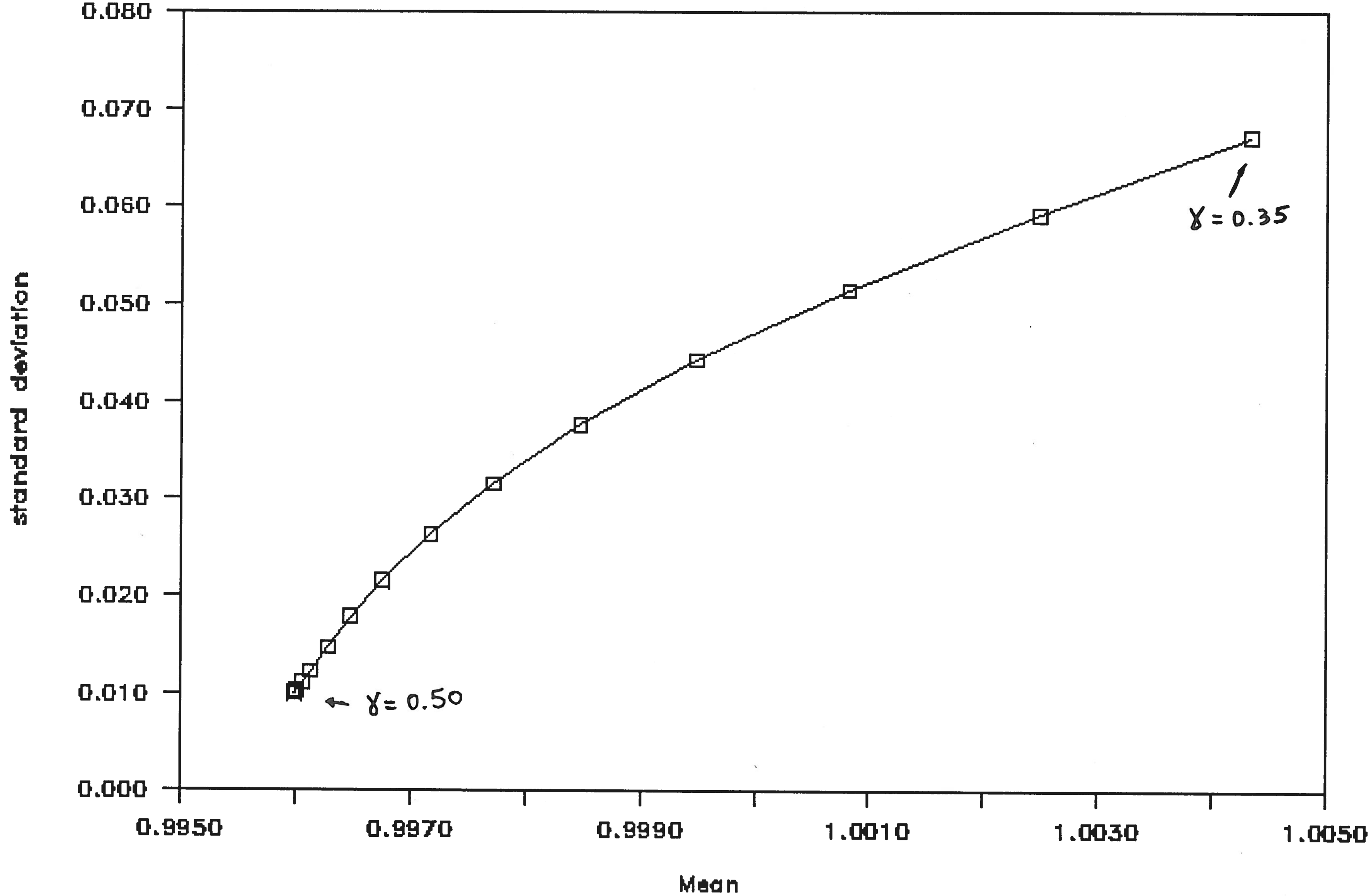


FIGURE 4

Mean Riskfree Rate

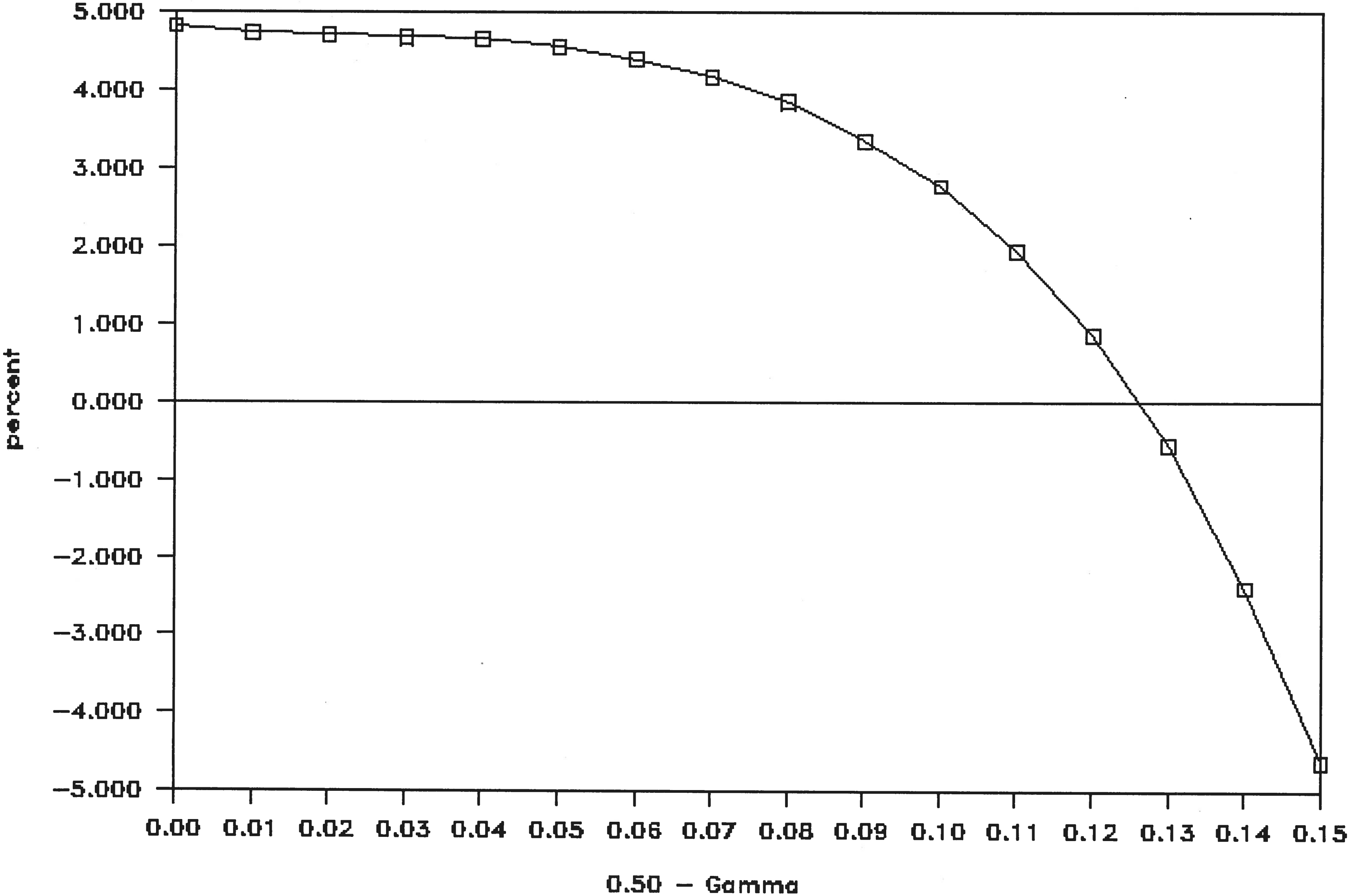


FIGURE 5

Consumption Growth – Standard Deviation

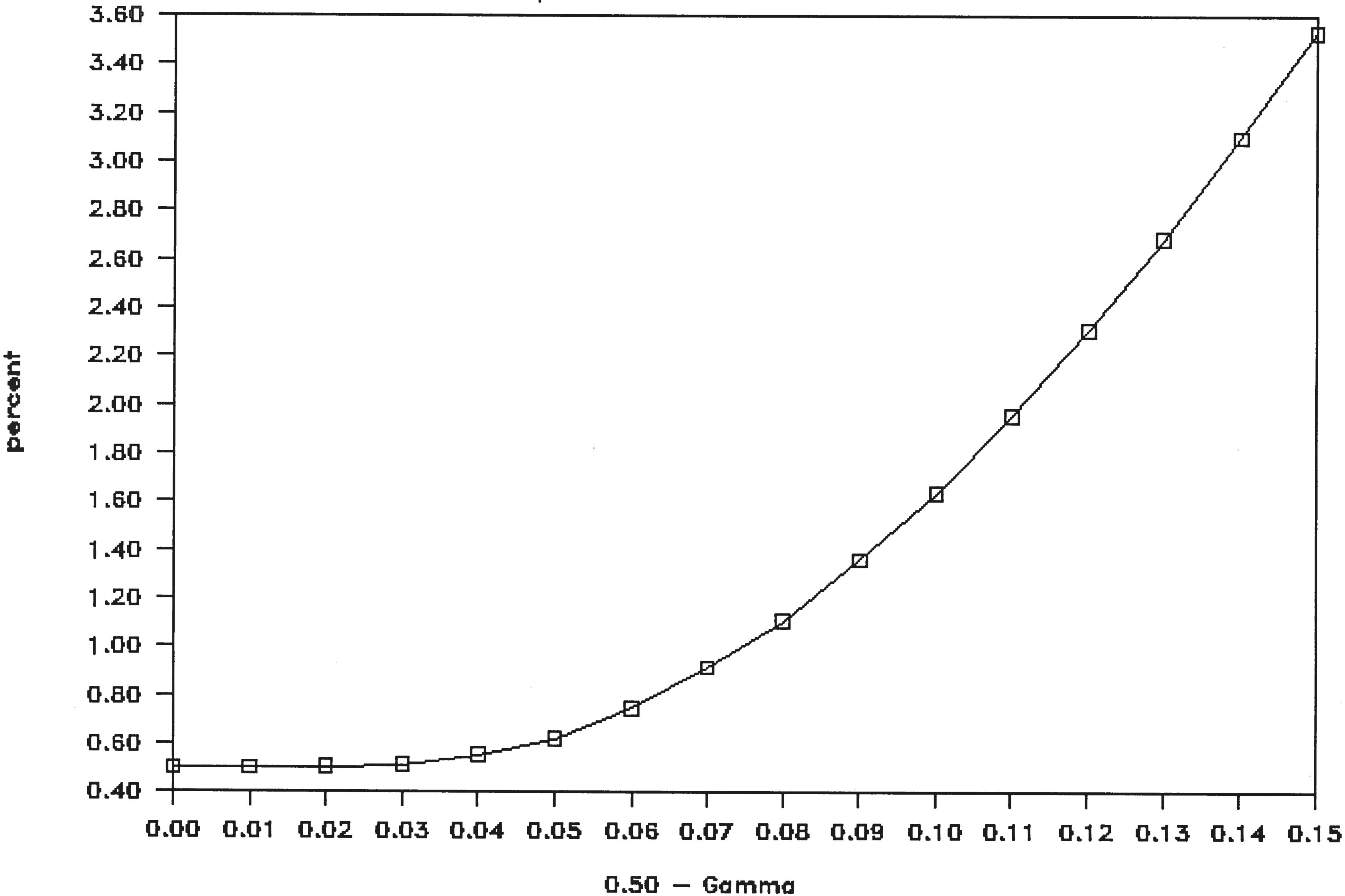
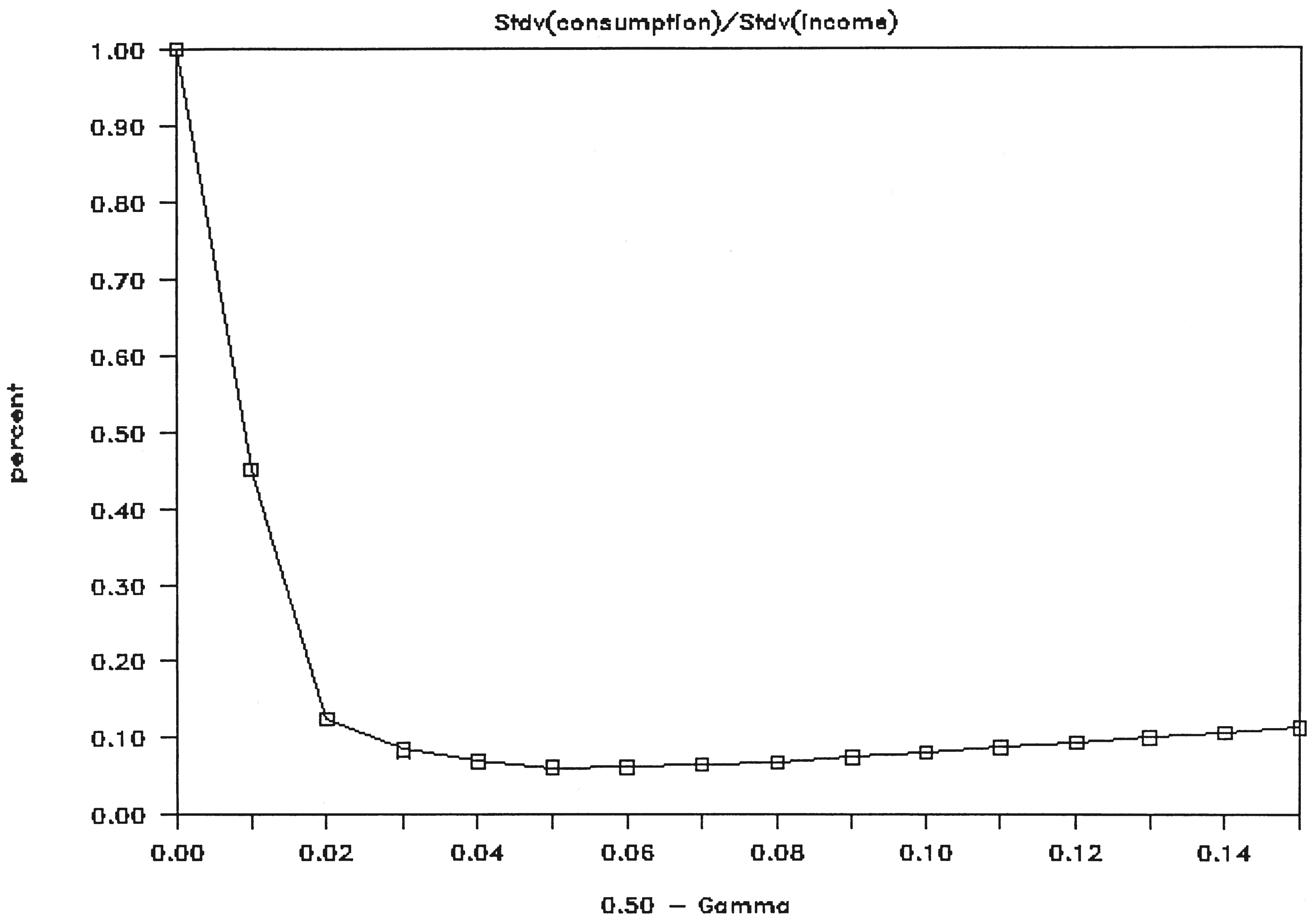


FIGURE 6


Stdv(consumption)/Stdv(income)
percent
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.50 – Gamma

Appendix - Computational Algorithm

The algorithm developed to solve the model is a modified version of that suggested by Bertsekas (1976) and used recently by Baxter, Crucini and Rouwenhorst (1990) to solve a version of the neo-classical growth model. The major difference is that the endogenous variables that are not state variables are allowed to take on a continuum of different values. A method based on linear interpolation is used to solve for market clearing values for these variables (prices).

The first thing to note is that the model must be completely parameterized. That is, numerical values must be chosen for all parameters associated with tastes and technology. The "solution" to the model referred to below is always contingent on a specific set of parameter values.

Consider substituting budget constraints and market clearing conditions into the equations [5]. The result is a system of two equations in two unknowns, $b_{1,t}$ and p_t, conditional on the state vector S_t. The objective is to characterize the control rules, $b_1(S_t)$,$p(S_t)$. Note that the asset payoff vector, X_{t+1}, is simply a vector of ones, since the only asset is a discount bond. The system of two equations can be written implicitly as follows.

$$P_t = E_t[A(S_t, P_t, S_{t+1}, P_{t+1})] \quad , \qquad [A1]$$

where A is some bounded non-linear function and the vector P_t is defined as $[p_t \;\; p_t]$. The vector P_t represents the fact that there are two moment conditions, one for each agent, that are defined in terms of the single price. The main difficulty, ignoring the non-linearity, is that of computing the conditional mean of the endogenous variables contained in S_{t+1} and P_{t+1}. These variables are $[b_{1,t+1}, p_{t+1}] \equiv C_{t+1}$. However, note that if the control rules, $C_{t+1}(S_{t+1})$, were known, this would not be a problem since $S_{t+1} = [b_{1,t}, \lambda_{t+1}]$, all of which are either known at time t or have a well defined conditional density function (the joint Markovian transition matrix).

The algorithm proceeds as follows. First consider the final period of a T period economy. $C_T(S_T)$ is clearly well defined, since bonds have no value. Next, consider period T-1. Again, ignoring the non-linearity in A1, the system can be solved for $C_{T-1}(S_{T-1})$ since the conditional mean of $C_T(S_T)$ is known. Finally, consider period T-2. Again, conditional on some arbitrary S_{T-2}, [A1] can be solved since the rules $C_{T-1}(S_{T-1})$ are known: the expression $E_{T-2}[B(C_{T-1}(S_{T-1}))]$ can be computed (conceptually) for some function B. This is just a standard backward recursion that is fundamental to the theory of dynamic programming. The only difference is that we are iterating on the first-order conditions instead of the the more common approach of iterating on the value function. When the control rules converge according to some suitable metric (defined below), the equilibrium rules for the infinite problem have been found.

What remains to be explained is how the non-linearity is handled and how the functionals $C_t(S_t)$ are characterized. The first step is discretize the values that the vector S_t is allowed to take on. The method used in this paper is simply to choose a sufficiently general upper and lower bound for the wealth variables and then define a uniform partition based on an arbitrarily chosen number of points (as large as is computationally feasible). Tauchen (1990) describes more sophisticated techniques for choosing the points in such a grid. The discrete approximation is the major source of approximation error associated with this approach. Note however, that control rules calculated in this way converge pointwise to the true rules as the partition becomes arbitrarily fine.

Now consider the recursive algorithm described above for the T-1 period in the iteration. Take some point in the state grid as given. For instance, if λ represents the set of values that λ_t is allowed to take on (there are 2), and $\mathcal{B}$ represents the set for $b_{1,t-1}$, this is a point in $\lambda \times \mathcal{B}$. The objective is to find a value for p_t and a point in $\mathcal{B}$ that most closely satisfies [A1]. This is where the linear interpolation comes in[1]. A function is defined that can loosely be interpreted as the excess demand for bonds. Given some

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[1]Thanks to Dan Bernhardt for suggesting the linear interpolation component of this algorithm.

arbitrary price, p_1, it is easy to solve for some excess demand: a point in $\mathcal{B}$. This can actually be done in a rather efficient way since, as can be shown analytically, the equations in [A1] are monotonic in $b_{k,t}$.

Given some excess demand point in $\mathcal{B}$, the fact that the excess demand function appears to be (based on computational results) monotonic in p_1 is very useful. The implication is that if excess demand is positive, there exists a price, $p_2>p_1$, such that the function will take on a negative value. The opposite holds if the given point in $\mathcal{B}$ is negative. A third price, $\bar{p}$, is then calculated as that which would set excess demand to zero, should the function be linear: hence the term "linear interpolation". Since the function will not in general be linear, this procedure starts over by equating p_1 to $\bar{p}$ and proceeding as above. The 'pseudo-equilibrium' price is defined as that which results in excess demand being zero. This also defines a pseudo-equilibrium value for $b_{1,t}$ (a point in $\mathcal{B}$).

This procedure is repeated for every point in the state space in order to completely characterize $C_{T-1}(S_{T-1})$ as a vector mapping from $\lambda\times\mathcal{B}$ to either $\mathbb{R}$, or $\mathcal{B}$. Finally, given $C_{T-1}(S_{T-1})$, the backward recursion algorithm proceeds in exactly the same manner until $b_{T-j}(S_{T-j}) = b_{T-j-1}(S_{T-j-1})$.