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ASPECTS OF ASSET BEHAVIOR IN CONTINUOUS TIME MACROECONOMIC MODELS

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I. INTRODUCTION

A widely held view, aided by the powers of hindsight, is that a great deal of unnecessary confusion has been created by attempts to treat economic events occurring over time (e.g. --accumulation processes) in Hicks-period-type models. A basic confusion arises from the need to convert all "flow" variables to stock dimensions, and from the inclusion of such "flow" variables and "genuine" stock variables in the single budget constraint inherent in the period-type formulation. Evidence that the period-type models give rise to serious stock-flow confusions is provided by Archibald and Lipsey's (A-L) criticism of Patinkin's first edition.[1]

Given these inherent problems, combined with the modern emphasis on problems related to economic growth, it is not surprising that the period-type models have given way to continuous time models as the most frequently employed framework for macroeconomic analysis. However, certain confusions derivative from this heritage of period analysis still persist in current macro-model formulations, and it would seem useful to spell out exactly what changes in specification are necessitated in the transition from discrete to continuous time models.

In a noteworthy recent article, Josef May has attempted just that--in particular he makes clear the distinction between

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1. See especially the first section of A-L.

stock and flow concepts inherent in a continuous time model, emphasising the distinction between the stock (or wealth) constraint and the flow (or dynamic budget) constraint, both of which are present in such a model.[2]

Important as that contribution is, it is deficient in that it fails to specify carefully which stock variables can change instantaneously and which must change over time. Such a specification is obviously of interest in any analysis concerned with short-run macroeconomic phenomena; i.e., in virtually any analysis ultimately concerned with the formulation of macroeconomic policy. An immediate manifestation of the above deficiency is the failure of most models to distinguish between various concepts of asset demands. While most writers (e.g. May, p. 2) recognize the need for any such demand to be defined by two time indices--the decision date and the objective date; few indicate how the actual asset demand specifications are affected by such considerations.

At the risk of belaboring the obvious, it may be worth elaborating on this point. Consider an individual in "full stock equilibrium" with respect to his holdings of a single asset, say real balances. He holds static expectations over all future values of all relevant variables and fully expects his current actual stock to equal his future desired stock. Now consider some unanticipated parametric change at time to which causes his desired stock to rise, (say from M^* to M^{**}

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2. See below, pp. 5-6; also Sidrauski (1967).

in Figure 1) for all future times, but that some costs (e.g. foregone consumption) are involved in changing his actual stock instantaneously. He will then change his holdings of real balances along some optimal path such as the dashed line in Figure 1, achieving full stock equilibrium at time t_1.

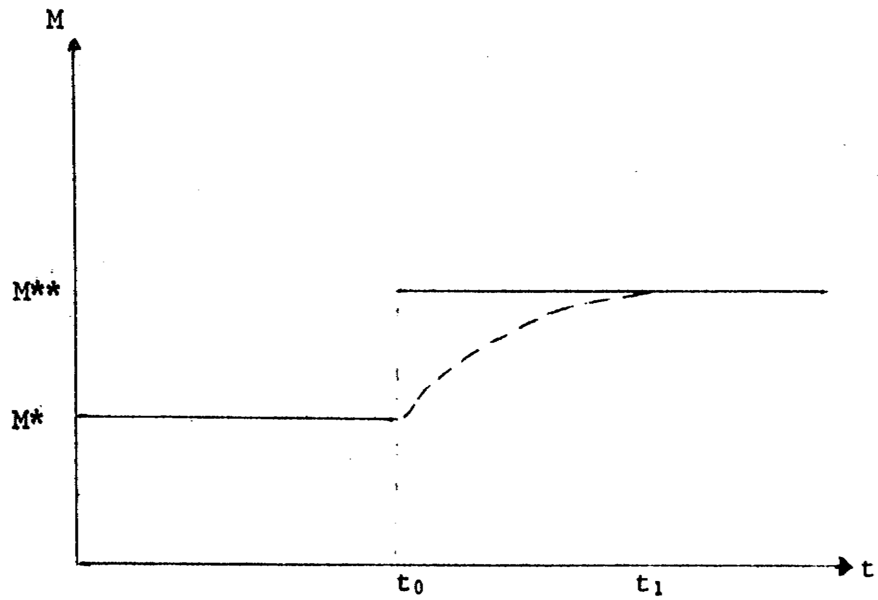

Fig. 1

Then at time t_0 there exists an infinite number of "demands for real balances"; $M_{t_0,\lambda}$; corresponding to each point of time, λ, in the future.[2a] Most current analyses consider only the ultimate demand M** or alternatively only the instantaneous demand $M_{t_0 t_0}$. I argue that in order for the formulation to be useful for short-run analysis, it is necessary to carefully specify and distinguish between at least both of the above; i.e., the instantaneous demand ($M_{t_0 t_0}$) and the ultimate demand (M_{t_0,t_1} where t_1 is the time at which full stock equilibrium is achieved).

2a. Note that a model of intertemporal optimizing behavior yields as its solution the time path of desired asset holdings for all time τ, $\tau \geq t_0$. The appendix presents such a model.

It is the purpose of the present paper to consider, in the context of a continuous time macroeconomic model, how the explicit consideration of the existence of adjustment costs (including foregone consumption) which preclude instantaneous changes in the level of certain stock variables modifies our interpretation of the 'conventional' macroeconomic analysis. In the following section of the paper we consider the interpretations of asset demand functions and of the traditional stock constraint. Attention is given to the implications of different possible specifications of adjustment costs--in particular, two specifications which might be thought to correspond to the theory of the individual asset holder and to the theory of the economy as a whole will be treated. The distinction is drawn between positions of short run, or transitory, equilibrium characterized by flow equilibrium in assets, and positions of long run, or full, equilibrium characterized in addition by (long-run) stock equilibrium.[3] Implications for the specification of asset demand functions are considered. Finally, implications for the dynamics of adjustment from one position of full equilibrium to another will be derived.

3. Michael Mussa makes a similar distinction, and draws the analogy between the short-run and long-run distinction made here, and the Marshallian distinction between short- and long-run supply. The "differences" between May and Patinkin as outlined in Fn. 8 on page 5 in May, can be easily attributed to the failure to distinguish between the long- and short-run asset demands. The confusions become even more apparent when we recognize that Patinkin's use of his asset demands as only short-run demands led to the A-L criticism whereas May criticizes him for essentially only considering (the same functions) as long-run demands.

II. ADJUSTMENT COSTS AND THE STOCK CONSTRAINT

The Basic Model

Following May,[4] we note the period model has the following Budget Constraint:

$$\bar{L} + \bar{B} + \bar{Y} + r \cdot B \cdot \Delta t = L + B + C, \qquad (1)$$

i.e., consumption during the period, C, plus asset holdings at the end of the period, L and B, are constrained by the initial value of assets, $\bar{L}$ and $\bar{B}$ plus income received during the period, $\bar{Y} + rB\Delta t$. Hence the budget constraint contains both "genuine" stock variables such as $\bar{M}$, $\bar{B}$ and modified flow variables such as $\bar{Y}$ and C which represent the integral of a flow over the period.[5] We can approximate these by assuming the actual flows to be constant over the period, so (1) becomes

$$\bar{L} + \bar{B} + \bar{y} \cdot \Delta t + rB\Delta t = L + B + \bar{c}\Delta t. \qquad (2)$$

Taking the limit as Δt goes to zero and rearranging yields

$$(\bar{L}-L) + (\bar{B}-B) = 0, \qquad (3)$$

the stock constraint, or what might be termed Walras' Law for stocks. It is immediately obvious that in terms of our earlier discussion it is the instantaneous demands $L_{t_0 t_0}$ and

4. The static part of the model derives primarily from that used by May. Where possible, I use May's notation, where an upper case letter denotes a stock, a lower case a flow, and a bar over a variables indicates that its value is given at a point in time, and where the mnemonics are fairly obvious.
5. The value of the flow variables of course changes

$B_{t_0 t_0}$ (henceforth simply L and B) which are constrained by the stock constraint. Equation (3) simply tells us how a given stock of wealth can be allocated between the various assets at a moment in time.

Another interpretation of equation (2) can be achieved by collecting terms involving Δt and dividing by Δt, yielding

$$\frac{L-\tilde{L}}{\Delta t} + \frac{B-\tilde{B}}{\Delta t} = \tilde{y} + rB - c \tag{4}$$

Upon taking the limit as Δt goes to zero, we now have

$$\frac{dL}{dt} + \frac{dB}{dt} = \tilde{y} + rB - c, \tag{5}$$

or the dynamic budget constraint which tells us that desired saving $(\frac{dL}{dt} + \frac{dB}{dt})$ must equal expected income less desired consumption. This then tells us how wealth can move over time--in a one asset world this describes the time path of M in Figure 1.

We write the long-run asset demands as follows, retaining the restrictive assumption of static expectations over the level of income, the rate of inflation (=0) and the rate of interest in order to focus attention on the stock-adjustment mechanism central to the argument. Denoting the current level of real income by $\tilde{y}$ and the interest rate by r, we have the long-run demand for a stock of real balances as

$$L' = f(\tilde{y},r), \tag{6}$$

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with the length of the period. The following analysis suggests that L and B might also.

and the long-run demand for real bonds as

$$B' = g(\tilde{y},r). \tag{7}$$

As is well known, these cannot be specified independently, but must satisfy the constraint imposed by the long-run demand for wealth,[6]

$$A' = h(\tilde{y},r). \tag{8}$$

It is perhaps worthwhile to stop at this point in order to compare our model with the more common ones (e.g., May, Foley and Sidrauski) which include actual wealth along with expected income and interest rates as an argument in the asset demand functions. Our model has the adding-up features that

$$\frac{\partial L'}{\partial y'} + \frac{\partial B'}{\partial y} = \frac{\partial A'}{\partial y},$$

and

$$\frac{\partial L'}{\partial r} + \frac{\partial B'}{\partial r} = \frac{\partial A'}{\partial r},$$

whereas the usual formulations constrain the sums on the left hand side of the above equations to zero (wealth now being held constant in the partial derivative). That result, I contend, is more appropriate for the

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6. Alternatively, we could write (6) and (7) as

$$L' = \tilde{L}(\tilde{y},r,A'), \tag{6'}$$

and

$$B' = \tilde{B}(\tilde{y},r,A'). \tag{7'}$$

where the interdependence of (6') and (7') is implicit in the inclusion of A' as an argument in the asset demands and the identity $A' \equiv L'+B'$.

instantaneous asset demands L and B.[7] Indeed, that is surely the way the above authors intend them to be interpreted--except they fail to make the distinction between L and L′, B and B′. As we will argue below, some of the above partial derivatives on L and B become meaningless under some specifications of the model when the instantaneous demand functions are explicitly introduced.

The dynamic side of the model includes a flow demand for commodities,

$$c = c(\bar{y}, r, A', A), \qquad (9)$$

where it is the difference $(A'-\tilde{A})$, the (long-run) excess demand for wealth which becomes important for our analysis. In particular, an increase in the excess demand for wealth leads to increased savings and hence reduced consumption, i.e. $\frac{\partial c}{\partial(A'-\tilde{A})} < 0$.[8] Completion of the specification of the dynamic part of the model would involve flow demands for assets and reference to the dynamic budget (flow) constraint derived previously. Explicit specification of the flow side is presented below for the special cases considered.

It is obviously of interest how the difference $(A'-\tilde{A})$

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7. If an increase in $\tilde{y}$ causes the instantaneous desired stock of real balances to rise, $\frac{\partial L}{\partial \tilde{y}} > 0$, then the theory imposes a negative sign on $\frac{\partial B}{\partial \bar{y}}$. Note also that the asset demands functions used by both May and by Foley and Sidrauski, by including actual wealth as an argument, correspond more closely to our instantaneous demands, L and B.

8. Compare to A-L where $(A'-\tilde{A})$ becomes $(\frac{M}{P})^d - (\frac{M}{P})$. I have also used a similar formulation recently in a paper attempting to resolve the controversy between Keynes-Wicksell and Neo-Classical Approaches to money and growth.

moves over time, and it is to this end that explicit recognition of adjustment costs is made.[9] As seems to be generally recognized, but seldom explicitly stated, there are two distinct aspects of adjustment costs relevant here--the costs of changing the level of wealth and that of changing the composition of a given level of wealth.[10]

There seem to be two important cases. Case I, which might be thought of as pertaining to the individual asset holder, is when the former cost (i.e., foregone consumption) is a binding constraint while the latter is not. The individual asset holder acts as a price taker in asset markets, and thus might be considered as being able to (or at least, perceiving that he is able to) rearrange his portfolio costlessly at constant prices. Case 2, when both costs are operative, would then correspond to a theory of the economy as a whole.

Case I--The Individual Asset Holder

Consider the implications of a divergence between actual and desired wealth positions, recognizing that it takes time to change the level of wealth. Given current real income

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9. For most of our results, we need only preclude any infinite rate of change of any asset--any change must involve time. We assume smooth, continuous adjustment which implies convex adjustment costs. For discussions of other specifications of adjustment costs see Mussa. In particular, discrete time might be one method of introducing "adjustment costs" into the model.

10. For example, May recognizes the possible existence of both types of adjustment costs, but (p.3, fn. 5) chooses to ignore the latter, and fails to incorporate the former fully into his analysis.

$\tilde{y}$, and the interest rate r, both of which are then expected to prevail indefinitely in the future, equation (8) gives the individual's long-run desired wealth position, A′, and the equilibrium composition of A′ determines the long-run asset demands L′ and B′, given by equations (6) and (7).

However, given the assumed costs of changing wealth, there exists a second concept of desired wealth, that of the instantaneous desired wealth position. Given $\tilde{y}$ and r, given actual wealth, $\bar{A}$, which differs from long-run desired wealth A′, and given the costs of changing $\bar{A}$; there is a level of wealth that the asset holder will desire to hold at point in time. This we call the instantaneous demand for wealth, A; and its equilibrium composition determines (is determined by?) the instantaneous asset demands L and B.

Consistent with this instantaneous stock demand for wealth there is a flow demand for wealth, or savings relationship. That is, in this model, savings are related to the stock disequilibrium in wealth holdings, $A'-\bar{A}$ as implied by equation (9). Following Friedman, the instantaneous demand at time t, A_t (then) would depend upon the rate of change of $\bar{A}$.

Formally, letting D be the operator d/dt, we have

$$A_t = \alpha(\tilde{y}, r, DA),$$

$$L_t = \psi(\tilde{y}, r, A_t) \quad (10)$$

$$B_t = \gamma(\tilde{y}, r, A_t).$$

The latter then correspond closely to May's demand functions,

10a. Formally, we specify that the rate of change, DA, depends on the asset disequilibrium, or $DA = T(A'-A) = T[h(\tilde{y}, \hat{r}) - \hat{A}]$.
Inverting, we get $T^{-1}[DA] = h(\tilde{y}, \hat{r}) - \bar{A}$. Now, by definition, $A_t = \bar{A}$, so (converting)

especially when we recognize below that $A_t = \widehat{A}_t$.

These considerations have implications for the usual treatment of the stock constraint facing the individual. From the definitions of A and $\widehat{A}$, we can write

$$(L'-\widehat{L}) + (B'-\widehat{B}) = A'-\widehat{A} \ , \qquad (11)$$

where the usual procedure might be to set the right-hand side of (11) identically equal to zero, yielding one version of Walras' Law for stocks--that the sum of the excess demands for assets equals zero. However, this procedure implies that whereas the individual can change the level of his wealth costlessly $[(A'-\widehat{A}) \equiv 0]$, he faces costs in adjusting the composition of his portfolio such that at any point in time his individual asset excess demands are <u>not</u> constrained to zero--precisely the opposite of the specification argued for in this paper. If one accepts the present specification, then $(A'-\widehat{A})$ will not be treated as identically zero, and (11) tells us only the obvious: that the sum of the excess demands for assets equals the excess demand for wealth. Current wealth is not an effective constraint on long-run asset demands.

As argued above, it is obvious that at any point in time, it is A, not A', which is constrained by $\widehat{A}$, so that we may write

$$(L-\widehat{L}) + (B-\widehat{B}) = A-\widehat{A} \ , \qquad (12)$$

where it is now true that at any point in time the right-

10a cont'd: solving: $A_t = \bar{A} = h(\tilde{y}, \tilde{r}) - T^{-1}(\delta A)$

$= \alpha(\tilde{y}, \tilde{r}, \delta A)$.

hand side of (6) is identically equal to zero. However, since there are no costs to changing the composition of $\widehat{A}$, both individual asset excess demands on the left hand side of (12) also are identically zero. In fact, for the individual, given $\widehat{A}$ (and A), it is L and B which at any point in time determine $\widehat{L}$ and $\widehat{B}$. Thus the stock constraint, whether interpreted in terms of (10) or (11), does not yield us any striking information about the individual asset holder: interpreted in terms of long-run asset demands "Walras' Law" becomes an equilibrium condition and is not an identity ; interpreted in terms of instantaneous demands it becomes trivial. From the former we can at least determine whether or not the individual has an excess demand for wealth, but of course this can always be determined directly from savings behaviour.

Thus, in terms of equation (5) above, given expectations and given $\widehat{A}$, the individual at time t_0 chooses c [according to equation (9)] and this serves to determine the R.H.S. of equation (5); the L.H.S. is thus also determined but there is no determination of the individual components of the L.H.S. The flow demands for the individual assets are indeterminate.[11]

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11. Mussa also presents an excellent treatment of a model similar to our Case I in which one of the striking results is that the flow demands for each individual asset are indeterminate. Since the asset holder can "later" rearrange his portfolio costlessly, he is not concerned with what form growth in the portfolio takes, but only with changes in total wealth. This corresponds closely to our results.

Case 2--The Aggregate Economy

Case 2, where both types of adjustment costs are considered to be operative, also has implications for our interpretations of the asset demands and the stock constraint. Again, given $\tilde{y}$ and r, equations (6) to (8) determine the long-run equilibrium values L', B', and A'; any two determining the third. And again, the distinction between long-run demands and instantaneous demands is paramount--however, the added feature of costs to rearranging a portfolio changes our previous analysis somewhat.

Since there are now these costs associated with changing the composition of a given level of wealth, insofar as specification of the instantaneous stock asset demands and flow demands are concerned, the current position is no longer adequately described by current wealth $\widehat{A}$. The individual asset flow demands will now be determinant and will vary, given $\widehat{A}$, with the composition of $\widehat{A}$. Similarily, the instantaneous stock demands, L and B, will vary, for given $\widehat{A}$, with the composition of $\widehat{A}$. Hence knowledge of the current position requires knowledge of $\widehat{L}$ and $\widehat{B}$ (actually, any two of $\widehat{A}$, $\widehat{L}$, and $\widehat{B}$ are sufficient). Then, given $\tilde{y}$ and r, and hence the long-run demands A', L', and B'; and given $\widehat{L}$ and $\widehat{B}$ and the assumed costs of changing $\widehat{L}$ and $\widehat{B}$, we can determine the instantaneous asset demands L and B, and the resulting instantaneous demand for wealth A, $A = L+B$.

Note that in Case 2 there will be an effective instantaneous demand for each asset, and a corresponding effective flow demand for <u>each</u> individual asset at each instant, with the flow demand for wealth, or savings function, equal to the sum of the two individual flow demands. This implies a more general formulation of the consumption function.[12]

$$c = c(\tilde{y}, r, L', \widetilde{L}, B', \widetilde{B}). \qquad (9')$$

That is, given $\widetilde{L}$ and $\widetilde{B}$, there exists a time path of desired holdings of each asset--the desired rates of adjusting. Once these rates are determined there is nothing inherent in the model to prevent them from being attained--the flow demands now become effective demands that serve to determine the endogenous variables in the system.

Hence, at any point in time, the actual stocks will be changing at the (effective) desired rates, and so actual stocks will equal short-run (instantaneous) desired stocks. Flow equilibrium implies short-run stock equilibrium, and the flow demands will equal the time derivatives of the actual stocks, or alternatively, of the instantaneous stock demands.

Again, we can refer to equation (3) to examine the traditional stock constraint, and again, if we interpret this as applying to long-run asset demands, we find only that the sum of the excess demands for assets equals the excess demand for wealth. The special case, sometimes inappropriately referred to a Walras' Law, that requires that sum to be identically zero is a result of the unwarranted assumption that the

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12. This is essentially the formulation I used in "Short-Run Adjustment in Models of Money and Growth," <u>op. cit.</u>

desired long-run level of wealth always obtains.[13] Interpreted in terms of long-run demands, "Walras' Law" is again an equilibrium condition, not an identity.

Again, we can recognize that it is the instantaneous demands that are constrained by current asset holdings, so reproducing equation (12)

$$(L-\widetilde{L}) + (B-\widetilde{B}) = (A-\widetilde{A}), \tag{12}$$

where now it is the case that, by definition, the two individual excess demands on L.H.S. of (12) identically equal zero; that their sum also equals zero is as much a result of the model as a restriction on it. Again, in a position characterized by long-run stock disequilibrium, consideration of the stock constraing does not yield us any important information helpful in analysing the behaviour of the economy over time.

This is not to say that the values of the various stock variables are not important, only that the stock constraint is not a binding one in the adjustment process. It is perhaps worthwhile exploring the relationships between the instantaneous stock demands and the flow demands in some detail.

Assuming the simplest possible case of constant stock adjustment, the flow demands for money and bonds, respectively, are given by equations (13),

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13. Such an assumption characterizes the Neo-Classical model of money and growth. For an analysis of the implications of relaxing that assumption in that context see Purvis, op. cit.

13a. It is interesting to compare this results to those obtained by Donald Tucker.

$$l^d = \lambda_2(L'-\widehat{L}) = \tilde{l}(\bar{y},r,\widehat{L}),$$

and (13)

$$b^d = \lambda_2(B'\widehat{B}) = \tilde{b}(\bar{y},r,\widehat{B});$$

and the respective flow supplies are

$$l^s = -\widehat{L}\pi,$$

and (14)

$$b^s = D\widehat{B} = D\widehat{K} ,$$

where the adjustment coefficients λ_1 and λ_2 have dimensions time^{-1}, π is the rate of change of prices, and K is the (value of the) capital stock, identically equal to $\widehat{B}$.[14]

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14. Under our assumptions, $\pi \neq 0$ only out of equilibrium, and $\pi^c \equiv 0$.

Bonds in the present model represent a claim on a real income stream, i.e., they are equities, and represent claims on the capital stock. This allows us to include the value of bonds in our definition of aggregate wealth. In the ensuing analysis, no attempt will be made to distinguish between the capital stock itself and the financial assets giving claims to ownership of that capital stock.

May erroneously retains the value of bonds in his wealth concept when he aggregates from his model of the individual to his model of the economy, failing to note the netting out effect of the inside bonds of his model [i.e., $\widehat{B}$ in the stock constraint and $r\widehat{B}$ in the flow constraint both go to zero with aggregation. B then is the net excess demand for bonds.] In his model, then, we are left with the unsatisfactory definition of wealth as only outside money--there seems to be no capital stock underlying the given output $\widehat{y}$. Hence our use of equity bonds seems more satisfactory.

A more general formulation would have each flow demand depending upon both stock excess demands. The present case is adhered to for simplicity--a more complete analysis incorporating optimizing behavior is presented in the appendix.

Flow equilibrium in the assets markets requires the respective equality of the flow demands and supplies given in (13) and (14) above, so we have two equations

$$l^d(\tilde{y},r,\tilde{L}) = -\tilde{L}\pi$$
and (15)
$$b^d(\tilde{y},r,\hat{B}) = DK \ ,$$

in two unknowns, π and DK. That is, at a moment in time the capital stock and the rate of interest are given, and flow equilibrium serves to determine the rate of inflation (i.e., the rate of change of actual real balances) and the rate of change of the capital stock.[15]

Alternatively, we could have looked at the instantaneous markets for stocks. From our stock-flow postulates (see also Friedman) we could write

$$L = \tilde{L}(\tilde{y},r,D\tilde{L});\ \partial\tilde{L}/\partial D\tilde{L} < 0,$$
and (16)
$$B = \tilde{B}(\tilde{y},r,D\hat{B});\ \partial\tilde{B}/\partial D\hat{B} < 0.$$

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15. By assumption, $p_k = 1$; so $K = B = (s/p.r)$ where s is the number of equities outstanding and p is the price level. Hence the flow supply of bonds could also be represented as $b^s = \hat{B}(n-\pi-Dr/r)$ where r is the rate of change of s, and $n-\pi$ is then rate of change of real income streams, and again we have two equations in two unknowns, π and Dr, given some determination of n.

That is, given the discrepancy between the actual and ultimate desired stock, the faster the real stock is growing, the smaller is the actual stock the asset holder may be willing to hold at that instant. The long-run asset demands given by equations (6) and (7) then may be treated as special cases of the above instantaneous demand functions arising when $D\tilde{L} = D\tilde{B} = 0$, i.e., arising in positions of long-run asset equilibrium.

Instantaneous stock equilibrium gives rise to the following conditions:

$$L = \psi(\tilde{y}, r, -L\pi)$$

and (17)

$$B = \gamma(\tilde{y}, r, DK) ,$$

where we have substituted for $D\tilde{L}$ and $D\tilde{B}$. Again, treating r as given at an instant of time, we have two equations in the two unknowns, π and DK.

However, from our previous analysis of the flow aspects of the market, and from our simple stock-flow relationships contained in (13), we might write the instantaneous asset demands as

$$L = \tilde{\tilde{L}}(\tilde{y}, r, \tilde{L}),$$

and (18)

$$B = \tilde{\tilde{B}}(\tilde{y}, r, \tilde{B}).$$

However, now analysis of the instantaneous demands does not lead to any solutions to the model as there are no unknowns involved. In addition, the usual adding up properties as applied to the short-run demands, i.e., $\partial\tilde{\tilde{L}}/\partial\tilde{y} + \partial\tilde{\tilde{B}}/\partial\tilde{y}$ are no longer interesting--those partial derivatives now refer to the change in the instantaneous demand for some asset due to a change in the expected level of income--holding the actual asset holdings constant.

Concluding Remarks on the Stock Constraint

When consideration of stock disequilibrium such that assets adjust to their long-run desired levels over time is introduced into the analysis, we have seen that in positions of other than full equilibrium that the stock constraint is not a binding constraint in the usual sense. Instead, the flow constraint given by (19) becomes the binding constraint on the wealth holders:[16]

$$s^d = l^d + b^d = \tilde{y} + rB - c \qquad (19)$$

where s^d is total planned savings and c is consumption.[17] In case I, that corresponding to the individual, only the total flow demand for wealth given by s is determinate, whereas in

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16. $\pi^e = 0$, so R.H.S. of (19) is the excess of expected income over planned consumption, and it is this which constrains planned asset accumulation. Of course, the expost constraint says that actual asset accumulation equalled the excess of actual income, $\tilde{y} + r\tilde{B} - \pi\tilde{L}$, over actual consumption, $\tilde{c}$.

17. It is interesting to compare this to analysis of the theory of investment. For example Foley and Sidrauski state (p. 93):

> The cost of adjustment view emphasizes equilibrium between a flow demand for investment and a flow supply. Our view emphasizes a stock demand for capital and a stock supply.

where "their view" is essentially that presented by Witte whereby the demand for a capital stock determines the price of capital, and investment is simply the output of the capital goods sector forthcoming at that price. --Cont'd--

case 2 both the individual flow demands 1^d and b^d are. When full equilibrium is attained, the stock constraint in terms of the long-run asset demands becomes "effective" whereas the flow constraint, in one limited sense, will no longer be "ineffective" since consumption will identically equal output (i.e., $s^d \equiv 0$, $\pi \equiv 0$ -- the "stationary state").

Finally we conclude that when explicit recognition to existence of costs of adjusting the composition of a portfolio is made, the usual method of analysis (employed by, say, May and by Foley and Sidrauski) is not legitimate without also giving some explicit attention to the ultimate asset demands and the necessary stock-flow adjustments inherent therein. As is argued in Appendix B, it is preferable to use models of dynamic optimization when consideration of short-run behavior is central to the analysis.

It is interesting that the cost of adjustment view emphasizes flow equilibrium in both cases.

III. IMPLICATIONS FOR DYNAMIC ANALYSIS

The considerations presented above are capable of rendering insights into how the economy moves from one position of full equilibrium to another, and in this section we will use a diagrammatic analysis to illustrate the process.

Consider first case I, where at any point in time the asset holders current position is fully described by knowledge of $\widehat{A}$. Given $\widehat{A}$, there is an equilibrium composition which is a function of r--this composition determines the instantaneous demands L and B, and since the composition can, by assumption, be attained, this also serves to determine the actual stocks $\widehat{L}$ and $\widehat{B}$. Thus, for any $\widehat{A}_0$, we can draw the negatively sloped instantaneous asset equilibrium curves A_0A_0 depicted in Figure 2--this curve depicts the instantaneous demand for real balances L as a function of the interest rate, given $\widehat{A}_0$. There exists a whole family of such curves, one for each possible $\widehat{A}$. An increase in $\widehat{A}$ increases the quantity of real balances demanded at each r, hence AA shifts to the right and in the diagram, A_1A_1 corresponds to some $\widehat{A}_1 > \widehat{A}_0$.

Also, for each given value of $\widehat{A}$, there is an interest rate r for which that $\widehat{A}$ is the long run desired wealth, A', i.e., $\widehat{A}_0 = \bar{A}(\bar{y},r_0)$. The locus of such points traces out the long run asset equilibrium curve A'A' in Figure 2.[18]

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18. Note again the analogy to the Marshallian distinction between short- and long-run supply.

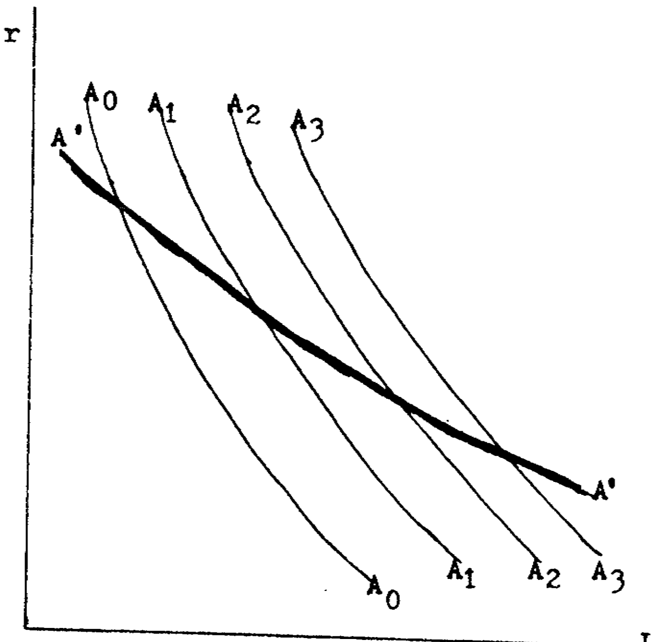

FIGURE 2
STOCK EQUILIBRIUM FOR THE INDIVIDUAL

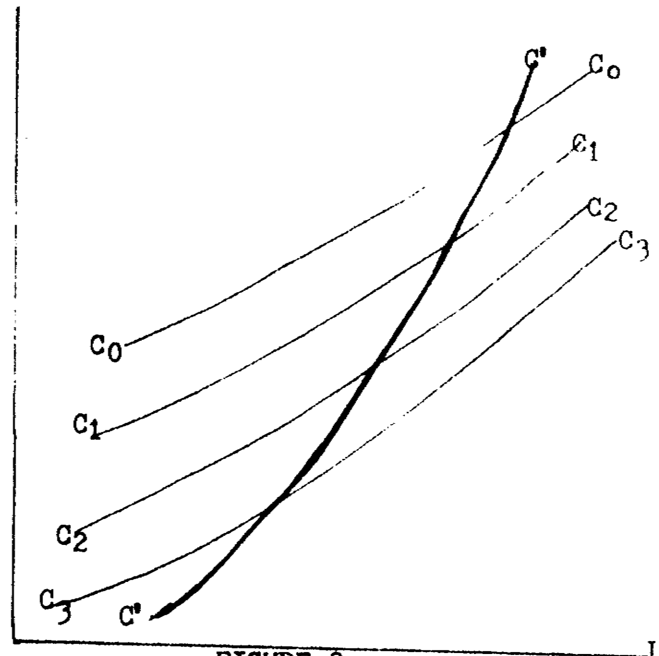

FIGURE 3
FLOW EQUILIBRIUM FOR THE INDIVIDUAL

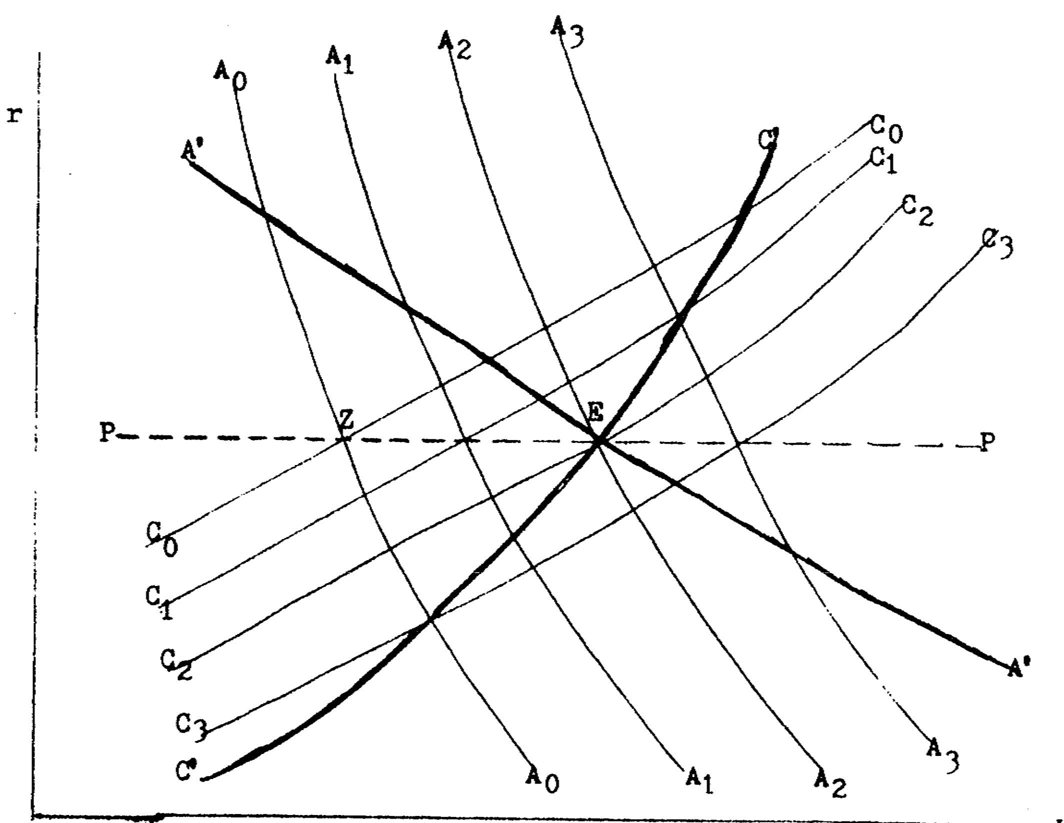

FIGURE 4
DETERMINATION OF INTERTEMPORAL EQUILIBRIUM FOR THE INDIVIDUAL

Flow equilibrium for this individual can also be depicted in r-L space, and is given by the upward-sloping curve C'C' (see for example, May, p. 6) in Fig. 3. An increase in C from such a point as W, generates an excess supply of wealth and serves to reduce saving (possibly becomes negative). This excess supply is eliminated when L falls back to its original level at the original interest rate (back to W), or when the interest rate rises to "rationalize" this new higher stock (point V). At the new higher interest rate at Z wealth has adjusted to its long-run equilibrium level. If, however, we fixed the level of wealth initially, the interest rate would only rise to equilibrate the composition of wealth. At this level savings would then be such as to adjust the level of wealth over time as given by equation (9). Hence for any value of wealth, $\bar{A}_0$ say, we would have a flatter "flow equilibrium" curve C_0C_0, and again there would be a family of such curves, one for each possible level of wealth--see Figure 3.

The complete picture is then seen by superimposing Figure 3 on Figure 2, as is done in Figure 4. Equilibrium at any instant is given by the intersection of the AA and CC curves corresponding to the given $\bar{A}$; the long-run equilibrium is determined by the intersection of the C'C' and A'A' curves. The individual adjusts over time along the locus of intersections of corresponding AA and CC curves--that adjustment path is labelled PP in Figure 4. At a point such as Z there is excess demand for wealth and the individual will be saving;

this causes $\tilde{A}$ to rise and the respective AA and CC curves shift to the right, and he approaches his long-run equilibrium position E.

Case 2, the case for the aggregate economy, is somewhat more complicated--in particular, we can no longer use a single curve to represent either the short-run stock equilibrium or flow equilibrium. In fact, the analysis of short-run stock equilibrium becomes singularily unhelpful (as argued above, pp. 18-19), hence we turn our attention to the flow equilibrium conditions.

As we have seen above, the flow demands for each asset are determinate, so that at any point in time we can conceive of a set of three flow equilibrium curves as depicted in Figure 5. The six quadrants represent conditions in the three markets as summarized in the following table.

Quadrant \ Market	goods	money	bonds
I	excess demand	excess supply	excess demand
II	excess supply	excess supply	excess demand
III	excess supply	excess demand	excess demand
IV	excess supply	excess demand	excess supply
V	excess demand	excess demand	excess supply
VI	excess demand	excess supply	excess supply

By the flow constraint, the three curves must have a common intersection which determines the instantaneous equilibrium ~~An increase in $\tilde{L}$ creates an excess supply of money and excess~~) compatible with the given value of $\tilde{L}$. Hence from equation (15) the rate of inflation at E_1 must be such as to render $\widetilde{L}$ consistent with flow equilibrium in the money market while simultaneously the rate of interest and rate of capital accumulation must be such as to render the given $\widetilde{B}$ compatible with flow equilibrium in Bonds.

Consider an initial configuration of $\widetilde{y}$, r, $\widetilde{L}$ and $\widetilde{B}$ such that there is long-run excess demand for both assets. The average individual will then be refraining from consumption (see equation (9′)) in order to accumulate assets--thus there will be some rate of fall of prices at which the goods market will clear. Note that we are in a strict neo-classical world where no separation of investment-savings decisions is considered--the savings decision to accumulate capital is directly channelled into a flow demand for capital goods. The equilibrium E_1 depicted in Figure 5 is, of course, only a "snapshot" of the economy at a moment in time--as the price-level changes the value of $\widetilde{L}$ changes and the intersection of the flow equilibrium curves moves to the right and the long-run or stationary state equilibrium is attained when that intersection coincides with some point on the long-run asset equilibrium curve A′A′ derived above. This is depicted as E* in Figure 6 below. At this point, all stock variables are constant and all income is consumed--the flow demands for both assets are zero.

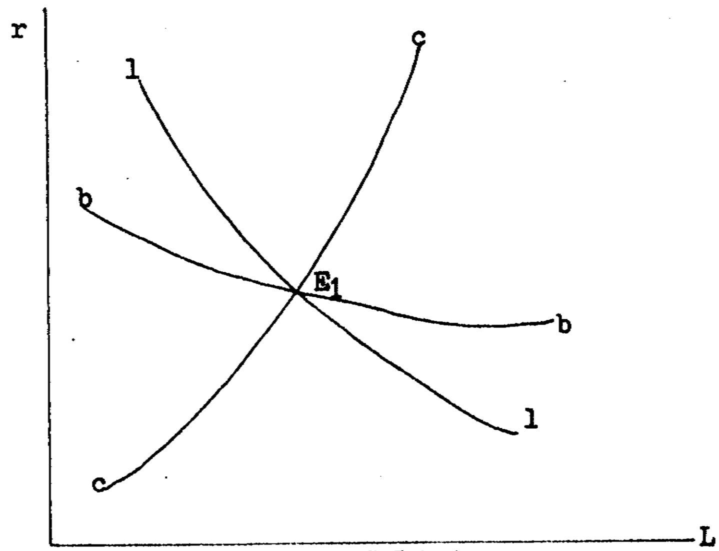

FIGURE 5
INSTANTANEOUS EQUILIBRIUM FOR THE ECONOMY

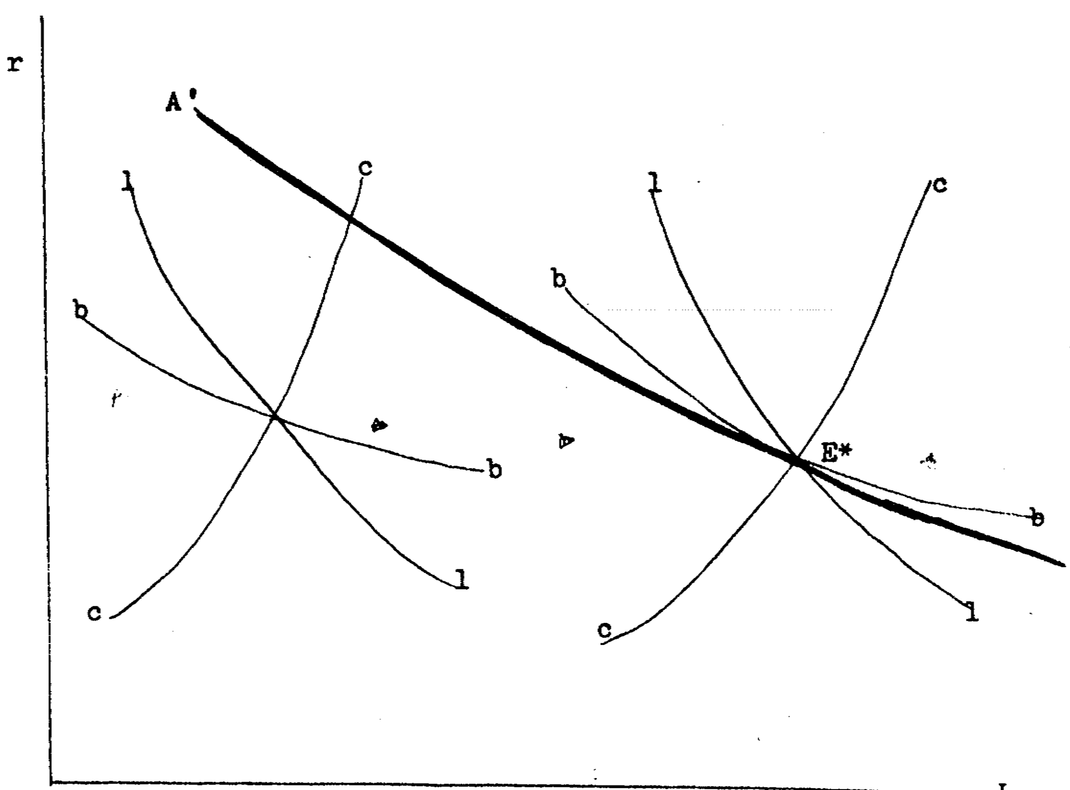

FIGURE 6
INTERTEMPORAL AND FULL EQUILIBRIUM FOR THE AGGREGATE ECONOMY

APPENDIX

Optimizing Behavior and the Specification of Asset Demands*

This appendix represents an attempt to provide rigorous underpinnings for the asset demand functions derived and used in the text. A well recognized common deficiency of recent macro-models incorporating asset behavior is the lack of such underpinnings.[1] In attempting to utilize the calculus of variations to "solve" for the time path of desired asset holdings (see Figure 1, p. 3), we hope to alleviate some of the confusions prevailing in the current literature that we discuss above, and to provide support for the specifications argued for in the text.[2] We consider in turn the two cases corresponding to the different specifications of adjustment costs used in the text.

Case I. Theory of the Individual

Consider the representative individual having preferences defined over goods and money so that his welfare at any point in time is given by a (time invariant) utility function.

(A1) $$U_t = U(c_t, Z_t),$$

where c_t and Z_t are respectively the flow of consumption and the flow of services from holding (a stock of) real

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* Highly tentative and preliminary; comments anxiously solicited.

1. See, for example, Foley and Sidrauski, p. .

2. Our indebtedness to the pathbreaking work of Miguel Sidrauski (and earlier, Frank Ramsey) will become apparent.

balances. For simplicity we assume a unit factor of proportionality between the flow of services and the stock held, and hence we can rewrite (A1) as

$$U_t = U(c_t, L_t). \tag{A2}$$

Now, we employ the common technique of adding up all utility streams as a construct to measure welfare;[3] thus we consider this individual as acting to maximize the present value of his future utility stream given by the utility functional

$$W = \int_0^\infty [U(c_t, L_t)]e^{-\delta t}dt, \tag{A3}$$

where δ is his constant subjective rate of time preference. As indicated in the text, this maximization process is subject to two constraints--the stock constraint (3) and the flow constraint (5); so we form the maximand

$$\text{(A4)} \quad I = \int_0^\infty \{U(c_t, L_t) + \lambda_t[y+rB_t-c-DA] + q_t[A_t-L_t-B_t]\}e^{-\delta t}dt$$

$$= \int_0^\infty F(c_t, L_t, B_t, A_t, DA_t, t)dt.$$

This specification reflects the arguments made in the text whereby the individual is free to allocate his existing stock of wealth costlessly between the two assets and hence his flow decision at a moment of time concerns only

3. See Koopmans for an excellant discussion.

his rate of change of total wealth, DA, and not the rate of change of the individual assets. That is, at any moment, he makes two decisions: a stock decision relating to the allocation of his existing wealth; and a flow decision concerning the decision of his income between consumption and overall wealth accumulation.

Then, following that argument, we get the well known Euler equations (A5-A8) as the first order conditions for a maximum (of A4) and the transversality condition (A9):

$$\text{(A5)} \quad \frac{\partial F}{\partial c_t} = 0 \quad \Rightarrow \quad U_c(c_t, L_t) = \lambda_t$$

$$\text{(A6)} \quad \frac{\partial F}{\partial L_t} = 0 \quad \Rightarrow \quad U_L(c_t, L_t) = q_t = \lambda_t \cdot \rho_t$$

$$\text{(A7)} \quad \frac{\partial F}{\partial B_t} = 0 \quad \Rightarrow \quad r = \rho_t$$

$$\text{(A8)} \quad \frac{\partial F}{\partial A_t} - D\left(\frac{\partial F}{\partial DA_t}\right) = 0 \quad \Rightarrow \quad D\lambda/\lambda = \delta - \rho_t$$

$$\text{(A9)} \quad \lim_{t \to \infty} A_t \lambda_t e^{-\delta t} = 0$$

where λ_t is the implicit price of consumption and $\rho_t = q_t/\lambda_t$ is the implicit return on holding bonds. The transversality condition (A9) arises due to the infinite time horizon assumption, and, roughly, requires that a zero value be attached to any wealth in existence at time t, as t approaches infinity.

From the equations (A5) and (A6) we can write the flow demand for consumption goods and the (instantaneous) demand for real balances as functions of λ_t and ρ_t, or

(A10) $$c_t = c(\lambda_t, \rho_t)$$

(A11) $$L_t = L_0(\lambda_t, \rho_t),$$

and equation (A7) says that the individual will hold bonds to the point where his implicit return ρ_t equals the market rate of interest r, or

(A12) $$B_t = B_0(r)$$

These equations are all derived for the given condition of the initial stock of wealth[4] and for the given rate of income $\bar{y}$, so from the stock constraint we can solve to write the demands as

(A13) $$\begin{aligned} c_t &= c_1(A_t, \lambda_t, r) \\ L_t &= L_1(A_t, \lambda_t, r) \\ B_t &= B_1(A_t, \lambda_t, r). \end{aligned}$$

Equations (A13) give the time path of optimal asset holdings and consumption given the initial wealth, A_t, and the level of income $\bar{y}_t$ and interest rate r. However these are not suitable for inclusion in a macro model to be empirically implemented due to the non-quantifiable argument λ_t. Thus, to write the demands in a full macro model re-

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4. Note that knowledge of the total wealth position A_t adequately describes the individual's stock position at any point in time.

quires the heuristic argument that, due to the assumptions of time invariant preferences and a constant rate of time preference, the implicit price of consumption depends on wealth, income and interest rates. Hence we write

$$c_t = c(\bar{y},\bar{r},A_t),$$

(A14) $$L_t = \psi(\bar{y},\bar{r},A_t),$$

$$B_t = \gamma(\bar{y},r,A_t),$$

which are essentially the demand specifications argued for in the text. The last two give the "instantaneous" asset demands as a function of the parameters $\bar{y}$ and $\bar{r}$ and the current level of wealth whereas the first equation gives the optimal division of the current income stream between consumption and accumulation; this division again depending on current wealth.[5] These are essentially the specifications which appear in the text.

5. A problem arises in terms of "steady state" analysis which requires the costate variable λ_t to be constant. This in turn requires $\rho_t = \delta$. But $\rho_t = r$ so steady state requires the equality of the two parameters, $\delta = r$; clearly a very special case. For the theory of the individual one might let δ be a function of actual wealth and the above would then determine the equilibrium wealth position. For the aggregate case analysed below the rate of interest becomes endogenous. (An alternative approach which occurs in the well known life-cycle model is to consider the various individuals in the economy having different rates of time preference and the rate of interest determined by their interactions in the market place.)

Case 2. Theory of the Aggregate Economy

Before proceding to consider the role played by adjustment costs, it will be useful to analyse further the basic model. We wish to analyse the system developed in Case I above modified to allow for the interest rate to be endogenously determined rather than simply a parameter.

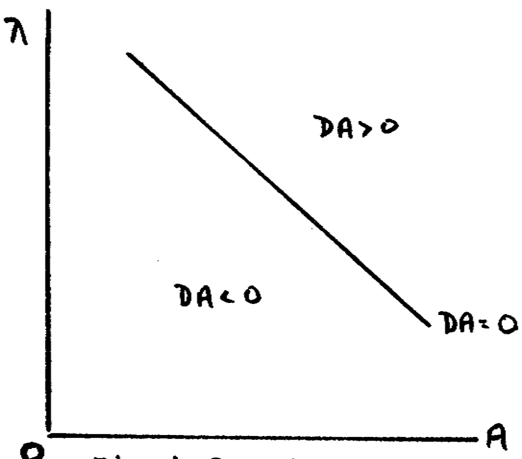
The laws of motion of the system are illustrated in Figure 7 where the state variable A_t is represented on the horizontal axis and its co-state variable along the vertical. The basic differential equations are reproduced here as

(5) $$DA = y+rB-c,$$

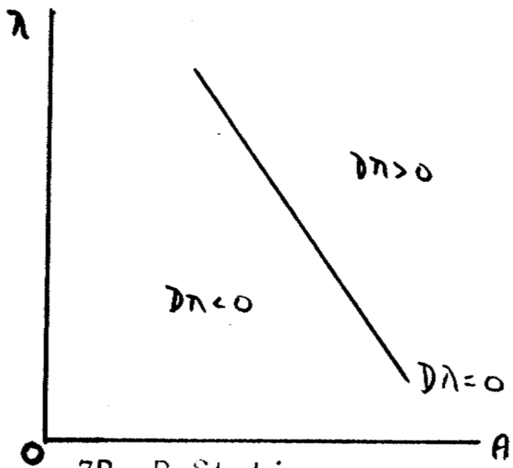
(A8′) $$D\lambda = \lambda(\delta-\rho_t)$$

In Figures 7A and 7B we draw the A stationary and the λ-stationary respectively. In 7A we argue that an increase in consumption reduces the implicit price of consumption, λ_t, and also the rate of accumulation by (5). Hence in order to restore a zero rate of accumulation at the higher rate of consumption, income must be higher-- i.e. wealth must be higher and hence the A stationary is negatively sloped. Above the A-stationary there is positive accumulation, and below, decumulation.

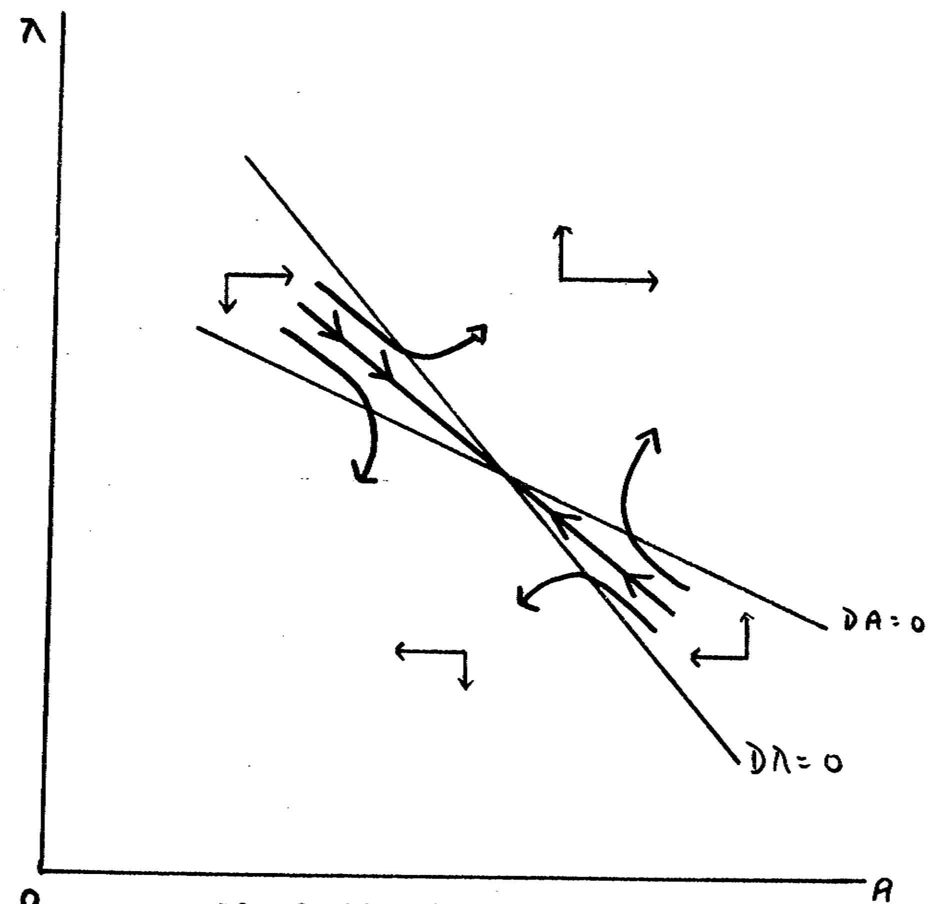
In Figure 7B, we argue that an increase in λ_t causes ρ_t to fall and hence $D\lambda$ to rise. In order to restore $D\lambda = 0$ at the new higher λ requires a fall in q_t - "the implicit price of wealth", so wealth rises and the λ-stationary is



7A A Stationary



7B B Stationary



7C Saddle Point

also negatively sloped. Above the λ-stationary $D\lambda$ is positive and conversely below.

In Figure 7C the unstable case occurring when the A stationary is steeper than the λ stationary is ignored, and we observe that we get a saddle point solution where the stable branch is "picked" by the transversality condition. The equilibrium corresponds to the steady state solution for A that all income is consumed, and the division of A between money and capital gives rise to a capital stock such that the marginal product of capital equals the discount rate.

The Role of Adjustment Costs

We now wish to modify the above analysis to allow for the role of costs of adjustment in attaining the desired portfolio mix. As argued in the text, this implies that a complete specification of the model would involve initial values of both assets individually, and in turn, we expect to achieve individual flow demands for each asset. Two possible models are examined--the first being the case of fixed per unit costs (the case of, say, brokerage fees) and the second, derivative from the theory of investment, the case of a quadratic cost function.

A. Brokerage Fees

Here we assume that the purchase of bonds requires payment of a fixed fee, τ, per unit value of transaction, so that the flow constraint becomes

(A15) $$DA = y+rB_t-c-\tau DB_t$$

where it is now obvious that the rate of asset accumulation not only is constrained by consumption but also by the composition of the additions to wealth. We form the new maximand

(A16) $$\mathcal{I}_1 = \int_0^\infty \{U(c_t,L_t)+\lambda[y+rB_t-c-DA-\tau DB_t] + q_t[A_t-L_t-B_t]\} e^{-\delta t} = \int_0^\infty F(c_t,L_t,B_t,DB_t,A_t,DA_t,t)dt.$$

The first order conditions then give rise to the following Euler equations:

(A17) $$\frac{\partial F}{\partial c} = 0 \quad \Rightarrow \quad U_c(c_t,L_t) = \lambda_t$$

(A18) $$\frac{\partial F}{\partial L_t} = 0 \quad \Rightarrow \quad U_L(c_t,L_t) = q_t = \lambda_t \cdot \rho_t$$

(A19) $$\frac{\partial F}{\partial B_t} - D\left(\frac{\partial F}{\partial DB_t}\right) \quad \Rightarrow \quad r = e_* + \tau(\delta-\dot{\lambda}/\lambda)$$

(A20) $$\frac{\partial F}{\partial A_t} - D\left(\frac{\partial F}{\partial DA_t}\right) \quad \Rightarrow \quad \rho_t = \delta - \dot{\lambda}_t/\lambda_t$$

and the transversality condition

(A21) $$\lim_{t \to 0} \lambda_t A_t e^{-\zeta t} = 0$$

Substituting from A20 we can rewrite A19 as

(A19') $$r = \rho_t(1 + \Upsilon)$$

which indicates the "distortion" in the bond market caused by the introduction of transactions causes. Again, from (A17) and (A18) we could write demand functions for consumption and money as functions of the initial conditions, B_0 and L_0, and λ and ρ. And again from the bond equation and the stock constraint we could now substitute r for e_t, except now we notice that λ_t and ρ_t are not constants but will change along the optimum adjustment path according to (A19) and (A20). This change in the implicit price of consumption and the implicit return to capital reflects the change in the own rates of interest as the rates of change of the two assets change--hence our specification of the instantaneous stock demand functions given by equations (16) in the text, reproduced here as (A22) and A23).

(A22) $$L = \psi(y, r, \widehat{DL}),$$

and

(A23) $$B = \gamma(\bar{y}, r, \widehat{DB}),$$

where implicitly the assumption has been made that $\partial\psi/\partial DB = \partial\gamma/\partial DL = 0$.

Together, these of course imply an instantaneous demand for wealth, $A = L+B$, and the consumption function

(A24) $$c = c(\bar{y}, r, L, B),$$

which derives from the postulate that instantaneous asset demands are held, i.e. $L = \tilde{L}, B = \hat{B}$.

B. Quadratic Costs[6]

In this section we wish to postulate that the price of bonds (capital) be an increasing function of the rate of accumulation--this is analogous to the Penrose Effect as used by Uzawa in his analysis of aggregative growth models. In this model we formulate the flow constraint as follows:

(A25) $$DL = y + rB_t - c - E(DB)$$

where $E(B)$ is the "expenditure" on bonds at any moment. The usual specification is that $E(0) = 0$, $E(DB) > 0$, $E'(DB) > 0$, $E''(DB) > 0$, which implies the quadratic

(A26) $$E(DB) = e_0 DB + e_1 (DB)^2$$

so the price of bonds, π_{B_t}, depends on the rate of accumulation,

(A27) $$\pi_{B_t} = e_0 + e_1 DB.$$

Finally we note that from the stock constraint,

(A28) $$DA = DL + DB.$$

This, combined with the fact that our initial conditions are given by L_0 and B_0, means that the problem is reduced to one involving only one constraint; (A25) above;

6. For a similar approach as applied to the theory of investment, see J. Gould.

hence, we form the maximand.

(A29) $$I = \int_0^\infty \{U(c_t, L_t) + \lambda[y + rB_t - c_t - DL - E(DB)]\} e^{-\delta t} dt$$

$$= \int_0^\infty F(c_t, L_t, DL, B_t, DB, t) dt.$$

Again, the first order conditions for a maximum are given by the following Euler conditions:

(A30) $$\frac{\partial F}{\partial c_t} = 0 \quad \Rightarrow \quad U_c(c_t, L_t) = \lambda_t$$

(A31) $$\frac{\partial F}{\partial L_t} - D\left(\frac{\partial I}{\partial DL_t}\right) = 0 \Rightarrow U_L(c_t, L_t) = \lambda_t[\zeta - \dot{\lambda}/\lambda]$$

(A32) $$\frac{\partial F}{\partial B_t} - D\left(\frac{\partial I}{\partial DB_t}\right) = 0 \Rightarrow r = [\zeta - \dot{\lambda}/\lambda][e_0 + 2e_1 DB] - 2e_1 D^2 B$$

Before trying to use these results to derive demand specifications, it may be worthwhile attempting to deal with (A32) in more detail.

(A32) is equivalent to the condition that

(A33) $$\int_0^\infty \left(\Delta B \frac{\partial F}{\partial B} + \Delta DB \frac{\partial F}{\partial DB}\right) dt = 0,$$

which states that in order for the existing capital stock to be an equilibrium one, the discounted value of an increment to that capital stock equals the increase in the discounted value of the adjustment costs associated with the increased rate of accumulation.

Again, by reasoning similar to that used above, we can solve for the consumption function and the instantaneous demand for real balances as functions of λ_t, the initial condition $\widehat{B}_0$ and $\widehat{L}_0$, and the rate of change of λ, $D\lambda$. Then using (A32) we can substitute for r, yielding demand equation in terms of B_0, L_0, λ, r, DB and D^2B. Ignoring D^2B and making the same heuristic heroic arguments as above, we derive specification of asset demand of the kind in equations (16) in the text.

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