Queen’s Economics Department Working Paper No. 6

INVESTMENT FUNCTIONS:WHICH PRODUCTION FUNCTION?

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11-1969

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Discussion Paper No. 6

November, 1969

Recent studies of the determination of real fixed capital formation have been dominated by the approach suggested by Dale W. Jorgenson.[1] One deficiency of this approach is the need to specify an explicit form for the production function. This specification is not an integral feature of the neo-classical theory of optimal capital accumulation[2] but it is an essential pre-requisite for econometric analyses. Jorgenson and his associated authors have always chosen the Cobb-Douglas form and they have argued that this specification is consistent with the results established in other areas of econometric research. Other economists have disputed the specification[3] and, in particular, they have suggested that the use of the Cobb-Douglas form must lead to an unjustified emphasis on the role of relative prices in the determination of investment expenditures. In this paper, an approximation is suggested whereby the alternative C.E.S. form can be fitted as a number of additional corrections to the final expression associated with the Cobb-Douglas form. Some numerical results are tabulated and these indicate that, for the British Economy at least, the specification of a Cobb-Douglas form for the production function is inappropriate.

This research was completed at the London School of Economics in May, 1969. The topic was suggested by J. D. Sargan as part of an investigation into the determinants of real fixed capital formation within the post-war British Economy.

1. 'The Brookings Quarterly Econometric Model of the United States', North-Holland, 1965. Edited by J. S. Duesenberry, et. al., Chapter 2. 'Capital Theory and Investment Behaviour', The American Economic Review, May 1963.

2. 'Determinants of Investment Behaviour', National Bureau of Economic Research, 1967. Edited by R. Ferber. Pages 129-156. This outline of "The Theory of Investment Behaviour" by Jorgenson contains no reference to the specification of a production function.

3. R. Eisner and M.I. Nadiri, Review of Economics and Statistics, August 1968.
R. Eisner, American Economic Review, June 1969.
R. Coen, American Economic Review, June 1969.

Desired Capital Services

We shall use the following notation:

X	Real Output
K	Capital Services
L	Labour Input
p	Price of Final Output
q	Price of Investment Goods
c	"User-Cost" of Capital Input
ϕ	Adjustment factor, taking account of the influences of tax provisions and a discount rate.

The C.E.S. form of the production function can be written as the following expression.

$$(1) \qquad X^{\frac{\rho}{\gamma}} = \alpha_0 K^{\rho} + \beta L^{\rho} ,$$

where ρ, γ, α_0 and β_0 are given parameters.

Differentiating partially, we obtain:

$$(2) \qquad \frac{\partial X}{\partial K} = \alpha_0 \gamma \; X^{(1-\frac{\rho}{\gamma})} \; K^{(\rho-1)} \text{ , so that}$$

$$(3) \qquad K = (\alpha_0 \gamma)^{\frac{1}{1-\rho}} \left(\frac{\partial X}{\partial K}\right)^{\frac{-1}{1-\rho}} \; X^{\frac{1}{1-\rho}(1-\frac{\rho}{\gamma})}$$

The following alternative expression would have resulted if we had used the Cobb-Douglas form.

$$(1a) \qquad X = MK^{\alpha}L^{\beta} ,$$

where M, and are given parameters.

$$(3a) \qquad K = \alpha\left(\frac{\partial X}{\partial K}\right)^{-1} X$$

We define user-cost as the price of investment goods after certain adjustments for tax provisions, longevity and discounting. These adjustments will be listed explicitly in a later section.

$$(4) \qquad c \equiv q/\phi$$

To obtain optimality, we set the marginal product of capital services equal to the adjusted price-ratio.

(5) $$\frac{\partial X}{\partial K} = \frac{c}{p} = \frac{q}{p\phi}$$ (Optimality Condition)

Let (K*) represent the optimal K when the Cobb-Douglas form is used and let (*K) represent the optimal K when the C.E.S. form is used. We shall speak of (K*) and (*K) as defining "desired capital services".

(6) $$(^*K) = \alpha_1 \left(\frac{p}{q}\right)^{m_1} X^{m_2} \phi^{m_3},$$

where $$m_1 = \frac{1}{1-\rho} = m_3$$

$$m_2 = m_1\left(1 - \frac{\rho}{\gamma}\right)$$

and $$\alpha_1 = (\alpha_0 \gamma)^{m_1}$$

In the special case specifying constant returns to scale for the production function, γ is unity and we obtain the following simplifications.

$$m_1 = m_2 = \frac{1}{1-\rho}$$ as before

$$m_2 = 1$$

$$\alpha_1 = \alpha_0^{m_1}$$

In the general case,

(7) $$\rho = \frac{m_1 - 1}{m_1} = \frac{n_1}{n_1 + 1}$$

and

(8) $$\frac{1}{\gamma} = \frac{m_1 - m_2}{m_1 - 1} = \frac{n_1 - n_2}{n_1}$$ if we define

$$n_i \equiv m_i - 1 \quad \text{for} \quad i = 1, 2, 3.$$

(9) $$(K^*) = \alpha\left(\frac{p}{q}\right) X\phi$$

The elasticity of substitution, σ, is equal to m_1, or $(n_1 + 1)$. As m_1 tends to unity and as n_1 tends to zero, the C.E.S. function may be replaced by the Cobb-Douglas function and (*K) may be replaced by (K*). Jorgenson's expression for desired capital services, (K*) may be considered as a special case of the variable (*K) defined by equation (6). It is obtained by setting m_1, m_2 and m_3 equal to unity, or n_1, n_2 and n_3 equal to zero. That is, Jorgenson's specification of the Cobb-Douglas form leads directly to the implication that the elasticities of desired capital services with respect to relative prices, output and the composite tax variable have a common value, unity. This is the feature of Jorgenson's work that has attracted most criticism.

Notice that there is no reason why comparisons should be restricted to the simple choice between the two characterizations of productive processes. We can consider the equation (6) as a definition of a family of concepts of desired capital service generated by setting the four parameters at different levels. The C.E.S. and Cobb-Douglas characterizations of the productive process provide constraints on 'suitable' members of this family.

Define (10) $\quad (*K*) \equiv \alpha_1 (\frac{p}{q}) X \cdot \phi$

Then, $\quad (*K) = (*K*) (\frac{p}{q})^{n_1} X^{n_2} \phi^{n_3}$

For n_1, n_2, n_3 sufficiently close to zero, we can approximate

$$(\frac{p}{q})^{n_1} X^{n_2} \phi^{n_3} \quad \text{by} \quad [1 + n_1 \log_e(\frac{p}{q}) + n_2 \log_e X + n_3 \log \phi]$$

Then,

$$(11) \qquad (*K) \simeq (*K*)[1 + n_1 \log_e(\frac{p}{q}) + n_2 \log_e X + n_3 \log \phi]$$

$$(12) \qquad \nabla(*K) \simeq \nabla(*K*) + n_1 \nabla[(*K*) \log_e(\frac{p}{q})] + n_2 \nabla[(*K*) \log_e X] + n_3 \nabla[(*K*) \log_e \phi] ,$$

where ∇ is the difference operator defined, for any variable Z_t, by the equation $\nabla Z_t = Z_t - Z_{t-1}$.

(13) $$\nabla(*K) \approx \nabla(*K^*) + n_1\nabla[(*K^*)\log_e(\frac{p\phi}{q})] + n_2\nabla[(*K^*)\log_e X]$$ if n_1 and n_3 are equal.

(14) $$\nabla(*K) \approx \nabla(*K^*) + n_1\nabla[(*K^*)\log_e(\frac{p\phi}{q})]$$ if n_1 and n_3 are equal and n_2 is zero.

Table 1 contains a summary of the prior restrictions imposed on the coefficients of this approximation (11) by the choice of production constraint and by the specification of constant returns to scale.

TABLE 1.

Specification	n_1	n_2	n_3
Cobb-Douglas	0	0	0
C.E.S.	$(\sigma - 1)$	NONE	$(\sigma - 1)$
C.E.S. with constant returns to scale	$(\sigma - 1)$	0	$(\sigma - 1)$

The desirable feature of the approximation (11) is its convenience for testing the validity of the prior constraints using conventional statistical methods.

Some Numerical Results

Two groups of equations were fitted to quarterly data[4] for the British Economy extending, for the dependent variable, from 1958 to 1965.

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4. A description of the series and an account of their derivation is available from Queen's University.

The first group ignored the prior restrictions on $(n_i : i=1, 2, 3)$ and the second group were based on the specification of a C.E.S. form for the production function.

$$(15)\qquad (I-R)_t = b_1(I-R)_{t-1} + \sum_{j=3}^{4} a_{1_{j-2}} \nabla(*K*)_{t-j} + \sum_{j=3}^{4} c_{1_{j-2}} \nabla(*K* \log \frac{p}{q})_{t-j}$$

$$+ \sum_{j=3}^{4} c_{2_{j-2}} \nabla(*K* \log X)_{t-j} + \sum_{j=3}^{4} c_{3_{j-2}} \nabla(*K* \log \phi)_{t-j}$$

$$(16)\qquad (I-R)_t = b_1(I-R)_{t-1} + \sum_{j=3}^{4} a_{1_{j-2}} \nabla(*K*)_{t-j}$$

$$+ \sum_{j=3}^{4} c_{4_{j-2}} \nabla(*K* \log \frac{p\phi}{q})_{t-j} + \sum_{j=3}^{4} c_{2_{j-2}} \nabla(*K* \log X)_{t-j}$$

$$(17)\qquad \phi = \frac{1 - u}{(1 - uv_3 - uv_4)(\delta + r) - uv_1\delta}$$

Additional Notation:

- R Replacement Investment
- u Corporate Tax-rate
- v_1 Rate of Annual, or Wear and Tear, Allowances
- v_3 Rate of Initial Allowances
- v_4 Rate of Investment Allowances
- r Dividend yield on industrial ordinary shares
- δ Index of capital longevity.

The form of the distributed lag imposes the following non-linear restrictions on the coefficients of the two groups but these restrictions were not used in estimation. They may be interpreted as over-identifying restrictions on the structural parameters $(m_i ; i = 1, 2, 3)$.

(m_1) $\qquad c_{11}/a_{11} = c_{12}/a_{12}$

(m_2) $\qquad c_{21}/a_{11} = c_{22}/a_{12}$

$$(m_3) \qquad c_{31}/a_{11} = c_{32}/a_{12}$$

$$(m_1, m_3) \qquad c_{41}/a_{11} = c_{42}/a_{12}$$

Whenever a zero restriction was imposed on b_1, the unconstrained estimates confirmed these over-identifying restrictions. That is, the suppression of lagged investment from the list of regressors produced two similar estimates for each of the structural parameters. When b_1 was non-zero, the first and fourth restrictions were confirmed by the unconstrained estimates.

Some of the empirical results for these groups of equations are tabulated in Table 2. Each cell of the table contains the estimated coefficients of the variable shown to the left of the table, its estimated standard error and the corresponding Student's t statistic. The estimated coefficients of the additional regressors are not direct estimates of n_1, n_2 and n_3 since they contain the scale factor α_1. Similarly the coefficients of the simple change-in-desired-capital variables must be reinterpreted as they do not correspond to the elasticity of output with respect to capital from the production function if the Cobb-Douglas form of this function is not used.

The C. E. S. hypothesis is that

(i) $c_{1j} = c_{3j}$ for $j = 1, 2$; and

(ii) both c_{1j} and c_{3j} $(j = 1, 2)$ are non-zero.

In the second group of equations, these two conditions imply that c_{41} and c_{42} are non-zero.

TABLE 2. Additional Regressors. Section One.

Variable	8.8	8.8a	8.8b	8.11	8.11a	8.11b
I_{t-1}	.6156	.3868	.3991			
	.1033	.1265	.1326			
	5.9564	3.0589	3.0088			
∇(*K*)						
	.2938	.4346	.4258	.6459	.6643	.6646
t-3	.0710	.0835	.0879	.0571	.0414	.0430
	4.1412	5.2062	4.8441	11.3161	16.0389	15.4675
	.2535	.3987	.3925	.6459	.6386	.6400
t-4	.0741	.0863	.0903	.0570	.0409	.0425
	3.4211	4.6191	4.3473	11.0163	15.6030	15.0706
∇(*K*log $\frac{p\phi}{q}$)		- .0614	- .0816		- .1013	- .0952
t-3		.0305	.0554		.0313	.0629
		-2.0157	-1.4744		-3.2425	-1.5138
		- .0666	- .0481		- .1204	- .0990
t-4		.0331	.0592		.0318	.0647
		-2.0102	- .8119		-3.7820	-1.5313
∇(*K*log X)						
			.0018			- .0006
t-3			.0042			.0047
			.4296			- .1267
			- .0018			- .0017
t-4			.0040			.0046
			- .4437			- .3728
$100R^2$	91.82	93.57	93.67	82.16	91.42	91.47

TABLE 2. Additional Regressors. Section Two.

Variable	8.8c	8.8d	8.8e	8.11c	8.11d	8.11e
I_{t-1}	.4946	.4047	.4298			
	.0985	.1209	.1318			
	5.0228	3.3477	3.2600			
∇(*K*)						
	.3549	.4108	.4104	.6324	.6524	.6685
t-3	.0652	.0804	.0879	.0470	.0415	.0440
	5.4396	5.1114	4.6709	13.4514	15.7277	15.1887
	.3176	.3755	.3737	.6138	.6278	.6430
t-4	.0685	.0830	.0908	.0471	.0410	.0435
	4.6383	4.5226	4.1157	13.0330	15.3219	14.7784
∇(*K*log $\frac{p}{q}$)						
	- .3084	- .2528		- .5107	- .2754	
t-3	.1191	.1263		.1519	.1480	
	-2.5888	-2.0016		-3.3620	-1.8603	
	- .3297	- .3160		- .4987	- .3458	
t-4	.1198	.1268		.1558	.1485	
	-2.7511	-2.4915		-3.2003	-2.3285	
∇(*K*log ϕ)						
		- .0488	- .0567		- .0915	- .1031
t-3		.0332	.0351		.0360	.0371
		-1.4680	-1.6149		-2.5416	-2.7809
		- .0203	- .0597		- .0812	- .1287
t-4		.0384	.0389		.0396	.0377
		- .5291	-1.5364		-2.0489	-3.4188
$100R^2$	94.08	94.57	93.00	88.75	92.23	90.34

The specifications indexed by 8.8 and 8.11 were used as the bases for the analysis of the production constraints. Five lists of additional regressors were combined with each base and the supplementary indices (a, b, c, d, e) were used. The first two indices correspond to lists of regressors indicated by the equation (16) and the remainder correspond to lists of regressors indicated by (15). Results for the former equation are listed in the first section of Table 2 and those for the latter equation are listed in the second section of that table.

Within the frameworks of (8.8, 8.8a, 8.8b) and (8.11, 8.11a, 8.11b), conventional significance tests indicate a choice of 8.8a and 8.11a. The same choice is indicated by the alternative standard-error criterion.[5] These results give overwhelming support for the choice of the C. E. S. form of the production function with constant returns to scale in preference to either the choice of the alternative Cobb-Douglas form or the choice of the C. E. S. form with the returns to scale in production to be determined. We obtain the following estimates of structural parameters.

$$(8.8a) \quad \left.\begin{aligned} c_{41}/a_{11} &= -.141 \\ c_{42}/a_{12} &= -.167 \end{aligned}\right\} \text{ imply } m_1 = m_3 = .86 \text{ or } .83,\ m_2 = 1.$$

$$(8.11a) \quad \left.\begin{aligned} c_{41}/a_{11} &= -.153 \\ c_{42}/a_{12} &= -.188 \end{aligned}\right\} \text{ imply } m_1 = m_3 = .85 \text{ or } .81,\ m_2 = 1.$$

The parameters m_1, m_2 and m_3 are the elasticities of desired capital services with respect to changes in the price-ratio, real output and the aggregate tax variable respectively. These results suggest that the significance of

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5. See: D. W. Jorgenson and J. A. Stephenson, Econometrica, April 1967.

R. Eisner and M. I. Nadiri, Review of Economics and Statistics, August 1968.

changes in relative prices for investment expenditures is less than that of changes in real output but that the former is far from negligible. The use of the theory of optimal capital accumulation to yield an expression for desired capital services always results in the equality of m_1 and m_3. If we ignore this specification, we obtain the equations for which results are presented on the second section of the Table 2. The coefficients of additional regressors in the equations 8.8c and 8.11c indicate that the imposition of equality may exaggerate the role of relative prices. These equations would suggest that this role is negligible since the differences $(a_{11} - |c_{11}|)$ and $(a_{12} - |c_{12}|)$ are very small. The remaining results confirm the proposition that m_1 is significantly less than unity. 0.39, 0.16, 0.58 and 0.45 are the four values implied by the estimated coefficients for the elasticity of desired capital with respect to changes in relative prices. These are markedly lower than the values implied for m_3, which ranged from 0.80 to 0.94, and gave little support for the specification of equality implicit in the theoretical models.

Two conclusions summarize the implications of these two groups of equations. If we accept a prior specification that the elasticities of desired capital with respect to changes in relative prices and to changes in tax factors are equal, the Cobb-Douglas form of the production function should be rejected in favour of a C. E. S. form with constant returns to scale in production. If we reject this prior specification, changes in taxation and in real output are far more significant factors in the determination of the level of investment expenditures than changes in relative prices. If the prior specification of equality is unjustified, its use will result in a severe exaggeration of the influence of prices. The success of the composite variables embodying the Cobb-Douglas assumption may be attributed to other arguments in the expressions. Our results indicate that the Investment Allowances, Initial Allowances, the corporate tax-rate and Annual Allowances may have been very significant instruments during the sample period.

APPENDIX

The logarithmic series is well-known.

$$\log_e(1 - s) = -s - \frac{s^2}{2} - \frac{s^3}{3} - \ldots \qquad \text{for } 0 \leqslant s < 1$$

Set $\quad r = 1 - s$

Then,

$$\log_e r = -(1 - r) - \frac{(1 - r)^2}{2} - \frac{(1 - r)^3}{3} - \ldots \quad \text{for } 0 < r \leqslant 1$$

For r sufficiently close to unity (s sufficiently close to zero), we can use the following approximation

$$\log_e r \simeq -(1 - r)$$

$$r \simeq 1 + \log_e r$$

Suppose $\quad r = \prod_{j=1}^{n} x_j^{\alpha_j}$

Then, $$\left(\prod_{j=1}^{n} x_j^{\alpha_j}\right) \simeq 1 + \log_e \left(\prod_{j=1}^{n} x_j^{\alpha_j}\right)$$

$$\simeq 1 + \sum_{j=1}^{n} \alpha_j \log_e x_j \quad ,$$

Where each (α_j) is sufficiently close to zero. Here sufficiency depends on the closeness of each $(x_j^{\alpha_j})$ to unity.

$$\left(\prod_{j=1}^{n} x_j^{1+\alpha_j}\right) \simeq \left(\prod_{j=1}^{n} x_j\right) + \sum_{j=1}^{n} \alpha_j \left(\prod_{i=1}^{n} x_i\right) \log_e x_j$$