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GENERAL SPATIAL PRICE EQUILIBRIUM

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1. Introduction

In 1952, Samuelson (1952) showed how the problem of determining the flows of a commodity moving between spatially separated points in an equilibrium when transportation costs were positive could be treated as an extremum problem. Samuelson's approach was to maximize gross economic rent (the sums of producers and consumers surpluses at all points) net of transportation costs in the system. The earliest use of his procedure of maximizing economic rent for characterizing equilibria is that of Cournot (1927; Chapter X) and Cournot dealt with the identical problem.

In this paper I will utilize the Cournot-Samuelson approach to characterize an equilibrium in Mosak's general equilibrium trade model (1944) for the case when transportation costs between countries are positive and the production of transportation goods is endogenous to the system. Elsewhere I have indicated that the Cournot-Samuelson approach could be used to generalize the Hitchcock-Koopmans transportation problem in linear programming, the von Thunen problem in pricing and allocating agricultural land, and the Weber problem in locating new firms. Mosak's model with transportation costs has been redeveloped with the aid

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I am indebted to Frank Flatters for helpful comments.

of the Cournot-Samuelson approach by Takayama and Judge (1964) (1970) but these latter authors did not make transportation costs endogenous nor did they develop economically meaningful dual problems. The explicit treatment of transportation costs as endogenous to the model reveals new properties of the Cournot-Samuelson spatial price equilibrium problem and focuses attention on some neglected issues in stability analysis and monetary theory.

The notion of maximizing economic rent will no doubt arouse consternation in the minds of modern economic theorists. A host of battles have been fought over the welfare significance of changes in the magnitudes of producers and consumers surpluses.[1] Samuelson was well aware that the maximization of economic rent could not be interpreted in a welfare sense and went so far as to give the magnitude, net economic rent, an alias, net social payoff. Of economic rent net of transport costs or net social payoff Samuelson noted "This magnitude is artificial in the sense that no competitor in the market will be aware of or concerned with it. It is artificial in the sense that after an Invisible Hand has led us to its maximization, we need not necessarily attach any social significance to the result" (1952, p. 288). In other words, in reaching an equilibrium the economic system acts AS IF it were maximizing economic rent. I find it useful to denote the Cournot-Samuelson approach as another in the economist's kit of

AS IF principles. Two other well-known AS IF principles are (a) the consumer reaches an equilibrium in allocating his budget as if he were maximizing a utility function and (b) an economy without transactions costs reaches an equilibrium as if the tatonnement were done "by means of tickets" Walras (1954, p. 37). Neither utility functions nor tickets are assumed to exist in the sense of being empirically observable and yet the assumption that they exist permits various empirical observations to be "explained".

From another standpoint, we know that demand and supply schedules can be derived in full economic general equilibrium as solutions of individual optimization problems.[2] Given these well-defined supply and demand schedules an equilibrium can be shown to exist. The Cournot-Samuelson AS IF principle considers sets of single-valued supply and demand functions and characterizes an equilibrium as the solution of an economic rent maximization problem; this equilibrium will be a special case of the static equilibrium defined under the most general conditions but it will nonetheless be an equilibrium. It is the conditions defining the equilibrium which are of interest and which emerge as outputs of the Cournot-Samuelson approach rather than as inputs as in the most disaggregated analysis of economic general equilibrium.

The Cournot-Samuelson approach is useful because it correctly characterizes equilibria in a relatively simple way. The approach is deficient in the sense that it fails to characterize welfare aspects of an equilibrium and also that it lacks generality in the sense that it requires single-valued supply and demand curves to be defined *a priori*.

Much of the analysis in this paper considers a situation in which diverse demand and supply schedules are linear. This permits the extremum problem to be posed as a quadratic program rather than as a more general type of non-linear program.

In Section 2, a two region-four commodity example of the Mosak model with endogenous transportation costs is presented in detail. In Section 3, the m region - n commodity case is presented. In Section 4 notes on the existence of a solution and the stability of a price adjustment procedure are developed. The analysis in Section 5 deals with the interpretation of transportation costs as transaction costs and the relevance of such an interpretation to monetary theory is considered. In Section 6, a duality theorem is presented which permits non-linear supply and demand schedules to be introduced into the general commodity - m region problem. Section 7 contains a summary of the results of this paper.

2. A Two Region - Four Commodity Example

The regions will consist of two spatially separated points. There will be three commodities labelled with subscripts 1, 2, 3 which are transported between the points at some positive transportation cost measured in dollars per unit transported between the points. Commodity 4 will be a produced transportation good whose price will be π_4. The transportation good will be produced in both regions and will be utilized or consumed in making round trips between the points.[3] It will be assumed that transportation goods can be transported at zero cost and so the unit price in the separate regions will be the same in equilibrium.

The reason the number three for the non-transportation goods was chosen was in order to provide enough scope to illustrate different supply and demand conditions for the commodities at different points. Ignoring transportation goods, there are three possible cases for market conditions for commodities: The Hitchcock-Koopmans case of fixed demands and/or supplies at various points, the generalized Hitchcock-Koopmans case of variable demands and/or supplies (but not both demands and supplies at one point) at various points; and the Cournot-Enke-Samuelson case of variable demands and supplies (both at each point) at various points. Figure 1 below illustrates diagrammatically the equilibrium for two point - four commodity example. This example includes the various possible cases for commodity markets.

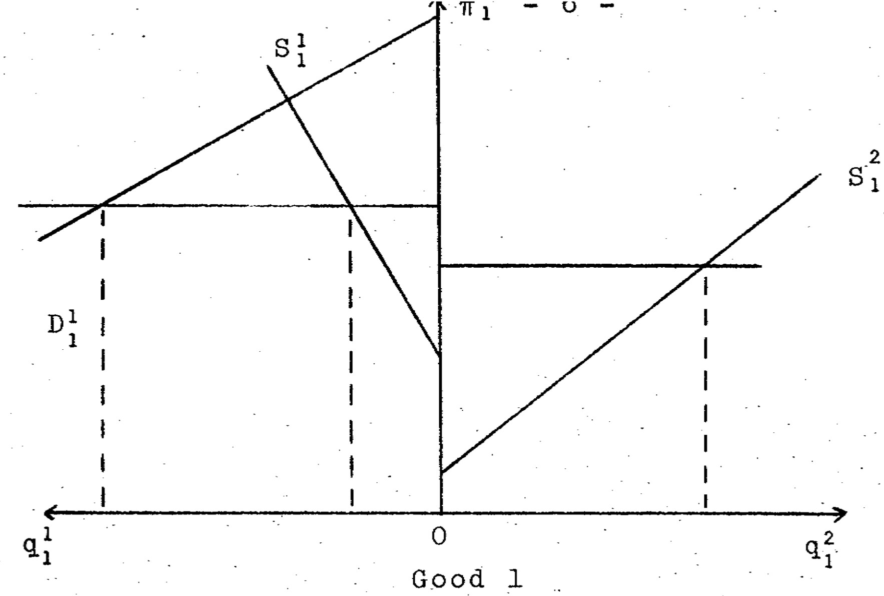
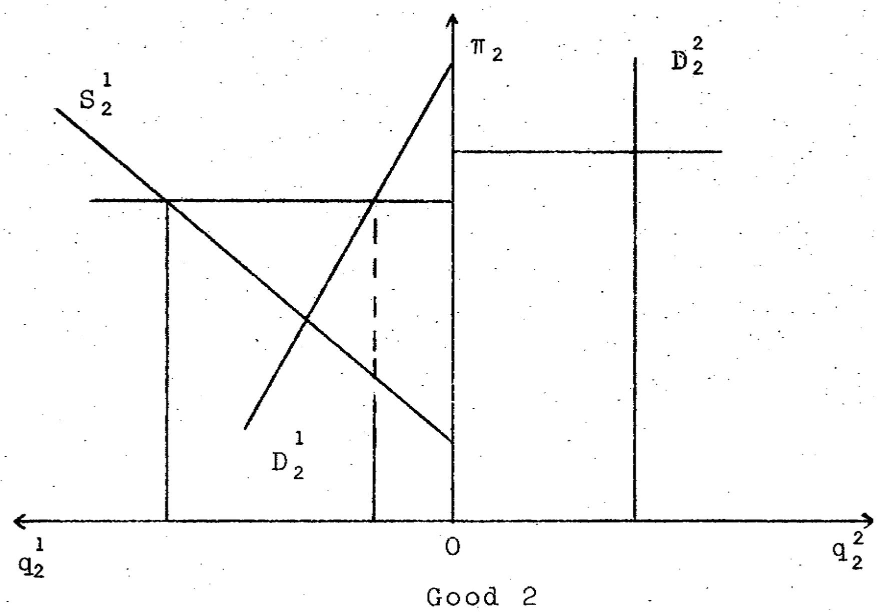
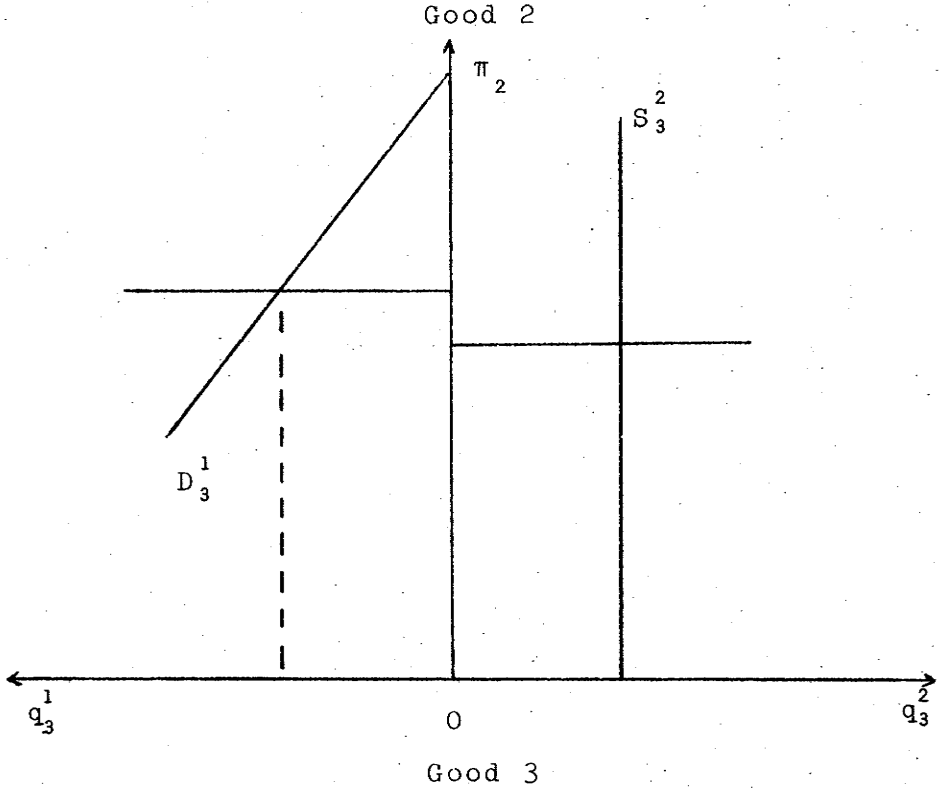


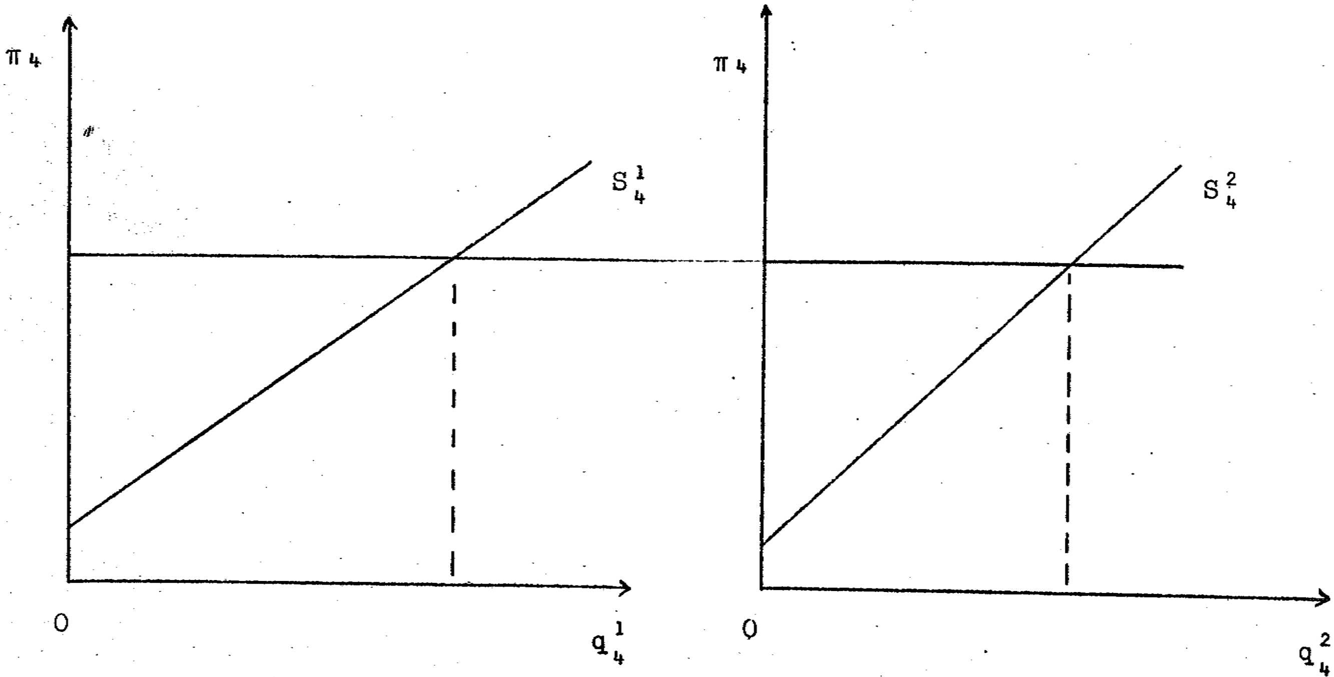
Figure 1 (a)



1 (b)



1 (c)



1(d)

Good 4

The first three back-to-back diagrams labelled Figures 1a, b, and c respectively each illustrate an equilibrium for one good, numbered 1, 2, and 3 respectively flowing between the two regions or points in our world. In Figure 1d, we have separate supply functions for the transportation good in each region. All prices in a region enter simultaneously in the determination of the equilibrium values for flows in a region. Hence the general equilibrium structure of our model.[4] In equilibrium, the flows of goods supplied from region k equal the flows of good i delivered to region 1. Also in equilibrium, the price of good i in the region of delivery minus the price of good i in the region of supply equals the cost of transporting a unit of good i between the regions if a flow of good i actually takes place between the regions in equilibrium.

We shall be dealing with only one schedule for each commodity in each region. Thus rather than dealing with demand and supply schedules for goods 1 and 2 in region 1, we shall define excess demand and excess supply schedules for goods 1 and 2 respectively. We define all our schedules to be linear in prices in order to make use of the quadratic program as opposed to a more general non-linear program.

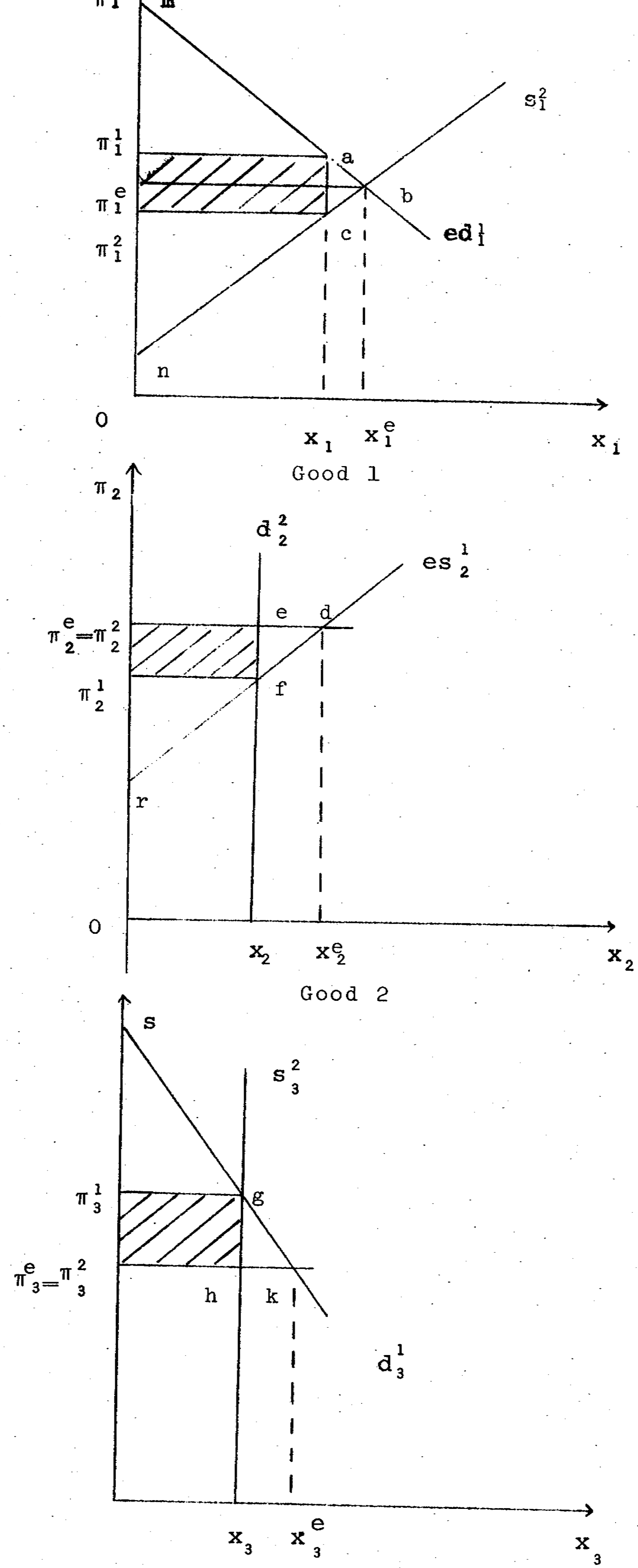
Figure 2 presents the equilibrium illustrated in Figure 1a, b, and c in an alternative way. In Figure 2 we have combined the equilibria in the separate regions for a good in one quadrant and we have redefined the equilibria for region 1 for goods 1 and 2 in terms of linear excess demand and excess supply curves.

Figure 2

(a)

(b)

(c)



Good 3

We have the following demand and supply relationships which were illustrated in Figures 2 and 1d. For region 1

$$ed^1_1 = -\delta^1_{11}\pi^1_1+\delta^1_{12}+\pi^1_2+\delta^1_{13}\pi^1_3+\delta^1_{14}\pi^1_4$$

$$es^1_2 = +\sigma^1_{22}\pi^1_2-\sigma^1_{21}\pi^1_1-\sigma^1_{23}\pi^1_3-\sigma^1_{24}\pi^1_4$$

$$d^1_3 = -\delta^1_{33}\pi^1_3+\delta^1_{31}\pi^1_1+\delta^1_{32}\pi^1_2+\delta^1_{34}\pi^1_4$$

$$s^1_4 = +\sigma^1_{44}\pi^1_4-\sigma^1_{41}\pi^1_1-\sigma^1_{42}\pi^1_2-\sigma^1_{43}\pi^1_3$$

where ed^1_1 is excess demand in region 1 for commodity 1

es^1_2 is excess supply in region 1 for commodity 2

d^1_3 is demand in region 1 for commodity 3

s^1_4 is supply in region 1 for commodity 4

π^1_i $(i=1,\ldots,4)$ is the price for commodity i in region 1

δ^1_{ij} is the parameter indicating the effect of a unit change in the price of the jth commodity in region 1

σ^1_{ij} is the parameter indicating the effect of a unit change in the price of the jth commodity on the supply of the ith commodity in region 1

For region 2,

$$s_1^2 = +\sigma_{11}^2\pi_1^2-\sigma_{14}^2\pi_4^2$$

$$d_2^2 = \underline{d}_2^2$$

$$s_3^2 = \underline{s}_3^2$$

$$s_4^2 = +\sigma_{44}^2\pi_4^2-\sigma_{41}^2\pi_1^2-\sigma_{42}^2\pi_2^2-\sigma_{43}^2\pi_3^2$$

where $\underline{d}_2^2$ is the fixed demand in region 2 for commodity 2.

$\underline{s}_3^2$ is the fixed supply in region 2 for commodity 3.

and the remaining terms have definitions analogous to those for similar terms above, defined for region 1.

If there is a flow between points in equilibrium, then x_i is the volume of the flow of good i measured in physical units.

We shall also require value x_i^e (i=1,2,3) indicating the interregional flow obtaining if trading relationships remained fixed and the system behaved as if transportation costs were zero. The x^{e}'s as well as the corresponding prices, π^{e}'s, are indicated in Figure 2.[5] Specifically in order to define π_i^e and x_i^e we are required to solve a constrained economic rent maximizing problem. We can proceed as if π_i^{e}'s and x_i^{e}'s are known because they can be solved for once the constrained economic rent maximization problem is solved; Takayama and Judge (1964) have indicated how to solve the relevant problem. Note that if in equilibrium a positive flow moves between two points then $\pi_i^{\,l}$ at delivery point l equals $\pi_i^{\,k}$ at supply point k plus transportation cost $t_i^{\,kl}$. Also the flow $x_i^{\,k}$ at supply point k equals the flow $x_i^{\,l}$ at demand point l.

The transportation good, numbered 4, is produced in flows x_4^1 and x_4^2 in regions 1 and 2 respectively. $r_i^{12} = r_i^{21}$ units of good 4 are required to transport a unit of good i between points 1 and 2 where i=1, 2, 3. The cost of transporting a unit of good i between region 1 and 2 is

$$\pi_4 r_i^{12} = \pi_4 r_i^{21} = t_i^{12} = t_i^{21}$$

where π_4 is the price of a unit of good 4, the transportation good.

Our primal problem will be the constrained minimization of the negative of the economic rent foregone because transportation costs are non-zero. The selection of this problem as primal is purely for convenience; Dorn treats the primal as a minimization problem. We shall minimize the negative of the sum of the areas $abc\pi_1^2\pi_1^1$, $edf\pi_2^1\pi_2^2$ and $gkh\pi_3^2\pi_3^1$ in Figure 2 subject to the condition that the differences in supplied and delivered prices be less than or equal to unit transport costs.

First we fix π_4 at some value[7] and determine p so as to

$$\begin{array}{ll} \text{minimize} & \tfrac{1}{2}pCp - rp \\ \text{subject to} & -Ap \geqq -t \\ & p \geqq 0 \end{array}$$

where

$$p = (\pi_1^1-\pi_1^e, \pi_2^e-\pi_2^1, \pi_3^1-\pi_3^e, \pi_1^e-\pi_1^2, \pi_2^2-\pi_2^e, \pi_3^e-\pi_3^2)$$

$$C = \begin{bmatrix} -\delta_{11}^1 & \delta_{12}^1 & \delta_{13}^1 & & & \\ \delta_{21}^1 & \sigma_{22}^1 & \delta_{23}^1 & 0\cdots & & \\ & & & \vdots & & \\ \delta_{31}^1 & \delta_{32}^1 & -\delta_{33}^1 & & & \\ & & & \sigma_{11}^2 & 0 & 0 \\ & 0\cdots & & 0 & 0 & 0 \\ & \vdots & & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & & 1 & & \\ & 1 & & & 1 & \\ & & 1 & & & 1 \end{bmatrix}$$

$$r = (x_1^e,\ x_2^e,\ x_3^e,\ x_1^e,\ \underline{d}_2^2,\ \underline{s}_3^2)$$

$$t = (\pi_4 r_1^{21}, \pi_4 r_2^{12}, \pi_4 r_3^{21})$$

Our dual problem, following Dorn (1960) is to find a vector (p,v) so as to

maximize $\quad -\frac{1}{2}pCp-tv$

subject to $\quad A^Tv+Cp=r$

$$v \geq 0$$

Vector v is

$$v = (x_1,\ x_2,\ x_3)$$

and A^T is matrix A transposed. Observe that we can convert this problem to an economic rent maximizing problem rather than a problem minimizing economic rent foregone.

We now have a set of prices and flows which optimize the quadratic program. These prices imply a flow of transportation goods $x_4^1+x_4^2$. The interregional flows imply a demand for a specific flow of transportation goods. That is the demand for transport is

$$T = r_1^{21}x_1+r_2^{12}x_2+r_3^{21}x_3$$

where (x_1, x_2, x_3) are optimal. If $x_4^1+x_4^2 \neq T$, then a new π_4 must be selected and the program optimized once again. From theorems of Debreu and Kakutani we are assured that a fixed point or an equilibrium price vector exists which satisfies the transportation flow equilibrium condition.[9] Thus an equilibrium for the model exists.

In general, it will not be possible for balance of payments equilibrium for each region to be satisfied when the transportation good flow equilibrium condition is satisfied. In short, balance of payments equilibrium will not obtain in a world with endogenous transportation goods. Also observe that there is a unique price for transportation goods which permits an equilibrium to obtain. Hence the equilibrium price vector is also unique. Recall that many conventional general equilibrium models have an equilibrium price vector unique up to multiplication by a positive scalar; that is these models are homogenous of degree zero in terms of prices.

3. The n Commodity, m Region Case with Linear Schedules

The jth region will be a point at which are defined n schedules, some for demand and some for supply. We shall order the commodities in each region the same way. In region j there will be a set I_j^{ed} of indices indicating linear schedules of excess demand, I_j^{es} for excess supplies, I_j^d for demands, I_j^s for supplies, $I_j^{\underline{d}}$ for fixed demands and $I_j^{\underline{s}}$ for fixed supplies. The sum of the numbers elements in the six sets will be n, or the total number of commodities. Note a region can have no schedule for some commodity i. This will be represented by a zero for the ith commodity in the set of either fixed demands of fixed supplies. Note also that in region j, commodity i can be represented by either a demand schedule (including excess demands or fixed demands) or a supply schedule (including excess supplies or fixed supplies) but never by both. In other words the sets of indices defined above are disjoint and any five of the six sets could be empty for a region. There are m regions.

Specifically, we have for commodity i in region j for excess demand

$$d_i^j = -\delta_{i1}^j \pi_1^j + \delta_{i2}^j \pi_2^j + \ldots + \delta_{in}^j \pi_n^j + \delta_{i,n+1}^j \pi_{n+1}^j$$

for excess supply

$$s_i^j = \sigma_{i1}^j \pi_1^j + \sigma_{i2}^j \pi_2^j + \ldots + \sigma_{in}^j \pi_n^j + \sigma_{i,n+1}^j \pi_{n+1}^j$$

for demand

$$d_i^j = -\delta_{i1}^j\pi_1^j + \delta_{i2}^j\pi_2^j + \ldots + \delta_{in}^j\pi_n^j + \delta_{i,n+1}^j\pi_{n+1}^j$$

for supply

$$s_i^j = \sigma_{i1}^j\pi_1^j + \sigma_{i2}^j\pi_2^j + \ldots + \sigma_{in}^j\pi_n^j + \sigma_{i,n+1}^j\pi_{n+1}^j$$

for fixed demand

$$d_i^j = \underline{d}_i^j$$

and for fixed supply

$$s_i^j = \underline{s}_i^j$$

where the variables and coefficients have been defined for the example in Section 2. We are of course assuming that demand curves have the usual appearance, that is $\frac{\partial d_i^j}{\partial \pi_i^j} < 0$ and also supply curves slope upward, that is $\frac{\partial s_i^j}{\partial \pi_i^j} > 0$.

In addition we require our basic matrix C to be positive semi-definite (to define a convex function) and to be symmetric. Hence if region j demands commodity i and supplies commodity k, then $\frac{\partial d_i^j}{\partial \pi_k^j} = \frac{\partial s_k^j}{\partial \pi_i^j}$.

For this m region, n commodity case, matrix C will be a symmetric positive semi-definite nmxnn matrix with the structure indicated below.

$$C = \begin{bmatrix} \overset{n}{\boxed{1}}\; n & & & \\ & \overset{n}{\boxed{2}}\; n & & \\ & & \ddots & \\ & & & \overset{n}{\boxed{m}}\; n \end{bmatrix}$$

The positive semi-definiteness assumption can be related to the notion of dynamic stability. For a two-region case where all schedules are defined for either demands or supplies (not excess demands or supplies nor fixed demands or supplies) and the price vector is the same in both countries (that is, no transportation costs) the symmetry and positive definiteness of matrix C implies that the price adjustment process (tatonnement) demonstrates true dynamic stability in the sense of Samuelson.[10] The geographic separation of suppliers and demanders introduces a new dimension to the tatonnement process. There must be arrangements so that prices are announced simultaneously in each of the two places. Prices must differ, with the spatial dimension, in equilibrium by the cost of transportation. Thus the price vector will not be the same in each of the two regions so that the positive definiteness of C is a more general case of stability conditions than the true dynamic ones.

Furthermore we are here dealing not with two but with m regions and with supply and demand schedules of a greater variety of forms than those discussed immediately above. Recall that matrix C contains elements of excess supply and demand functions as well as fixed demand and supply functions.[11]

In summary, the symmetry and positive semi-definiteness of matrix C has an interpretation as a stability condition for a price adjustment process when transportation costs are exogenous. For an equilibrium to exist, the conditions on demand and supply curves are the same as those for true dynamic stability (local stability).

In the general n commodity, m region case the primal problem which is the constrained minimization of economic rent foregone problem will be:

$$\begin{array}{ll} \text{minimize} & f(p) = \tfrac{1}{2}pCp - rp \\ \text{subject to} & -Ap \geqq -t \\ & p \geqq 0 \end{array}$$

where p is an mnx1 vector indicating prices for commodities in various regions. Elements of vector p will be two types. For a region k acting as a net supplier for commodity i the element will be $\pi_i^{e^k} - \pi_i^k$, and for a region k acting as a net demander for commodity j the element will be $\pi_j^k - \pi_j^{e^k}$.

It is easiest to consider the structure of matrix A as a number of separate units. First each row will have two entries, each being a positive number, namely one or unity. Let us consider vector p with all elements excised except those pertaining to commodity i. Let matrix A be shrunk to a size such that there are no gaps where the entries were excised. Now the part of matrix A corresponding to this new part of vector p is a classic "transportation matrix" with all positive entries.[12] Reordering the entries of p so that all entries corresponding to demand points are on the left and all corresponding to supply points are on the right, the corresponding submatrix of A will be:

$$
\begin{array}{r c}
 & \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{\substack{\text{number of}\\ \text{demanders for}\\ \text{commodity i}}} & \qquad\qquad \overbrace{\qquad\qquad\qquad}^{\substack{\text{number of}\\ \text{suppliers for}\\ \text{commodity i}}} \end{array} \\
\begin{array}{r}
\substack{\text{number of}\\ \text{suppliers for}\\ \text{commodity i}}\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\
\\ {''} \\ \\ \\ \\ \\ \\
{''} \\ \\ \\
\end{array}
&
\left[\begin{array}{cccc ccccc}
1 & & & & 1 & & & & \\
1 & & & & & 1 & & & \\
1 & & & & & & 1 & & \\
\vdots & & & & & & & \ddots & \\
1 & & & & & & & & 1 \\
 & 1 & & & 1 & & & & \\
 & 1 & & & & 1 & & & \\
 & 1 & & & & & 1 & & \\
 & \vdots & & & & & & \ddots & \\
 & 1 & & & & & & & 1 \\
 & & \ddots & & & & & & \\
 & & & 1 & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & & 1 & & & 1 & & \\
 & & & \vdots & & & & \ddots & \\
 & & & 1 & & & & & 1
\end{array}\right]
\end{array}
$$

Note that the matrix above can have its columns reordered to obtain the original subvector of p.

The complete matrix A consists of n such submatrices, each submatrix corresponding to a single commodity excluding transport. These n submatrices will be vertically stacked upon one another. Given that the elements of p are ordered in accord with the arrangement of elements of matrix C, we noted that the submatrices will not have all columns corresponding to demand price elements on the left and all columns corresponding to supply price elements on the right. The columns must be in an irregular pattern. In addition when we have all n submatrices stacked to form matrix A, the columns of a submatrix corresponding to commodity i will in general be separated by various numbers of columns of zeros. These columns of zeros will have positive entries corresponding to either demand or supply prices for commodity j.

To summarize, matrix A will consist of n submatrices of the form of the above submatrix, that is

$$\tilde{A} = \begin{bmatrix} \boxed{\begin{matrix}\text{Commodity}\\ 1\end{matrix}} & & & \\ & \boxed{\begin{matrix}\text{Commodity}\\ 2\end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}\text{Commodity}\\ n\end{matrix}} \end{bmatrix}$$

with the columns reordered to suit the ordering of commodities and regions in matrix C and in vector p.

Let q_i be the number of net demanders for commodity i and s_i be the number of net suppliers. Now the submatrix corresponding to commodity i will have $q_i s_i$ rows and $(q_i + s_i)$ columns. Hence matrix A will have $\sum_i^n q_i s_i$ rows and $\sum_i^n (q_i + s_i)$ columns.

Vector t will be composed of $\sum_i q_i s_i$ elements. Each element $t_i^{k\ell}$ indicates the cost of shipping a unit of commodity i from region k to region ℓ.

Vector r will be composed of $\sum_i^n (q_i + s_i)$ elements. Each element will be either a fixed demand or supply, or an element x^e for a projected demand or supply.

The dual to the general n commodity, m region case will be a constrained maximization of economic rent problem, that is

$$\begin{array}{ll} \text{maximize} & q(p,v) = \tfrac{1}{2}pCp - tv \\ \text{Subject to} & A^T v + Cp = r \\ & \qquad v \geqq 0 \end{array}$$

where A^T is the transpose of the matrix A from the problem immediately above and vector v is a $1 \times \sum_i q_i s_i$ vector of interpoint flows. An element of v is for example $x_i^{k\ell}$ or the physical flow of commodity i from region k to region ℓ.

There will be a unique value for the price of the transportation good which will result in the demand for transport equalling the supply required in equilibrium. The theorems of Debreu and Kakutani assure that such an equilibrium can exist.[13]

4. Aspects of Existence and Stability

We have seen that an equilibrium can exist for a general trade model with produced or endogenous transportation costs. The important economic issue is whether there exist forces which will drive the economy to the equilibrium. Samuelson (1947; p. 112) made formal Walras' "fiction" developed originally by Walras to indicate that the economy would move to an equilibrium. Samuelson's "true dynamic stability" is a well defined process which was invoked to assure that an equilibrium would be obtained for an economy.

Let us assume that if the price of the transportation good is fixed initially then a tatonnement process, perhaps formalized like "true dynamic stability" analysis, exists which results in economic rent maximization or the clearing of all markets except that for the transportation good. Asking for such a process is not demanding much more than is required of modern treatments of Walras' tatonnement. However, the fact that the transportation good market is not necessarily in equilibrium after the tatonnement is complete requires that the process of price adjustment be repeated with alternative initial prices for the transportation good.

We can illustrate the situation in Figure 3 below. We are considering the market for the transportation good.

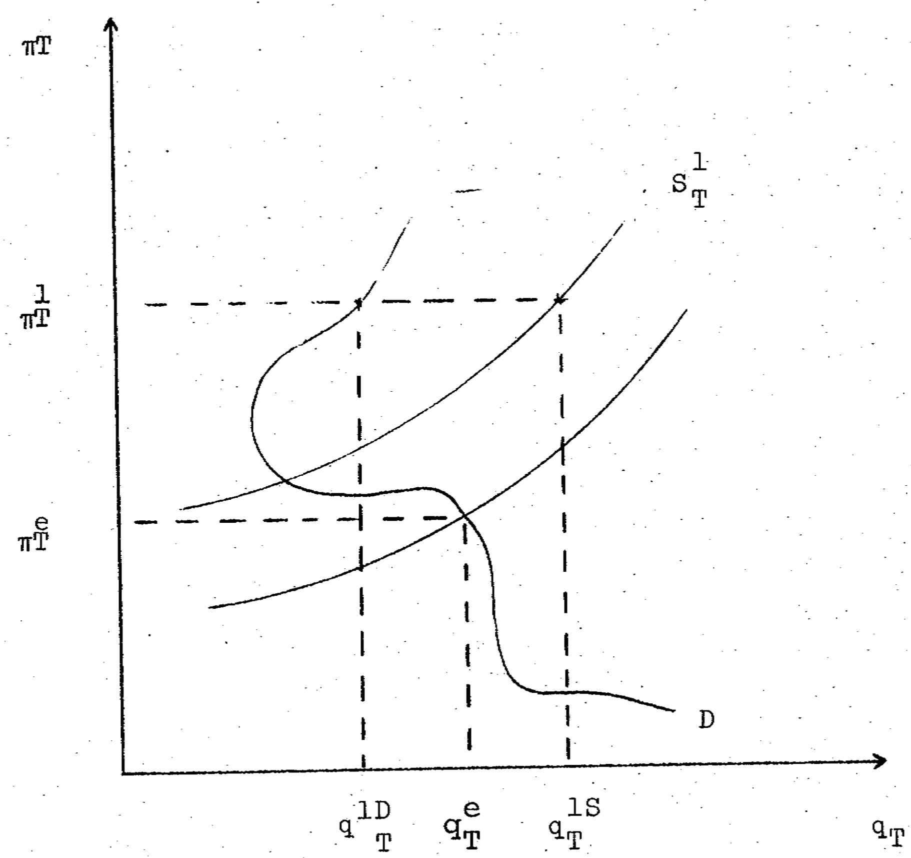


Figure 3

At initial price π_T^1, the demand for the transportation good is q_T^{1d} and the supply is q_T^{1s} in Figure 3. Observe that we can define an orthodox supply curve through point (π_T^1, q_T^{1s}) by assuming all commodity prices are held constant except that price for the transportation good. The demand curve is simply point (π_T^1, q_T^{1D}).

By varying p_T we trace out a series of demand points. Corresponding to each value for p_T there will be a supply curve. The schedule of demand points might be schedule D indicated in Figure 3. Equilibrium is indicated to be at point (π_T^e, q_T^e).

The nature of the D schedule depends on the nature of matrix C discussed in Section 3. The properties of the D schedule will determine whether the equilibrium is stable and unique. The nature of this model suggests that the concept of stability must be extended to account for feedback from the transportation good market. A family of tatonnements are required, one member of the family for each value of the price of the transportation good and then the question of stability becomes that of ascertaining whether the signals fed back from the non-transportation good markets will cause the transportation good market to tend toward a unique equilibrium value for the price and output of the transportation good. The development of a well-behaved tatonnement with feedback remains to be undertaken.[14]

5. Trade with Endogenous Transactions Costs

The general model developed in Section 3 can be used to illustrate some aspects of an economy with money and transactions costs. We can consider an economy to be characterized by a finite number of linear supply schedules for each of n goods and a finite number of linear demand schedules.[15] A fixed amount of an n+1 good is consumed in making a transaction. We shall consider this n+1 good to be tradable and shall call it the transactions good. This model is then a multi-commodity case of the generalized Hitchcock-Koopmans transportation problem with endogenous transactions (transportation) costs.

We can interpret transactions costs as brokerage fees which must be paid when each transaction takes place. We shall let the services of money be the transactions good. If we fix the price of a unit of the transactions good at say π_a, and determine an equilibrium by maximizing economic rent in the system, a specific quantity of the transactions good will be demanded, say q_a in Figure 4.

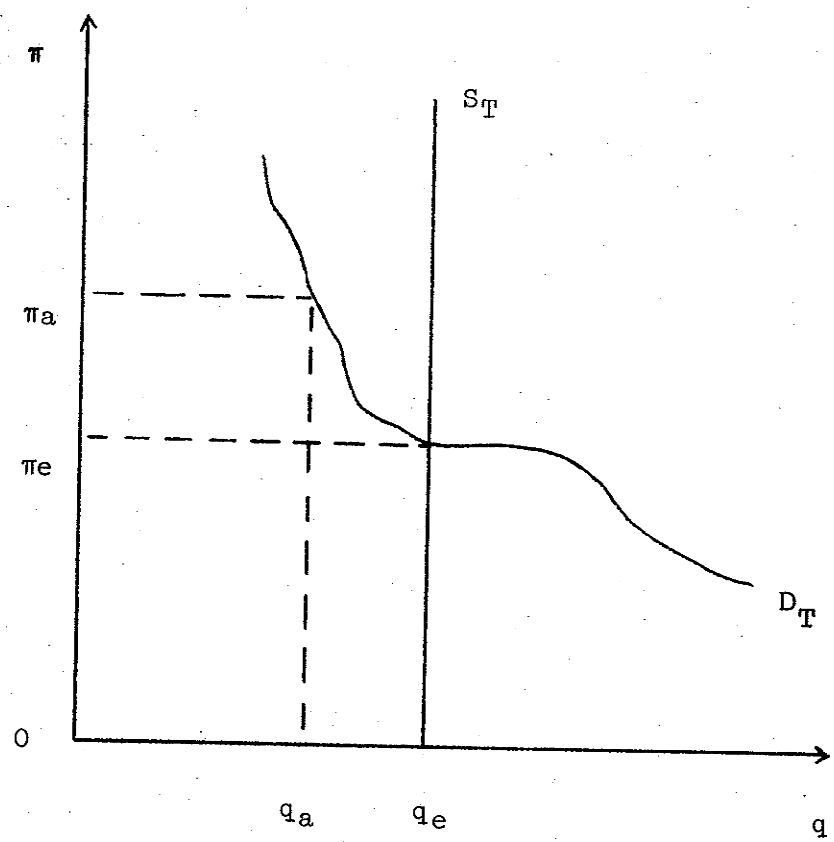


Figure 4

This procedure can be repeated with alternative initial π's and a demand schedule D_T is traced out. Let S_T be the supply of the transactions good as a function of the stock of money in the system. Money is supplied by decree at zero cost to the monetary authority. The unique price for the transactions good occurs at p_e in Figure 4.

The slope of the demand schedule for the transactions good will depend on the parameters of the demand and supply schedules for commodities or on the structure of a matrix analogous to matrix C in Section 3. We noted this result in Section 4. There exists the possibility of multiple intersections of D_T with S_T as well as the possibility of no intersections. The nature of the D_T schedule will determine the nature of the stability of the economic system. In the well-behaved case illustrated in Figure 4 the equilibrium is unique; the absolute price level is determined and for alternative stocks of money, alternative price vectors exist.

The controversies in monetary theory arising from Patinkin's development of the "real balance effect" are dealing with a different set of issues. In those debates, the focus was on the process whereby flows of cash balances and flows of other commodities traded were related.[16] Clower argues that "the conception of money economy implicit in modern accounts of the general equilibrium theory of money and prices is formally equivalent to the classical conception of a barter economy", Clower (1967). He cites Patinkin's opus as one of those modern accounts. I share Clower's opinion and feel, like Niehans (1969), that the basic problem with Patinkin-like analyses is that though money balances are introduced to overcome explicitly acknowledged transactions costs, yet those transactions costs never explicitly appear in the models. The model developed above in this Section has been developed with those transactions costs explicitly included.

We have assumed that the services of money can affect transactions costs. It is analogous with the fact that the amount of grease affects the amount of force needed to move a locomotive. However we also know that there does not exist a point where the addition of more grease eliminates friction entirely. Similarly we might expect that though the cost of producing money may be zero, there does not exist a point where the price of its services becomes zero.

Note also that changes in the stock of money have an effect on the flows of other goods in the system. The model thus has important "real balance effects" though these effects arise from very different sources than in Patinkin's model.

6. The Case of Non-linear Demand and Supply Schedules

The assumption of linearity in various schedules has permitted us to appeal to the well-developed mathematical literature on quadratic programming in order to formulate our Mosakian model of trade with a produced transportation good. Samuelson's original formulation of the problem of spatial price equilibrium for one commodity flows did not require linearity in either the demand and supply schedules or the excess supply or excess demand schedules. Below is an outline of our general model when nonlinearities are permitted. Our programs have general nonlinear objective functions with linear constraints.

The following two region - two commodity example with exogenous transportation costs will illustrate an equilibrium with nonlinear supply and demand schedules. With no loss of generality, we will restrict consideration to cases involving only supply and demand schedules rather than cases involving excess demand and supply schedules and fixed demands and supplies.

The following problem is analogous to the negative of the economic rent foregone minimizing problem of Section 2 and 3. Minimize

$$\begin{aligned}
&\int_{\pi_1^e}^{\pi_1^2} q_1^2(\pi_1^2, \pi_2^2)\,d\pi_1 - x_1^{e^2}(\pi_1^2 - \pi_1^e) \\
&+\int_{\pi_1^1}^{\pi_1^e} s_1^1(\pi_1^1, \pi_2^1)\,d\pi_1 - x_1^{e^1}(\pi_1^e - \pi_1^1) \\
&+\int_{\pi_2^e}^{\pi_2^1} q_2^1(\pi_1^1, \pi_2^1)\,d\pi_2 - x_2^{e^1}(\pi_2^1 - \pi_2^e) \qquad (6.1)\\
&+\int_{\pi_2^2}^{\pi_2^e} s_2^2(\pi_1^2, \pi_2^2)\,d\pi_2 - x_2^{e^2}(\pi_2^e - \pi_2^2)
\end{aligned}$$

subject to:

$$-\,(\pi_1^2-\pi_1^e) - (\pi_1^e-\pi_1^1) \geqq -t_1^{12}$$

$$-\,(\pi_2^1-\pi_2^e) - (\pi_2^e-\pi_2^2) \geqq -t_2^{21} \qquad (6.2)$$

$$p = (\pi_1^2-\pi_2^e,\ \pi_1^e-\pi_1^1,\ \pi_2^1-\pi_2^e,\ \pi_2^e-\pi_2^2) \geqq 0 \qquad (6.3)$$

where $\frac{\partial q_i^j}{\partial \pi_i^j}$ records the effect on the quantity of the ith commodity demanded in region j of a change in the price of the ith commodity in region j.

$\frac{\partial s_i^k}{\partial \pi_i^k}$ records the effect on the supply of the ith commodity demanded in region k of a change in the price of the ith commodity in region k.

t_i^{hk} is the fixed exogenously given cost of transporting a unit of the ith commodity between regions h and k.

$\theta(p)$ is the sum of the four integrals. We require the minimand $\theta(p) - rp$ or 6.1 to be convex. The Kuhn-Tucker conditions can be set down for the problem (6.1,2,3).

$$\begin{aligned}
&-\frac{\partial\theta}{\partial\pi_1^2} - x_1^{12} & & = -\,x_1^{e^2} \\
&-\frac{\partial\theta}{\partial\pi_2^1} & - x_2^{21} & = -\,x_2^{e^1} \\
&-\frac{\partial\theta}{\partial\pi_1^1} - x_1^{12} & & = -\,x_1^{e^1} \\
&-\frac{\partial\theta}{\partial\pi_2^2} & - x_2^{21} & = -\,x_2^{e^2}
\end{aligned}
\qquad \text{or} -\nabla\theta(\bar{p}) - \bar{u}A = -r \qquad (6.4)$$

and $$A\bar{p} - t = 0 \qquad (6.5)$$

$$\bar{u}(A\bar{p} - t) = 0 \qquad (6.6)$$

$$\bar{u} \geqq 0 \qquad (6.7)$$

Vector $u = (x_1^{12}, x_2^{21})$ is the shadow flow corresponding to the prices. An element $x_i^{k\ell}$ is the physical flow of commodity i between k and ℓ.

The vector $\nabla\theta(\bar{p})$ can be expressed as the product of a matrix and the price vector as indicated below.

$$\nabla\theta(\bar{p}) = \left[\begin{array}{cc|cc} \left.\frac{\partial s_1^1}{\partial \pi_1^1}\right|_{\bar{\pi}_1^1} & \left.\frac{\partial s_1^1}{\partial \pi_2^1}\right|_{\bar{\pi}_2^1} & & \\ \left.\frac{\partial q_2^1}{\partial \pi_1^1}\right|_{\bar{\pi}_1^1} & \left.\frac{\partial q_2^1}{\partial \pi_2^1}\right|_{\bar{\pi}_2^1} & & \\ \hline & & \left.\frac{\partial q_1^2}{\partial \pi_1^2}\right|_{\bar{\pi}_1^2} & \left.\frac{\partial q_1^2}{\partial \pi_2^2}\right|_{\bar{\pi}_2^2} \\ & & \left.\frac{\partial s_2^2}{\partial \pi_1^2}\right|_{\bar{\pi}_1^2} & \left.\frac{\partial s_2^2}{\partial \pi_2^2}\right|_{\bar{\pi}_2^2} \end{array}\right] \begin{bmatrix} \pi_1^{e^1} - \bar{\pi}_1^1 \\ \bar{\pi}_2^1 - \pi_2^{e^1} \\ \bar{\pi}_1^2 - \pi_1^{e^2} \\ \pi_2^{e^2} - \bar{\pi}_2^2 \end{bmatrix}$$

Observe that $\nabla\theta(\bar{p})$ above is analogous to expression Cp in the quadratic program. For duality to hold in quadratic programming in the sense of Dorn, it was necessary that C be positive semi-definite and symmetric. The symmetry assumption was required to prove that the value of the primal equals the value of the dual in equilibrium. Hence I conjecture that symmetry is required in the above matrix in the expression of $\nabla\theta(\bar{p})$.[17]

These results permit us to place a segment of the literature on spatial price equilibrium in perspective. Symmetry of demand and supply functions or excess demand and supply functions in the neighbourhood of equilibrium is a necessary equilibrium condition. The admissable models of spatial price equilibrium with linear schedules have symmetry for all ranges of the non-negative prices. Efforts have been made to dispense with the symmetry condition by Yaron, Plessner, and Heady (1965), Plessner and Heady (1970) and to define valid models of spatial price equilibrium. However, the above result indicates that symmetry of the coefficients is a necessary equilibrium condition and efforts to dispense with the symmetry condition are misdirected.

We have dealt with a nonlinear programming problem and its Kuhn-Tucker equilibrium conditions. The economics of the problem suggests that a valid dual problem must exist. For a two region - one commodity nonlinear spatial price equilibrium problem, the following Figure is illustrative.

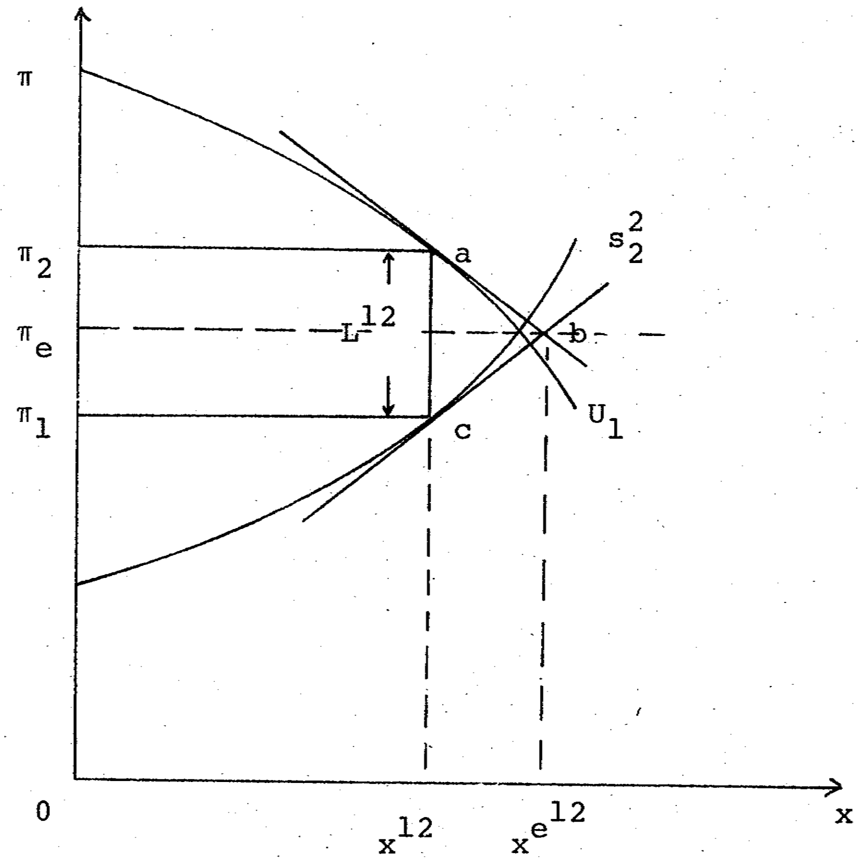


Figure 5

We observe that the $x^{e^{12}}$ and π^e are defined with respect to the net rent maximizing prices and quantities. It should be apparent that area $\pi_2 abc\pi_1$ has the identical structure to the corresponding area for the problem with linear demand and supply schedules. This leads me to conjecture the existence of the following valid <u>locally quadratic</u> non-linear programming duality relationship.

The primal is defined as

minimize $\theta\,(p) - rp$

subject to $-Ap \geqq -t$

where $p \in S = \{p \mid p \geqq 0$ and $\nabla\theta(p) = C(p)p$ where $C(p)$ is symmetric$\}$

and $\theta(p)$ is convex.

The dual problem is

maximize $-\theta\,(p) - tu$

subject to $\theta\,(p) + uA = r$

$u \geqq 0$

$p \in T$ and $T = \{p \mid \nabla\theta(p) = C(p)p$

where $C(p)$ is symmetric$\}$

The duality theorem is: If the primal has a solution, then the dual has a solution and the value of the primal equals the value of the dual.

This new duality relationship is more general than Dorn's in the sense that $\nabla\theta(p)$ need only be <u>locally quadratic and symmetric</u>. It is less general in the sense that the domain of p is restricted to set S for the primal and T for the dual. A proof of the theorem will make use of the

Kuhn-Tucker theorem as in Mangasarian's (1969; p. 126) exposition of Dorn's result. The difficult step involves the definition of the properties of S and the mapping of the set of admissable p vectors into the set of p vectors which minimize $\theta(p)$. A proof remains to be developed.

Finally, we should note that the introduction of endogenous transportation costs into the model with non-linear schedules requires no essentially new analysis. The analysis in Sections 2 and 3 and results in Sections 4 and 5 carry over to the non-linear case with no essential change.

7. Concluding Remarks

We have taken Mosak's general equilibrium trade model and introduced endogenous transportation costs by means of the Cournot-Samuelson economic rent maximization principle. With regard to trade and monetary theory, we noted the emergence of new balance of payments problems, new problems in the definition of a tatonnement, and new problems in the definition of the price level and a monetary equilibrium.

The Mosakian model with exogenous transportation costs and linear supply and demand schedules has been analysed in a rapidly growing literature on "spatial price equilibrium". We have been able to define an economically meaningful dual spatial price equilibrium problem and to focus on the effects of endogenous transportation costs on the nature of the specification of the model and of the equilibrium. Finally, we observed that a model with non-linear demand and supply schedules could be defined and that symmetry in the basic matrix defining a model of spatial price equilibrium played an indispensible role. A new duality relationship, generalizing Dorn's quadratic programming problem was presented.

FOOTNOTES

1. The most recent one is recorded in the current issue of the American Economic Review, where the protagonists are M. Krausse, D.M. Winch, and E.J. Mishan (1971; p. 199).

2. See for example Nikaido (1970; Sections 33 and 40).

3. One delivery will consist of a ship or vehicle transporting the commodity to the delivery point from the supply point and having that ship or vehicle return empty. Other settings could be envisaged but the accounting procedures will always remain the same in an economic sense.

4. The Hitchcock-Koopmans transportation problem in linear programming and the Cournot-Enke-Samuelson problem in spatial price equilibrium deal with only one commodity. These problems have been subsumed in a more general one commodity model in Hartwick (1970), (1971). Mosak treated the many-commodity-problem or the general equilibrium model but without transportation costs.

5. In a one commodity model, with exogenous transportation costs, the π^e's and x^e's defined for the system are indeed the prices and interregional flows obtaining for the case when transportation costs are zero. See Hartwick (1971). However in a multicommodity case the removal of transportation costs and subsequent maximization of economic rent will result in an equilibrium in which the prices and flows are different from the π^e'S and x^e's indicated in Figure 2. The π^e's and x^e's in our general model must be defined in terms of the equilibrium obtaining when transportation costs are not zero.

6. These remarks indicate that the primal-dual programs developed below are not suitable algorithms for solving general spatial price equilibrium problems but are suitable for elucidating the nature of an equilibrium once it is known to exist. Alternatively, we can consider a pair of primal-dual problems for this spatial price equilibrium problem as follows: a solution to the constrained economic rent maximization problem yields new variables, π_i^e's and x_i^e's which appear in the dual constrained economic rent foregone minimization problem. This phenomenon of having variables from the primal appear in the dual of a non-linear program is a general result although the particular case developed below is novel in the use that it makes of the variables carried over from one problem to the other.

7. Observe that in fixing π_4 we are fixing values for (i) the price of transportation between any two points for any commodity and (ii) the intercepts of the schedules in Figure 2. We are thus in a position to set up a constrained economic rent maximization problem as Takayama and Judge (1964) and (1970) have done. We proceed differently however in order to develop an economically meaningful dual problem to that of Takayama and Judge and to analyze the economic implications of having transportation costs endogenous.

8. For a two-region case with variable demands and supplies, the objective function is:

minimize $\frac{1}{2}pCp - rp$ or the negative of area

AERFD = $-\frac{1}{2}$EBCF + ABCD in Figure 1F below.

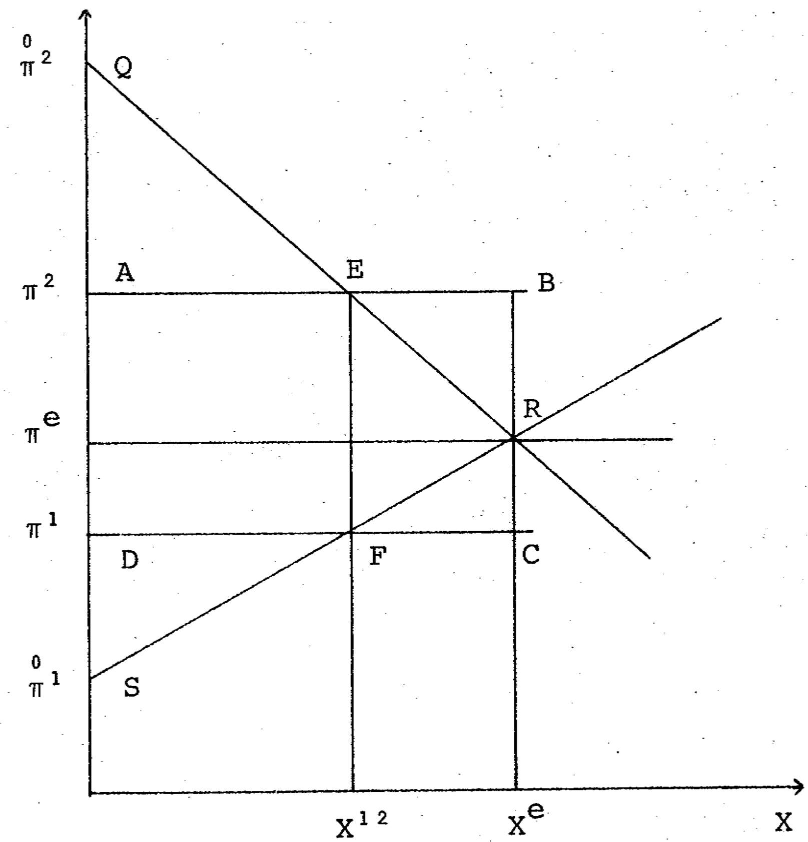


Figure 1F

The dual objective function is:

maximize $-\frac{1}{2}pCp - tv$

or the negative of area

$$AERFD = -\tfrac{1}{2}EBCF - AEFD$$

in Figure F1.

9. See Debreu (1959; p. 19) or the exposition of the same material in Lancaster (1968; pp. 349-352). Heuristically, what we are doing in the proof of a fixed point in this problem is to vary the price of the transportation good over all non-negative values and to check whether, for a particular value of the price π_4, the amount of the transportation good demanded equals the amount supplied. The proof of the existence of a fixed point assures us that an equilibrium exists although of course numerically, there is no way to solve for such an equilibrium. See Dorfman, Samuelson and Solow (1958, Section 13.4) for a lucid account of similar existence result namely the existence of a solution in the Walras-Cassel model.

10. See Samuelson (1947; pp. 270-274).

11. Woodland (1970) has developed a technique for numerically solving spatial price equilibrium problems which treats each point as an "individual" and the sum of all individual functions as market functions. Spatial price equilibrium problems are then treated as disaggregated market price convergence problems.

12. See for example Gale (1960; p. 15).

13. See footnote 9.

14. The problem raised in this model for a stability analysis are similar to those first discussed by Walras in a model with production as opposed to a model with only exchange. See Patinkin (1965; p. 534).

15. The assumption of linearity can be removed without changing the results as we shall note in Section 6.

16. See for example Archibald and Lipsey (1958).

17. In a recent paper dealing with a related model of spatial price equilibrium, Takayama and Woodland (1970) have demonstrated that symmetry in the above matrix must hold for the value of their dual to equal the value of their primal. One might examine the proofs of duality in quadratic programming to observe where the symmetry assumption is invoked.

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