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THE GENERALIZED TRANSPORTATION PROBLEM AS A QUADRATIC PROGRAM

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I. Introduction

In Hartwick [1970] I indicated that there existed an economic rent maximization problem which generalized the Hitchcock [1941]-Koopmans [1947] transportation problem in linear programming and that this new problem was a special case of the Cournot-Enke-Samuelson [1952] spatial price equilibrium problem. The latter problem was shown to be an economic rent maximization problem, also. My suggested generalization of the Hitchcock-Koopmans problem was to replace the fixed demands and supplies in the linear program by demand schedules and supply schedules which were functions of delivered and supplied prices respectively and to solve for the optimal interpoint flows by means of maximizing gross economic rent in the system.

I asserted that a dual problem to the generalized transportation problem (the Hitchcock-Koopmans, generalized Hitchcock-Koopmans, and Cournot-Enke-Samuelson problems) was a problem in which economic rent foregone was minimized, the economic rent being foregone because of the presence of non-zero transportation costs. That dual was presented and suggested to represent a new optimality principle in spatial price equilibrium.

In this paper I shall demonstrate that when all demands and supply schedules are either linear or are points or scalars, then the generalized transportation problem and its dual can be expressed as a primal-dual quadratic program, this latter program being due to Dorn [1960] and Mangasarian [1969]. The duality theorem of Dorn [1960] provides the mathematical foundation for the existence of a solution to the linear version of the generalized transportation problem and its dual.

2. A Primal-Dual Quadratic Program

The class of programs which we are concerned with is

$$\text{Minimize} \qquad f(z) = \tfrac{1}{2}zCz+rz \tag{1}$$

subject to

$$Az \geqq b \tag{2}$$

$$z \geqq o \tag{3}$$

where C is a symmetric, positive definite nxn matrix and where r is an nxl vector, z is an nxl vector and A is an mxn matrix. This problem will be referred to as Problem DI.

A dual problem to Problem DI is

$$\text{Maximize} \qquad g(z,v) = -\tfrac{1}{2}zCz+bv \tag{4}$$

subject to

$$A'v - Cz = r \tag{5}$$

$$v \geqq o \tag{6}$$

where v is an mxl vector. This problem is due to Dorn [1960] and will be referred to as Problem DII. Note that a transpose of a matrix is indicated by a prime and row and column vectors do not have a distinct symbol.

Below are the duality theorems without proofs for the Problems DI and DII due to Dorn [1960] and Mangasarian [1969], and two other theorems of interest for the generalized transportation problem in Section 3.

Theorem 2.1 (Dorn [1960]; Duality Theorem).

If $\tilde{z}$ solves Problem DI, then Problem DII has a solution $(\tilde{z},\tilde{v})$. Also

$$f(\tilde{z}) = g(\tilde{z},\tilde{v}) \tag{7}$$

Theorem 2.2 (Mangasarian [1969]; Converse Duality Theorem.)[1]

If $(\tilde{z},\tilde{v})$ solves Problem DII, then Problem DI has a solution $\hat{z}$. Also $C(\tilde{z}-\hat{z}) = 0$ and $f(\hat{z}) = g(\tilde{z},\tilde{v})$.

Theorem 2.3 (Optimality Criterion).

Let z_o and (z_o,v_o) be feasible solutions to Problems DI and DII respectively and

$$f(z_o) = g(z_o,v_o) \tag{8}$$

Then z_o and (z_o, v_o) are optimal solutions to Problems DI and DII respectively.

Proof: Let $\hat{z}$ be a feasible solution to Problem DI.

From (1)

$$f(\hat{z})-f(z_o) = \tfrac{1}{2}\hat{z}C\hat{z}+r\hat{z}-\tfrac{1}{2}z_oCz_o-rz_o$$

and from (8)

$$= \tfrac{1}{2}zCz+rz+\tfrac{1}{2}z_oCz_o-bv_o$$

and since C is positive semi-definite[2]

$$f(\hat{z})-f(z_o) \geqq \hat{z}Cz_o+r\hat{z}-bv_o \qquad (9)$$

From (5) and (3)

$$\hat{z}Cz_o = \hat{z}(A'v_o-r)$$

and from (2) and (6)

$$-bv_o \geqq -\hat{z}A'v_o$$

Substituting these last two relationships in (9)

$$f(\hat{z})-f(z_o) \geqq \hat{z}A'v_o-\hat{z}r+r\hat{z}-\hat{z}A'v_o = 0.$$

Thus z_o solves Problem DI.

Since DI has a solution, then by Theorem 2.1, DII has a solution (z_o, v_o).

In the following theorem, a_i will be the ith row of matrix A, v_i the ith entry in vector v, and β_i the ith entry in vector b.

Theorem 2.4 (Equilibrium Theorem)

Let $\hat{z}$ and $(\hat{z},\hat{v})$ be feasible solutions to Problems DI and DII respectively. These solutions are optimal if and only if

$$\hat{v}_i = 0 \quad \text{whenever } a_i\hat{z} > \beta_i \tag{10}$$

Proof: Suppose condition (10) holds. From (3) and (10)

$$\hat{v}A\hat{z} = \hat{v}b \tag{11}$$

From (5),

$$\hat{v}A\hat{z}-\hat{z}C\hat{z} = \hat{z}r \tag{12}$$

Substituting from (11) in (12), we get

$$\tfrac{1}{2}\hat{z}C\hat{z}+r\hat{z} = -\tfrac{1}{2}\hat{z}C\hat{z}+b\hat{v}$$

which by Theorem 2.3 indicates that $\hat{z}$ and $(\hat{z},\hat{v})$ are optimal solutions to Problems DI and DII respectively.

Suppose now that $\hat{z}$ and $(\hat{z},\hat{v})$ are optimal solutions. From Theorem 2.1

$$\tfrac{1}{2}\hat{z}C\hat{z}+r\hat{z} = -\tfrac{1}{2}\hat{z}C\hat{z}+b\hat{v}$$

or

$$-\hat{z}C\hat{z} = r\hat{z}-b\hat{v} \tag{13}$$

$\hat{z}$ and $(\hat{z},\hat{v})$ must satisfy (2)(3) and (5)(6) respectively.

From (5)

$$\hat{z}A'\hat{v}-\hat{z}C\hat{z} = \hat{z}r$$

Substituting from (13)

$$\hat{v}A\hat{z}-b\hat{v} = 0$$

or

$$\sum_{i=1}^{m} \hat{v}_i(a_i z-\beta_i) = 0 \tag{14}$$

Since $\hat{v}_i$ and $(\alpha_i z-\beta_i)$ are non-negative for all i, from (2) and (3), each component of the sum in (13) must equal zero. Hence condition (10) obtains.

3. The Generalized Transportation Problem

Our geographic world will consist of m spatially separated points of supply for one homogeneous commodity and n spatially separated points of demand for the commodity. The supply points will be subdivided into three non-empty sets. Each point 1,...,h will have a linear excess supply schedule defined as a function of supply price. Each point $h+1,\ldots,k$ will have a linear supply schedule defined as a function of supply price. Each point $k+1,\ldots,m$ will have a fixed supply defined. The inverse functions of the schedules at points $1,\ldots,k$ will also be defined, that is supply price as a function of quantity supplied.

In summary, the m supply points are defined by their quantities supplied.

$$\left.\begin{array}{l} es_i = -\omega_i\pi_i^o + \omega_i\pi_i \\ \text{or} \\ \pi_i = \pi_i^o + \dfrac{1}{\omega_i}\, es_i \end{array}\right\} \quad (i=1,\ldots,h)$$

$$\left.\begin{array}{l} s_i = -\hat{\omega}_i\pi_i^o + \hat{\omega}_i\pi_i \\ \text{or} \\ \pi_i = \pi_i^o + \dfrac{1}{\hat{\omega}_i}\, s_i \end{array}\right\} \quad (i=h+1,\ldots,k)$$

$$\left.\sigma_i\right\} \quad (i=k+1,\ldots,m)$$

where es_i is excess supply at point i $(i=1,\ldots,h)$

s_i is supply at point i $(i=h+1,\ldots,k)$

σ_i is supply at point i $(i=k+1,\ldots,m)$

π_i is the price at a supply point i $(i=1,\ldots,m)$

The n demand points are analogously defined in terms of three subsets of demand relations, excess demand schedules, demand schedules, and fixed demands.

$$\left.\begin{array}{l} ed_j = \eta_j \pi_j^{o\prime} - \eta_j \pi_j^{\prime} \\ \text{or} \\ \pi_j^{\prime} = \pi_j^{o\prime} - \dfrac{1}{\eta_j} ed_j \end{array}\right\} \quad (j=1,\ldots,h')$$

$$\left.\begin{array}{l} d_j = \hat{\eta}_j \pi_j^{o\prime} - \hat{\eta}_j \pi_j^{\prime} \\ \text{or} \\ \pi_j^{\prime} = \pi_j^{o\prime} - \dfrac{1}{\hat{\eta}_j} d_j \end{array}\right\} \quad (j=h'+1,\ldots,k')$$

$$\left.\delta_j \right\} \quad (j=k'+1,\ldots,n)$$

where ed_j is excess demand at point j $(j=1,\ldots,h')$
d_j is demand at point j $(j=h'+1,\ldots,k')$
δ_j is demand at point j $(j=k'+1,\ldots,n)$
π'_j is the price at a demand point j $(j=1,\ldots,n)$

We define a single excess supply or excess demand at a point on the grounds that know a priori whether a point with both demand and supply schedules will be a net exporter or importer of the commodity in equilibrium. A preliminary problem is solved before the quadratic program is set up. Our quadratic program is thus not a satisfactory algorithm for solving actual spatial equilibrium problems but is a device for elucidating the nature of a general spatial price equilibrium.

Figure 1 below will illustrate the nature of the ensuing quadratic program for a two point case. It is in fact Samuelson's spatial price equilibrium diagram.[3]

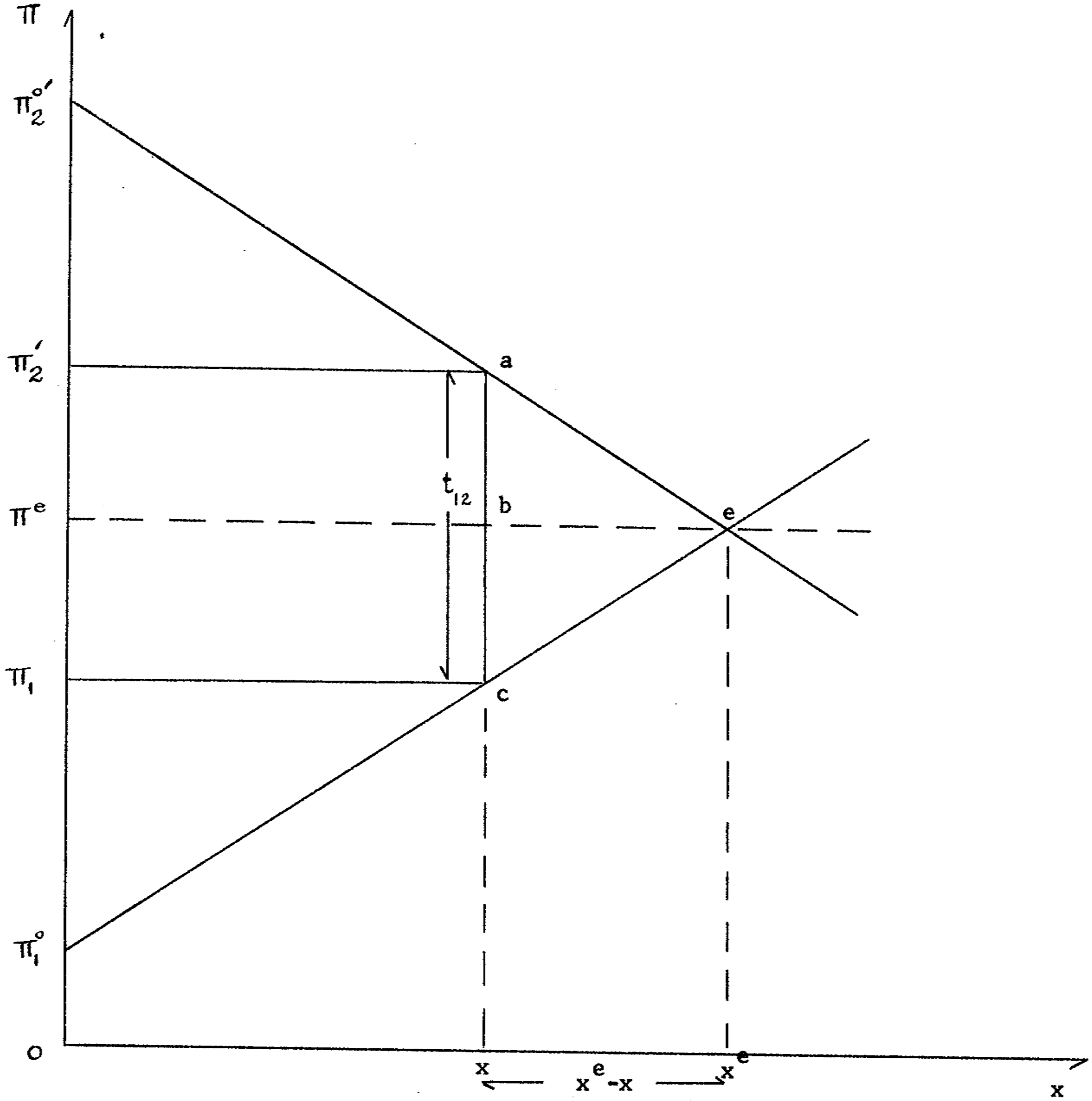

Figure 1

The x axis indicates interpoint flow. The p axis indicates price. Thus an excess demand is equal to an excess supply by definition since following Samuelson schedules $x=\pi_1^o+\omega_1\pi_1$ and $x=\pi_2^{o'}-\eta_2\pi_2'$ are excess supply and excess demand schedules respectively. Samuelson's two point spatial price equilibrium problem was to maximize area $\pi_2^{o'}ac\pi_1^o - \pi_2'ac\pi_1$ or what he termed net social payoff. Hartwick's dual was to minimize area ace subject to $\pi_2'-\pi_1 \geqq t_{12}$.

If we let the excess supply and excess demand schedules become simply supply and demand schedules at two points, we have the generalized Hitckcock-Koopmans transportation problem. Once again the maximization of area $\pi_2^{o'}ac\pi_1^o - \pi_2'ac\pi_1$ yields a flow equilibrium in the primal problem and the minimization of area ace subject to $\pi_2'-\pi_1 \geqq t_{12}$ yields a price equilibrum in the dual problem. We could of course combine a partial Samuelson problem and a partial generalized Hitchcock-Koopmans problem.

In Figure 1, π^e and x^e indicate the price and interpoint flow which would obtain if transportation costs were zero. Thus triangle ace indicates economic rent foregone because transportation costs are positive. For any point we can define quantity K_i as the area of a triangle

$$K_i = \pi_i^o \pi_i^e x_i^e \quad \text{and} \quad \begin{array}{l}(i=1,\ldots,k) \text{ for supply points} \\ (i=1,\ldots,k') \text{ for demand points}\end{array}$$

where $\pi_1^e = \pi_2^{e'} = \pi^e$ in Figure 1

and $x_1^e = x_2^e = x^e$ in Figure 1.

K_i thus represents the total economic rent which a point could reap if transportation costs were zero. K_i is a constant which once we have determined the general pattern of interpoint flows, does not vary with the precise solution values for prices or flows. We shall have occasion to make use of $K = \sum_{i=1}^{k+k'} K_i$.

We shall also introduce another constant

$$\tilde{T} = \sum_{j=1}^{k'} x_{ji}^e t_{ji}$$

where t_{ji} is the cost of transporting a unit of the commodity from supply point i to demand point j. $x_{ji}^e = x_j^e = x_i^e$.

Since the schedules in Figure 1 are linear, the two point primal-dual problems sketched above can be expressed separately as quadratic programs. We will now see that these two programs do in fact define a saddle point problem or that they are indeed dual to each other.

The primal problem to the generalized transportation problem will be defined as follows and labelled Problem GI. We are minimizing economic rent foregone.

Minimize $\quad f(p) = \frac{1}{2}pCp+rp$

subject to

$$Ap \geqq t$$
$$p \geqq 0$$

where p is an mnxl vector

$$p = (\pi_1^{\prime}-\pi_1^{\prime e}, \ldots , \pi_j^{\prime}-\pi_j^{\prime e}, \ldots , \pi_n^{\prime}-\pi_n^{\prime e}, \pi_1^{e}-\pi_1, \ldots , \pi_i^{e}-\pi_i, \ldots , \pi_m^{e}-\pi_m)$$

where C is a diagonal mnxmn positive semi-definite matrix

$$C = \begin{bmatrix} \eta_1 & & & & & & & & & & & & \\ & \eta_2 & & & & & & & & & & & \\ & & \ddots & & & & & & & & & O & \\ & & & \hat{\eta}_{k'} & & & & & & & & & \\ & & & & 0 & & & & & & & & \\ & & & & & \ddots & & & & & & & \\ & & & & & & 0 & & & & & & \\ & & & & & & & \omega_1 & & & & & \\ & & & & & & & & \omega_2 & & & & \\ & & & & & & & & & \ddots & & & \\ & O & & & & & & & & & \hat{\omega}_k & & \\ & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & & 0 \end{bmatrix}$$

where r is a lxmn vector

$$r = (0,\ldots,0_{k'},-\delta_{k'+1},\ldots,-\delta_n,0,\ldots,0_k,\sigma_{k+1},\ldots,\sigma_m)$$

where A is an mnxmn "transportation" matrix with two ones in each row, the remaining entries being zeros. The first entry in a row is associated with a demand at a point and the second with a supply. If in any row the relevant demand is a fixed value <u>and</u> the relevant supply is a fixed value, then the two entries are negative and positive respectively. These rows correspond with entries in the t vector with negative signs. The t vector is defined following the A matrix. Otherwise the entries are positive and negative from left to right in each row.

$$A = \left[\begin{array}{ccccccccccccccc}
1&&&&-1\\
1&&&&&-1\\
\vdots&&&&&&\ddots\\
&1&&&&&&-1\\
&1&&&&&&&-1\\
&\vdots&\vdots&&&&&&&\ddots\\
&&-1&&&&&&&&+1\\
&&-1&&&&&&&&&+1\\
&&&\vdots&&&&&&&&&\ddots\\
&&&-1&&&&&&&&&&+1\\
&&&-1&&&&&&&&&&&+1
\end{array}\right]$$

and where t is an mnx1 vector

$$t = (t_{11}, t_{21}, \ldots, t_{nk}, \ldots, t_{k',k+1}, -t_{k'+1,k+1}, \ldots,$$
$$-t_{n,k+1}, \ldots, -t_{n,m})$$

Note that if C were zero and the related part of A removed, we have the dual to the Hitchcock-Koopmans linear programming, transportation problem.[4] If on the other hand r were zero and the related part of A removed, we have the dual to the combined generalized Hitchcock-Koopmans and Cournot-Enke-Samuelson problem. Either of these two problems could be isolated if the elements of the other problem were removed from matrices C and A when r was still taken to be zero.

Problem GI is of the form of Problem DI and so if a feasible solution exists to Problem GI we know that it has a minimum and that its dual exists and also has a solution by Theorem 2.1. Following Dorn and Mangasarian, the dual to Problem GI is the following, Problem GII.

Maximize $\qquad g(p,v) = -\frac{1}{2}pCp+tv$

subject to

$$A'v-Cp = r$$

$$v \geqq 0$$

We now observe that since min $f(p)$ = max $g(p,v)$ when the problems are solved, vector v can be defined as

$$v = (x^e_{11}-x_{11}, x^e_{21}-x_{21}, \dots, x^e_{nk}-x_{nk}, \dots, x^e_{k'k+1}-x_{k'k+1},$$

$$+x_{k'+1,k+1}, \dots, +x_{n,k+1}, \dots, +x_{n,m})$$

and v is in fact the desired interpoint flow vector since it is also feasible.[5]

Now if we cancel the opposite signed constants x^e_{ji}'s from the constraints of Problem GII and alter the objective function of Problem GII as follows:

$$K-\tilde{T}+g(p,v)$$

where constants K and $\tilde{T}$ were defined above, we see that Problem GII is an interpoint pure flow problem which yields the flows which simultaneously maximize economic rent in the system and minimize transportation costs subject to quantities demanded equalling quantities supplied. We shall label Problem GII amended to result in a rent maximization problem, Problem GTII. A solution to Problem GI implies a solution to Problem GII which in turn implies a solution to GTII, the generalized transportation problem, the solution of which yields optimal interpoint flows.

If C is made equal to zero, the corresponding part of A′ removed, and the equality in the constraints changed to less than or equal to, we have the Hitchcock-Koopmans linear programming transportation problem. Alternatively, if the elements of C,t,A′, and r related to the supplies σ_i $(i=k+1,\ldots,m)$ and $\delta_j(j=k'+1,\ldots,n)$ were removed, we would have the primal problem for the combined Cournot-Enke-Samuelson problem and the generalized

Hitchcock-Koopmans problem. These problems could be isolated as distinct problems by further excisions of elements of C,t,A', and r.

Theorem 2.4 indicates that one of the price equilibrium relationships of the Hitchcock-Koopmans transportation problem carries over to the general problem in GI and GII. That is, if the equilibrium prices between any two fixed demand and fixed supply points differ by less than the cost of transport, then no shipment will be undertaken between those two points. On the other hand, if the equilibrium prices between any other two demand and supply points j and i differ by more than the cost of transport then $x_{ji} = x^e_{ji}$ or the actual flow between i and j will equal the flow taking place if there were no transportation costs. The economics of this last condition is clear. If in equilibrium profits can be made from transporting the commodity between two points, then transportation will be carried on to the point where demanders are all served or where the maximum possible flow takes place. That maximum flow is the flow obtaining when transportation costs are zero. As long as the program has feasible solutions and at least one demand, supply, excess demand, or excess supply schedule has some non-zero and finite slope, then we will not observe $x_{ji} = x^e_{ji}$ in equilibrium. By introducing some elasticity of demand and supply into the transportation problem, we rule out the necessity for corner solutions; excess supplies will not obtain if the problems are feasible. The familiar condition in the Hitchcock-Koopmans problem of an excess supply in equilibrium implying a zero price does not appear in the general problem.[6]

Theorem 2.4 indicates that if problems GI and GII are feasible and if the value of the objective function of Problem GI, plus the value of the objective function of Problem GTII, plus $\sum_{i,j} t_{ji}x_{ji}$ sum to K, then the problems are optimized. For the two-point Cournot-Enke-Samuelson case or the generalized Hitchcock-Koopmans case, Theorem 2.4 indicates if $\bar{p}=(\pi_2',\pi_1)$ and $\bar{x}=(x_{21})$ are feasible and if area $\pi_2^{o\prime}\pi_2'a+\pi_1^o\pi_1c+\pi_2'ac\pi_1+ace$ equal $\pi_2^{o\prime}\pi_1^oe$ in Figure 1 then a solution $\bar{p}$, $\bar{x}$ has been obtained.

4. The Dual and Analogies in Physics

Koopmans and Reiter [1951; pp. 258-259] indicated that Kirchhoff's Law on the distribution of current in an electrical network was derived from a constrained quadratic minimization problem in which total heat was minimized subject to certain constraints on the flows of electrical current and that this problem was analogous to the minimization of transportation costs subject to flow constants.[7] Since heat is associated with waste or a loss of energy in the transmission of electrical current, the analogy might be made instead with minimizing economic rent foregone rather than with minimizing transport costs. The problem analogous to the Kirchhoff quadratic program would then be the Problem GI or the constrained minimization of economic rent foregone because of the presence of positive transportation costs. Problem GI does not contain the Hitchcock-Koopmans constrained transportation cost minimization problem within it. Transportation costs in a social accounting sense are not a loss or waste in the same sense in which economic rent foregone is a loss.

The actual reason for subtracting transportation costs from the objective function of maximizing economic rent is that in a fully specified general equilibrium problem like that of Mosak [1944; Chapt. 5] except for the addition of transportation costs, or in other words in a Mosak model with endogenous transportation costs, there will be a sector producing transportation goods the value of whose output is identical in value to the costs of transportation. Hence when we net out transportation costs from rent maximization, we are actually netting out the value associated with a sector which is not producing any output for final consumption. Transportation cost minimization inheres in the maximization of the value of output and the former is in general not the objective of economic men. It is the latter goal which is the objective.

The treatment of problems like the generalized transportation problem as a quadratic program has been carried out by Takayama and Judge [1964][1970]. These authors dealt with a many commodity extension of the Cournot-Enke-Samuelson model. However in neither paper have they been successful in formulating a true dual problem.[8]

FOOTNOTES

1. Dorn [1960] enunciated this theorem but the proof was marred by an error. Mangasarian's proof is based on the Kuh-Tucker theorem and makes no reference to Dorn's theorem except to mention the origin of the general enunciation. Hence, we ascribe the theorem to Mangasarian.

2. Since C is positive semi-definite, for any two vectors x and y,

$$(x-y)C(x-y) \;=\; xCx + yCy - 2xCy \geqq 0$$

or

$$xCx + yCy \geqq +2xCy \;.$$

3. See Samuelson [1952; p. 288].

4. See Gale [1960; p. 15].

5. Cp becomes a flow vector equal to $x^e_{ji} - x_{ji}$.

6. An interesting equilibrium condition for the linear programming transportation problem which appears to have been overlooked by Gale [1960; p. 15] in his exposition is that an optimal interpoint flow vector must satisfy strict equality for all fixed demands. This is a natural economic condition, namely that in minimizing transportation cost, one must minimize the weighted flows from supply points to demand points and given an aggregate linear transportation cost function, this can be done by always not over supplying any demander. To prove this result, assume an optimal solution exists in which one demand δ_j is less than the flow received. Hence by the

equilibrium theorem for linear programming, the demand price π'_j at j must be zero. But if $\pi'_j=0$, the dual price constraint, $\pi'_j-\pi_i = t_{ij}$ $(t_{ij} \neq 0;\ \pi'_j,\pi_i = 0)$ can never be satisfied.

Hence equality for demands must be satisfied by an optimal solution.

7. For other problems in physics, see Dorn [1960] and references therein.

8. As far as I know, the economic rent foregone minimization dual originated in my paper Hartwick [1970]. The definition of the generalized Hitchcock-Koopmans derives from that paper also and hence so does the generalized transportation problem. In that earlier paper, the analysis relied heavily on the use of diagrams. Those diagrams make clear the nature of the Cournot-Enke-Samuelson model as an economic rent max-min problem as well as the nature of the generalized Hitchcock-Koopmans problem. Figure I is the basic diagram for the two point case of the latter problem.

REFERENCES

Dorn, W.S. [1960], "Duality in Quadratic Programming", Quarterly of Applied Mathematics (17), pp. 155-162.

Hartwick, John M. [1970], "A Generalization of the Transportation Problem in Linear Programming and Spatial Price Equilibrium", Queen's University Institute for Economic Research, Working Paper 30, October, pp. 1-25.

Hitchcock, F.L. [1941], "The Distribution of a Product from Several Sources to Numerous Localities", Journal of Mathematics and Physics, July, pp. 224-230.

Koopmans, Tjalling C. [1947], "Optimum Utilization of the Transportation System", in Proceedings of the International Statistical Conferences, Vol. 5, Washington, D.C.

Koopmans, T.J. and S. Reiter, [1951], "A Model of Transportation", in Activity Analysis of Production and Allocation, ed. T.C. Koopmans, New York, John Wiley, Chapter 14.

Mangasarian, Olvi L. [1969], Nonlinear Programming, New York, McGraw-Hill.

Mosak, Jacob [1944], General Equilibrium Theory in International Trade, Bloomington, Indiana, Principia Press, Chapter 5.

Samuelson, Paul A. [1952], "Spatial Price Equilibrium and Linear Programming", American Economic Review (42), June, pp. 283-303.

Takayama, T. and G.G. Judge [1964], "Equilibrium Among Spatially Separated Markets: A Reformulation", Econometrica (32), pp. 510-524.

__________, [1970], "Alternative Spatial Equilibrium Models", Journal of Regional Science, April, pp. 1-12.