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THE LOCATION OF FIRMS AND GENERAL SPATIAL PRICE EQUILIBRIUM

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Alfred Weber [17] developed a model to show how a firm might locate when entering production given geographically distinct markets for the firm's product and the firm's inputs. The optimal location for the firm entering was shown to result from the solution to a straightforward transportation cost minimization problem.[1] In a separate stream of development Cournot [2] and Samuelson [13] demonstrated that a problem of economic rent maximization could define a set of prices and interpoint flows in space which characterized a spatial price equilibrium in reality. In this paper I will indicate that Cournot-Samuelson principle of economic rent maximization and the extension to a dual problem in Hartwick [4] can be shown to generalize the Weber problem in a number of directions.[2]

In Section I, the problem of locating a single firm serving a single product market is developed in general and the special case of the Weber Problem is indicated. In Section 2, the single firm problem is extended to incorporate more general spatial constraints in the form of certain inelasticities in supply and demand and of rivals in fixed geographic positions. In Section 3, the simultaneous location of many firms producing a homogeneous product is analyzed. In Section 4, the simultaneous location of many firms producing *different* products as analyzed. A general equilibrium model of location is outlined in Section 5. Section 6 contains comments and conclusions.

The utilization of changes in levels of economic rent, the sum of producers' and consumers' surpluses, as measures of changes in social welfare dates from Cournot's *Mathematical Principles*. There is also a vast literature dealing with the quality of economic rent as a social welfare measure. Hotelling [7] rigorously investigated this latter issue in his investigation of the welfare effects of an excise tax. However his paper antedated the celebrated Bergson paper on the Social Welfare Function. Nonetheless changes in economic rent do have normative significance.[3] We will speak of optimal location in the sense of economic rent maximizing location.

An alternative interpretation of a problem of economic rent maximization is simply to assert following Samuelson that "net social payoff" or economic rent "is artificial in the sense that after an Invisible Hand has led us to its maximization, we need not necessarily attach any social welfare significance to the result". [13; p.288]. In this sense our primal-dual problems whose solutions yield a location equilibrium provide a useful fiction. The rent maximization problem is then of positive rather than normative significance.

The elucidation of the nature of a locational equilibrium is an essentially mathematical exercise and it particularly turns on the proof of existence of solutions to a class of mathematical problems. The emphasis in this paper is on revealing the nature of a locational equilibrium for alternative classes of firm entry problems and presenting notes on the underlying mathematical nature of the equilibrium. Rigorous proofs of the existence of solutions to the fundamental non-linear primal-dual problems are not attempted here. Proofs of existence for analogous problems have been provided by Takayama and Judge [14],[15] and Takayama and Woodland [16]. Only the rudiments of a proof of the existence of a general equilibrium are presented in Section 5. The basic locational problems are readily amenable to graphical presentation and many figures are presented.

1. The Case of One Firm, One Product Market and Many Suppliers

We shall first consider the familiar Weber landscape with a market located at a point in the plane, say a town, and numerous input markets located at points in the plane. The markets as points are all geographically separated from each other by homogeneous space. The points can all be considered as towns. At the product market is a demand schedule where quantity demanded is a decreasing function of delivered price. At each supply market is a supply schedule where quantity supplied is an increasing function of price fob or price net of transportation costs. Our problem is to determine the optimal location for a firm which supplies a finished product to the product market, incurring some transportation costs in the shipment of the product, and which receives its inputs for production from the diverse supply markets, where inputs are shipped to the firm at some transportation cost.

This problem differs from the classic Weber Problem in the respect that we have supply and demand schedules in markets which have some elasticity other than infinite for supplies and zero for demands.[4]

The technology of production is of the fixed-coefficient or Leontief type. Each unit of output requires a certain physical amount of each input, that is the output of x units of product requires $y_i = b_i x$ units of input i.[5]

Let us first consider the case of two inputs from geographically distinct sources of supply and one market as in Figure 1. The market is at 1. The supply locations are at 2 and 3.

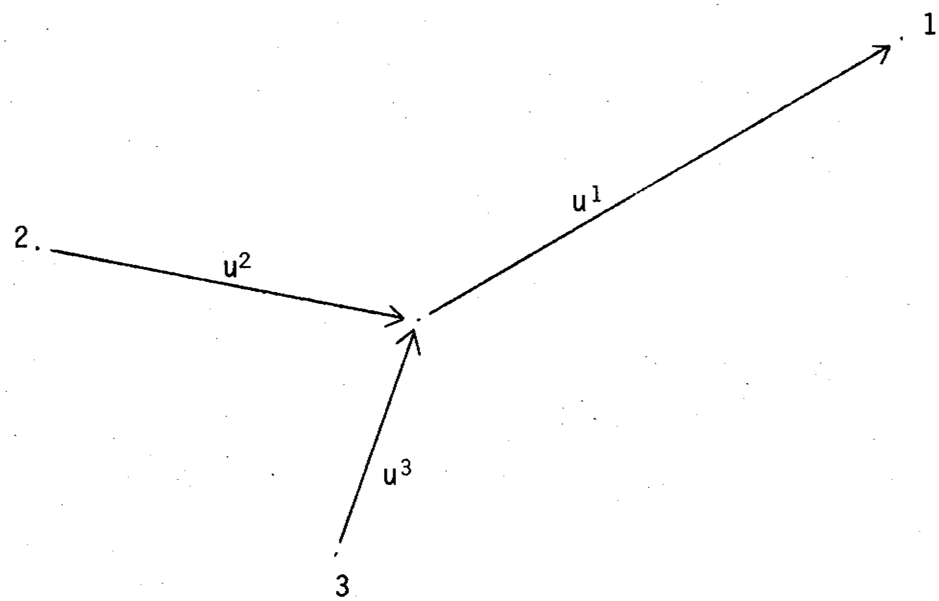


Figure 1.

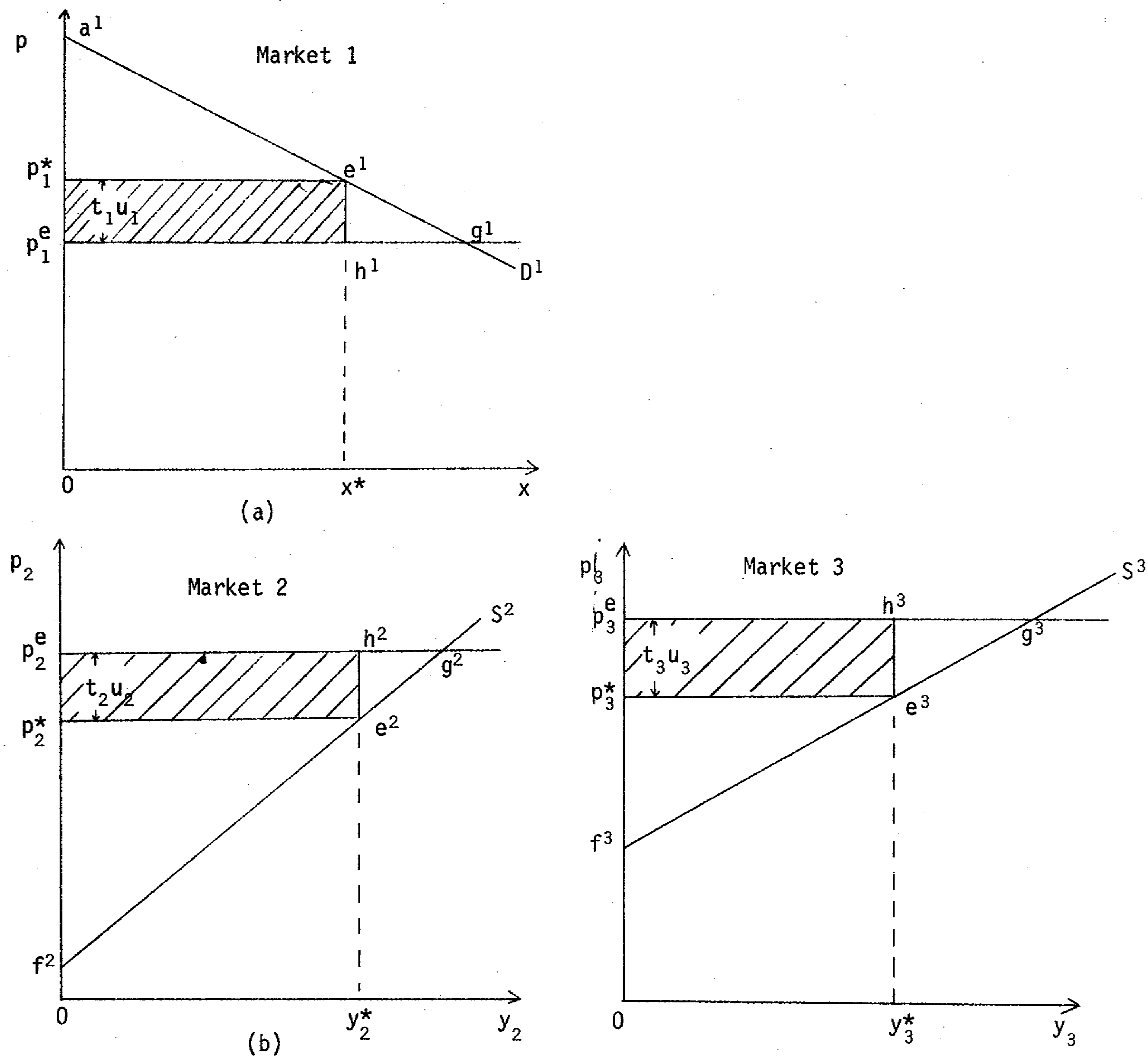
u^i is the distance from the location of the firm to point i. In order to find the optimal location, we test the level of economic rent at each potential site for the firm. The site (that is with co-ordinates u_1, u_2, u_3 with respect to fixed markets) which results in the maximization of economic rent as compared with all other sites will be optimal. We will also observe that in the limiting case of infinitely elastic supplies and inelastic demands, economic rent is zero and our problem reduces to the classic Weber Problem.

Let us consider the equilibrium conditions being satisfied by any feasible location for the firm. We select an arbitrary location with (u_1, u_2, u_3) specified. Now the price in the market for the product must equal the sum of the prices charged for inputs at the markets plus transportation costs on inputs plus the transportation good on the final product. That is

$$p(x) = (p_2(y_2)+t_2u_2)+(p_3(y_3)+t_3u_3)+t_1u_1 \qquad (1)$$

where $p(x)$ is the inverted demand schedule expressing price as a function of quantity demanded, $p_i(y_i)$ is the inverted supply schedule expressing price as a function of quantity supplied, $(i=2,3)$, and $t_i(i=1,\ldots,3)$ is the transport cost per unit distance per unit weight for commodity i.

Since $y_2=b_2x$ and $y_3=b_3x$, we can substitute in (1) and have an equation in the single variable x. The x which satisfies (1) is the equilibrium output at the chosen site. $p(x)$ will be the price of the commodity in the market at I $p_2(y_2)$ and $p_3(y_3)$ will be the prices at the supply points. Graphically, our equilibrium will appear as in Figure 2 below.



p
a^1
Market 1
p_1^*
e^1
t_1u_1
p_1^e
g^1
h^1
D^1
0
x*
x
(a)
p_2
Market 2
S^2
p_2^e
h^2
g^2
t_2u_2
p_2^*
e^2
f^2
0
y_2^*
y_2
(b)
p_3
Market 3
S^3
p_3^e
h^3
g^3
t_3u_3
p_3^*
e^3
f^3
0
y_3^*
y_3

Figure 2.

In Figure 2, the hatched areas indicated total transportation costs associated with transporting x^* to market and y_2^* and y_3^* to the firm from the supply markets. Triangle $a^1e^1p_1^*$ in Figure 2a is consumer surplus in the product market. Triangle $p_2^*e^2f^2$ in Figure 2b is producers' surplus in the market for input 2 and triangle $p_3^*e^3f^3$ is producers' surplus in the market for input 3.

To this point, we selected an arbitrary position for the firm given the other three markets geographically fixed. Cournot and Samuelson have demonstrated that a situation in which economic rent is maximized will replicate an actual equilibrium for the case of interpoint commodity flows. In other words the market adjustment is such as to maximize simultaneously the sum of consumer and producers' surpluses or economic rent. The optimal location for the firm will be where economic rent is maximized.

Formally, our three market economic rent maximization problem or optimal location problem is as follows:

Primal: Determine non-negative u_1, u_2, u_3 so as to maximize

$$\int_0^{x} p_1(x')dx' + \int_0^{b_2x} p_2(x_2)dx_2 \int_0^{b_3x} p_3(x_3)dx_3$$

$$- xt_1u_1 - b_2xt_2u_2 - b_3xt_3u_3$$

subject to:

u_1, u_2, u_3 emanating from a common vertex where the entrant is situated.

This constraint could be expressed mathematically in terms of the geographically fixed markets as is done in Hartwick [5]. We are then able to assign shadow prices to the geographically fixed sites. However this constraint is largely a mathematical formalism common to both primals and duals throughout this paper and will be left as an implicit constraint in all subsequent development. The optimal solution is in fact at a saddle-point in u_1, u_2, u_3 and there exists a dual problem which will also yield the optimal solution. The dual problem has the following economic interpretation: if we minimize the economic rent foregone because transportation costs are non-zero subject to the condition that the delivered price exceeds the price fob by at least the cost of transport, then we shall determine the optimal location for the firm. The economic rent foregone for the arbitrary location illustrated in Figure 2 is the sum of the areas of triangles $e^1g^1h^1$ plus $h^2g^2e^2$ plus $h^3g^3e^3$. Formally our dual problem for the three market case is:

Dual: Determine non-negative u_1, u_2, u_3 and prices $p_1^*, p_2^*, p_3^*, p_1^*, p_2^*, p_3^*$ so as to minimize

$$\int_{b_2p_2^e+b_3p_3^e}^{p_1^*} x(p_1)dp_1 + \int_{p_2^*}^{p_2^e} y(p_2)dp_2 + \int_{p_3^*}^{p_3^e} y(p_3)dp_3$$

$$-\ (p_1^*-(b_2p_2^e+b_3p_3^e))x(p_1^*)-(p_2^e-p_2^*)y(p_2^*)-(p_3^e-p_3^*)y(p_3^*)$$

subject to:

$$b_2p_2^e+b_3p_3^e - p_1^*+t_1u_1 \geqq 0$$

$$p_2^e - p_2^*+t_2u_2 \geqq 0$$

$$p_3^e - p_3^*+t_3u_3 \geqq 0$$

where p_i^e $(i=2,3)$ is a price of input i at the firm's or at the vertex of co-ordinates u_1,u_2, and u_3. If the solution has all u's positive, we will have literally, an interior solution and all prices and flows will be positive or alternatively strict equalities will hold in equilibrium in the constraints to the dual.

As one might expect, since the earliest problems of Cournot [2], Samuelson [13], Takayama and Judge [14], [15] and Hartwick [4],[6] are extremals dealing with economic rent, the formal structure of the above generalized Weber Problem is the same as those previously developed. Note that the primal reduces to the familiar Weber Problem when the integrals are all zero or when no economic rent figures in the solution or alternatively when the demand schedule has zero elasticity throughout and the supply schedules have infinite elasticity and we are minimizing total transport costs for a unit of the delivered product.

The extension of the case of firm entry with one product market and two input markets to the case with m product and n input markets is simply a matter of introducing more terms of consumers' and producers' surpluses into the primal-dual problem.

2. Extensions of the Location Problem for the Single Firm

(i) The case of fixed demands and supplies.

Consider the case of m geographically separate markets for a product with fixed demand requirements and n geographically separate markets for n inputs required to fabricate the product in question. We have production coefficients b_i, relating the amount of input i required per unit of output $(i=1,\ldots,n)$. The ith supply will be denoted by σ_i $(i=1,\ldots,n)$. The demand in the jth town or point will be denoted by $\delta_j (j=1,\ldots,m)$. It is clear that for the problem of supplying the demanders to have a solution $\sum_{j=1}^{m} \delta_j$ must be less than or equal to σ_i/b_i for $(i=1,\ldots,n)$.

Our problem is to locate a firm that will fabricate the product and supply the demanders and to locate that firm so as to minimize total transportation costs. Formally, our problem is:

Primal: Determine non-negative $u_i (i=1,\ldots,n)$, $u'_j (j=1,\ldots,m)$ and $x_j (j=1,\ldots,m)$ so as to

maximize

$$- \sum_{j=1}^{m} x_j t'_j u'_j - \sum_{i=1}^{n} b_i \left(\sum_{j=1}^{m} \delta_j \right) t_i u_i$$

subject to:

$$x_j \geqq \delta_j \qquad (j=1,\ldots,m)$$

$$b_i \left(\sum_{j=1}^{m} x_j \right) \leqq \sigma_i \qquad (i=1,\ldots,n)$$

Dual: Determine non-negative $u_i (i=1,\ldots,n)$, $u'_j (j=1,\ldots,m)$ $p'_j (j=1,\ldots,m)$, $p_i (i=1,\ldots,n)$, $p^e_i (i=1,\ldots,n)$ so as to

minimize

$$- \sum_{j=1}^{m} \left[p'_j - \left(\sum_{i=1}^{n} b_i p_i^{\,e} \right) \right] \delta_j - \sum_{i=1}^{n} \left(p_i^{\,e} - p_i \right) \left[\frac{\sum_{j=1}^{m} \delta_j}{b_i} \right]$$

subject to:

$$\sum_{i=1}^{m} b_i p_i^{\,e} - p'_j + t_j u_j \geqq 0 \qquad (j=1,\ldots,m)$$

$$p_i^{\,e} - p_i + t_i u_i \geqq 0 \qquad (i=1,\ldots,n)$$

where primes indicate firm to product market distances and transportation rates.

Observe that this problem is essentially a location problem as opposed to an allocation problem in the sense that the Hitchcock-Koopmans transportation problem is. Demands will always be met with strict equality implying positive shadow prices in the dual for product delivered, p_j' $(j=1,\ldots,m)$. Supplies will generally be underutilized since we have fixed production coefficients and by producing enough output to just satisfy demands, transportation costs on inputs can be minimized.[6] Hence we can expect zero shadow prices in the dual on most supplies arising mathematically and economically from the price equilibrium conditions or conditions of complementry slackness. The p_i^{e}'s are in a sense dummy prices which emerge after the price equilibrium conditions have resulted in shadow prices emerging on the fixed demands and supplies. Note finally that the objective function in the primal is non-linear and concave in u's and x's.

An example of an actual location problem of the above type might be the determination of a site for a temporary producton process, such as the construction of a building or bridge. Of course we are not restricted to having all schedules of zero elasticity for a solution to emerge. Some points of supply might have non-zero elasticities and the problem could be solved by minimizig economic rents.

(ii) The case of geographically-fixed rivals at supply and demand points

Consider the case of one market for the product at a point and two geographically distinct supply points. Let us follow the assumptions of Section I, namely that the demand schedule slopes downward and supply schedules slope upward. In addition, however, we shall consider the case when a firm is supplying the product to the demanders from a plant at the demand point. This product supplier's supply schedule will be assumed to slope upward. Also let there be a rival firm located at supply point 2 utilizing part of the input available at supply point 2. We shall assume this rival buyer's demand curve is downward sloping.

Our problem is to locate a new firm which will supply some product to the demand point I utilizing inputs from supply points I and 2. Transportation costs are fixed and are non-zero. The optimal solution will derive from primal-dual economic rent maximize-minimize problem. Graphically, the equilibrium location can be illustrated in Febure 3 below.

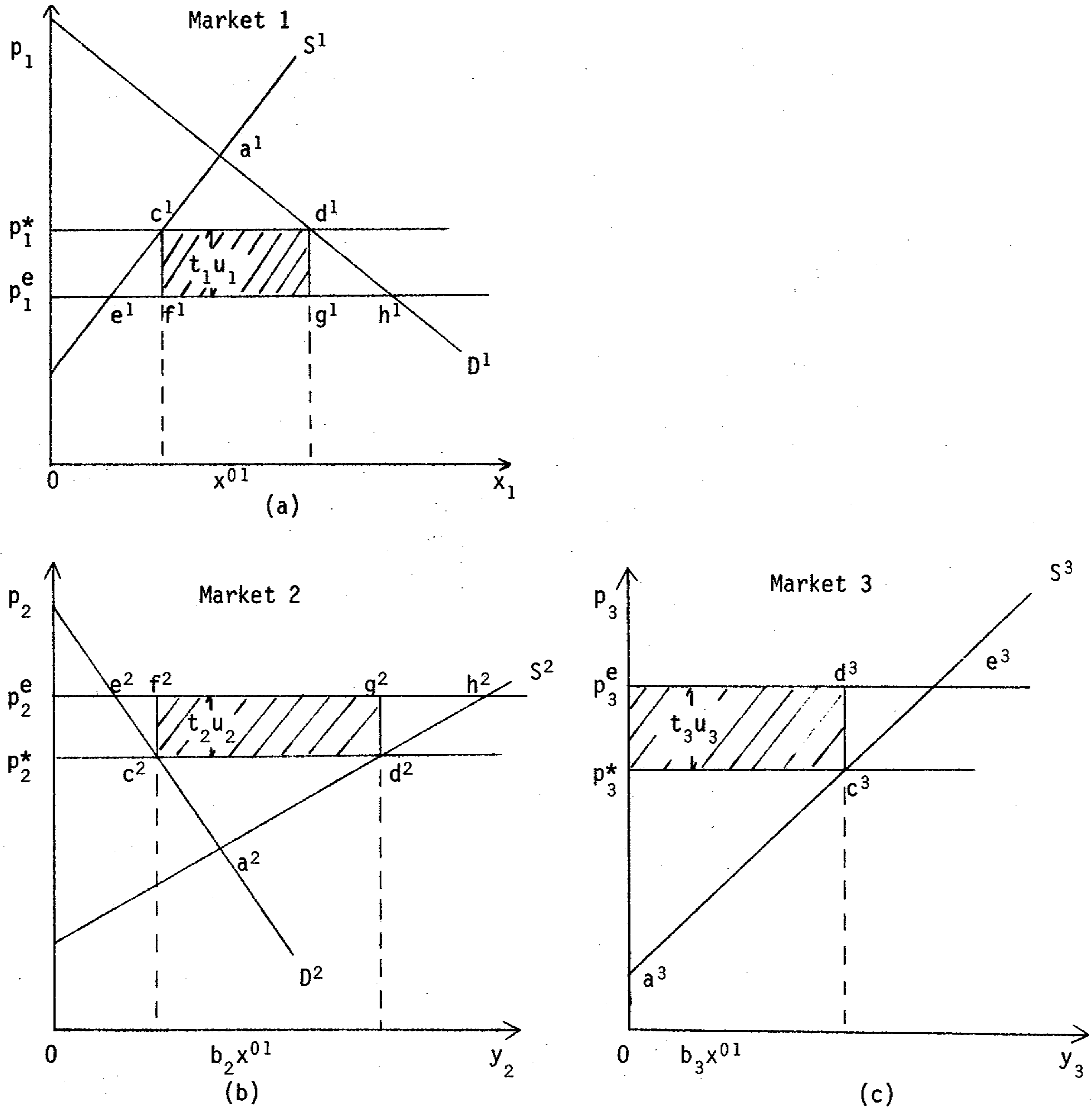

Market 1
p_1
S^1
a^1
c^1
d^1
p_1^*
t_1u_1
p_1^e
e^1
f^1
g^1
h^1
D^1
0
x^{01}
x_1
(a)
Market 2
p_2
e^2
f^2
g^2
h^2
S^2
p_2^e
t_2u_2
p_2^*
c^2
d^2
a^2
D^2
0
b_2x^{01}
y_2
(b)
Market 3
S^3
p_3
e^3
d^3
p_3^e
t_3u_3
p_3^*
c^3
a^3
0
b_3x^{01}
y_3
(c)

Figure 3.

Figure 3 differs from Figure 2 in the respect that there is a supply curve in Figure 3a and a demand curve in Figure 3b; neither curve was present in Figure 2. In an equilibrium product flows from the new firm to the product market will be x^{01}, input flows to the new firm from market 2 will be $y_2=b_2x^{01}$ and input flows from market 3 will be $y_3=b_3x^{01}$. For the product, the difference between the shadow price at the firm and the delivered price will equal the unit transportation cost t_1u_1. For input i, the difference between the supply price fob and the shadow price at the firm will be t_iu_i (i=2,3). The cross-hatched areas in Figure 3 indicate total transportation costs on the flows related to the respective markets.

The optimal location will occur when distances u_1, u_2, and u_3 and the implied interpoint flows are determined to maximize economic rent or the sums of areas of triangles $a^1c^1d^1$, $a^2c^2d^2$, and $a^3c^3p_3^*$.

The dual problem which will yield the same location as the primal is to determine u_1, u_2, and u_3 and prices so as to minimize the areas of trangles $(c^1e^1f^1+d^1g^1h^1)+(e^2f^2c^2+g^2h^2d^2)+c^3d^3e^3$ subject to the conditions that prices between the two points including the firm's location are greater than or equal to the unit transportation costs between the points.

(iii) The case of geographically fixed rivals at points other than those of supply and demand

This case differs from that analyzed immediately above in the respect that rivals for demands and supplies are geographically fixed at points separate from the principal supply and demand points. In other words our firm entering will be receiving net supplies from some points as in the above case and supplying a net demand. Once again let us take the case of one rival in the single product market and one rival in supply market 2. An equilibrium for a firm entering is illustrated in Figure 4.

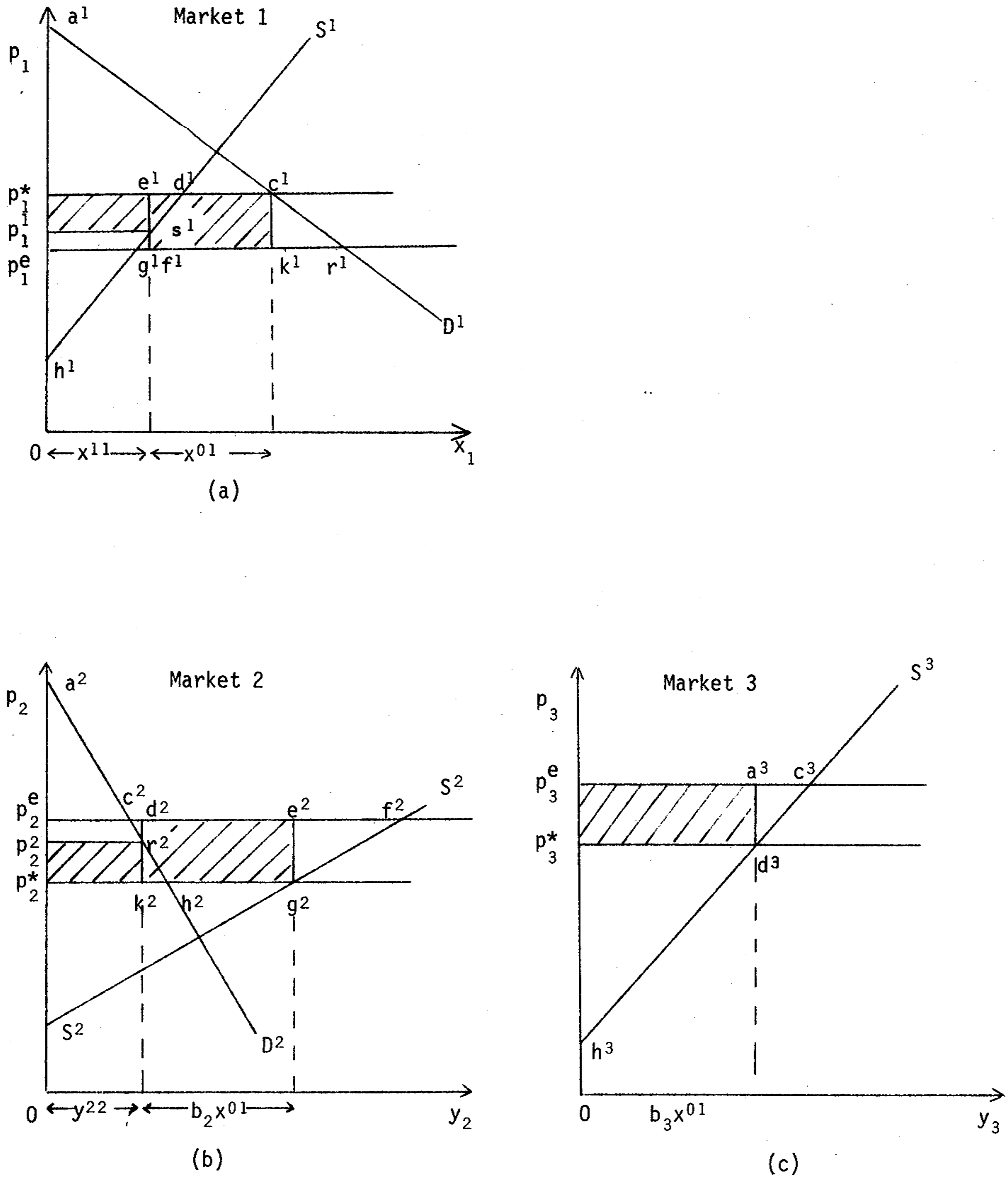

Market 1
a^1
S^1
p_1
p_1^*
p_1^1
p_1^e
e^1 d^1 c^1
s^1
g^1 f^1 k^1 r^1
D^1
h^1
$0 \leftarrow x^{11} \rightarrow \leftarrow x^{01} \rightarrow$
x_1
(a)
Market 2
a^2
p_2
S^2
p_2^e
p_2^2
p_2^*
c^2 d^2 e^2 f^2
r^2
k^2 h^2 g^2
S^2
D^2
$0 \leftarrow y^{22} \rightarrow \leftarrow b_2x^{01} \rightarrow$
y_2
(b)
Market 3
S^3
p_3
p_3^e
p_3^*
a^3 c^3
d^3
h^3
0
b_3x^{01}
y_3
(c)

Figure 4.

The optimal location will be the one which maximizes economic rent or the sum of areas of triangles or $(a^1c^1p_1^*+h^1s^1p_1^1)+(a^2r^2p_2^2+p_2^*g^2s^2)+p_3^*d^3h^3$. The dual problem is to determine the location which minimizes economic rent foregone or the areas of triangles $(c^1k^1r^1+g^1f^1s^1+e^1d^1s^1)+(c^2d^2r^2+r^2k^2h^2+e^2f^2g^2)+a^3c^3d^3$ subject to the condition that the difference between prices at the delivery point and at the shipping point are greater than or equal to the unit transportation costs between the points.

In Figure 4a x^{01} is the amount produced by the entrant, x^{11} is produced by the established firm. p_1^* is the price of the product market, p_1^1 is the price for the product at the established firm and p_1^e is the price at the entering firm for the product. In 4b $b_2x^{01}=y_2^*$ is the amount of input 2 supplied to the entrant by the supplier at point 2, y^{22} is the amount supplied by the supplier at 2 to another established firm. p_2^e is the price for supplier 2's input at the entrant's point, p_2^2 is the price for supplier 2's input or product at the other established firm and p_2^* is the price for the input at supplier 2's point. Figure 4c is similar to 3c in all respects.

It is apparent that the models developed under cases (ii) and (iii) can be readily amended to handle situations when either demands or supplies have zero elasticity. Elements of case (i) will be combined with elements of cases (ii) or (iii) to yield new slightly different models.

3. Entry of Many Firms and Alternative Supply Sources

When there is either a multiplicity of markets for a product or multiplicity of sources of supply for an input or both gross economic rent can in general be increased by simultaneously locating two or more firms rather than a single firm. The optimal locations for the prescribed number of firms entering will be determinable from a the solution to a primal-dual economic rent maximizing-minimizing problem of the form we have seen in Sections 1 and 2.

Let us consider the case of two geographically separate product markets, two geographically separate input markets, and the simultaneous location of two firms entering. We shall consider the simplest case of no existing rivals and assume the supply schedules slope upward to the right with some non-zero and non-infinite elasticity and that demand curves slope down to the left with some non-zero and non-infinite elasticity. An equilibrium is illustrated in Figure 5 below.

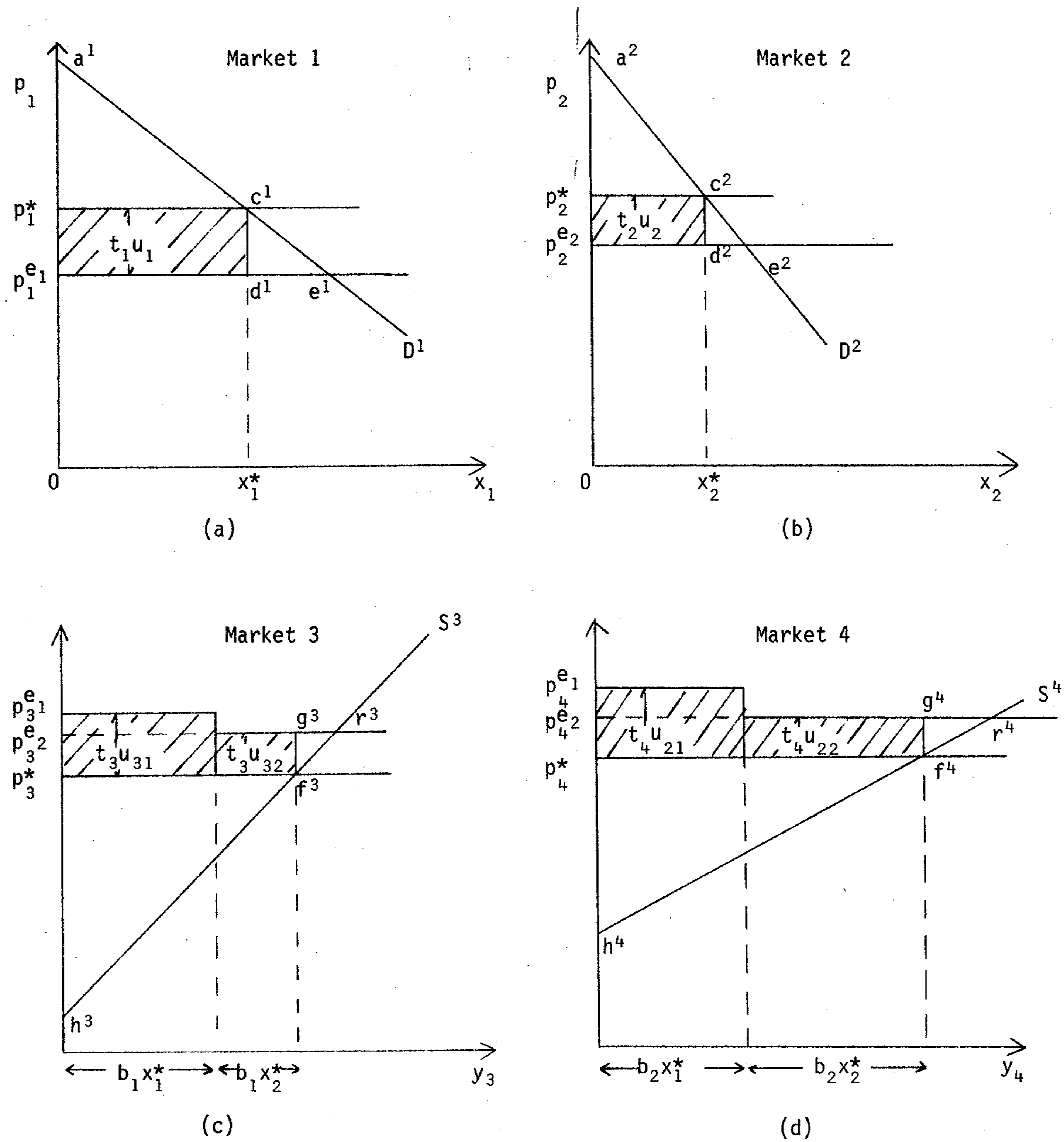

Market 1
a^1
p_1
c^1
p_1^*
t_1u_1
$p_1^{e_1}$
d^1
e^1
D^1
0
x_1^*
x_1
(a)
Market 2
a^2
p_2
c^2
p_2^*
t_2u_2
$p_2^{e_2}$
d^2
e^2
D^2
0
x_2^*
x_2
(b)
Market 3
S^3
$p_3^{e_1}$
$p_3^{e_2}$
g^3
r^3
t_3u_{31}
t_3u_{32}
p_3^*
f^3
h^3
$b_1x_1^*$
$b_1x_2^*$
y_3
(c)
Market 4
S^4
$p_4^{e_1}$
$p_4^{e_2}$
g^4
r^4
t_4u_{21}
t_4u_{22}
p_4^*
f^4
h^4
$b_2x_1^*$
$b_2x_2^*$
y_4
(d)

Figure 5.

We assume each producer has the same technology with fixed coefficients as in the previous models. At points or markets 1 and 2 are demanders. At points or markets 3 and 4 are suppliers of the two respective inputs required in the production of a unit of the product. The optimal locations for the two firms will obtain when the sites are adjusted so as to maximize the areas of triangles $u^1c^1p_1^* + a^2c^2p_2^* + h^3f^3p_3^* + h^4f^4p_4^*$. The same solution will obtain from the dual problem: minimize the areas of triangles $c^1d^1e^1 + c^2d^2e^2 + g^3r^3f^3 + g^4r^4f^4$ subject to the condition that the difference between the delivered price and shipping price along any actual or potential flow route is less than or equal to the unit transportation cost along that route. We now have two geographically separate sites where the two neww firms locate Prices on products and inputs at these sites are indicated by a superscript e. p_1^{e1} is the product price at the firm which supplies the product market at 1. p_2^{e2} is the product price at firm which supplies the product market a 2. The equilibrium distances are defined from the illustrated spatial equilibrium in Figure 6, the companion diagram to Figure 5.

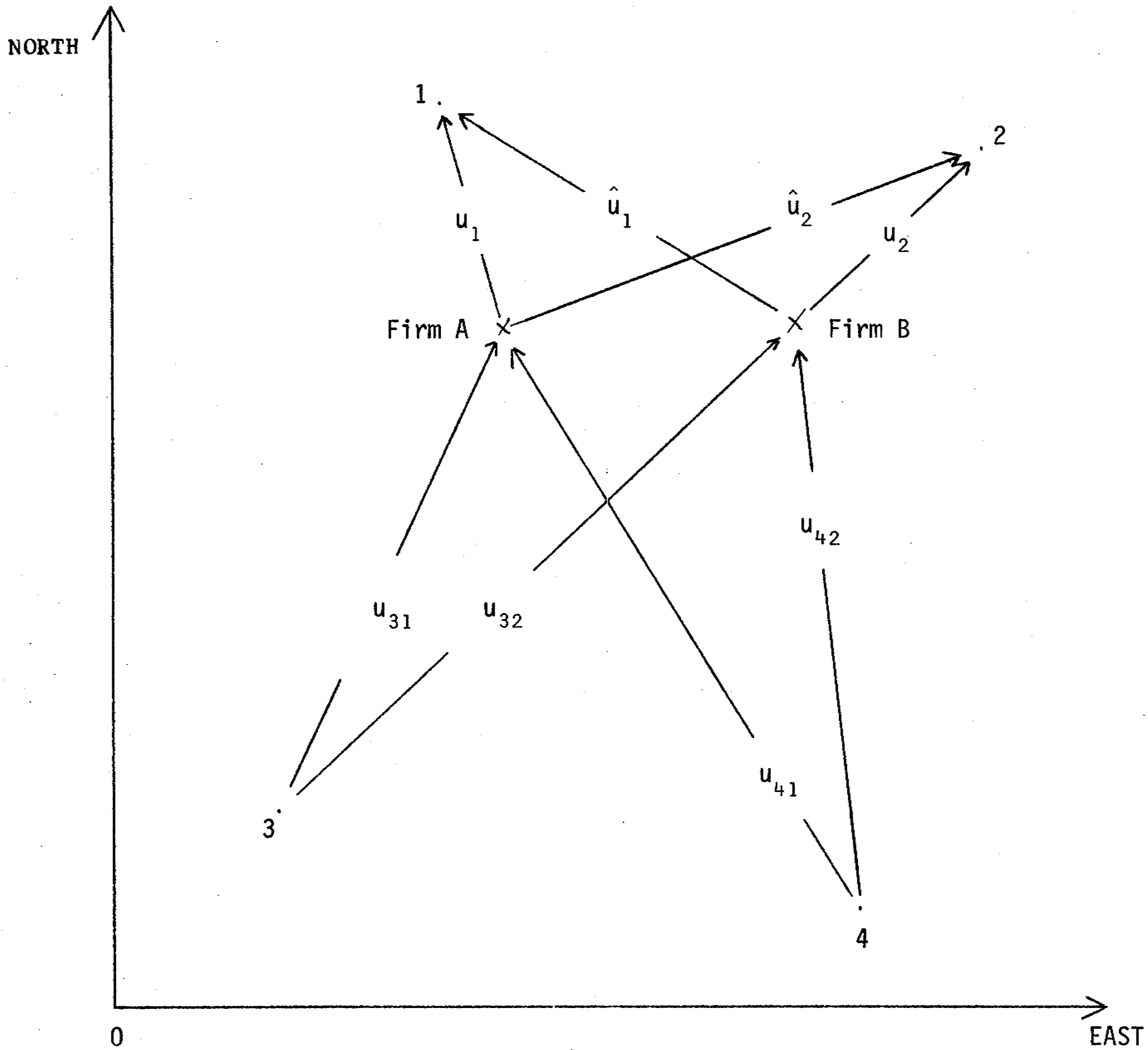

NORTH
1
2
u_1
$\hat{u}_1$
$\hat{u}_2$
u_2
Firm A
Firm B
u_{42}
u_{31}
u_{32}
u_{41}
3
4
0
EAST

Figure 6.

The mathematical price equilibrium conditions emerge in the primal-dual formulation of the optimal location problem and can be illustrated in Figure 6. We have in Figure 6 Firms A and B optimally located with respect to markets 1,2,3, and 4. The u's are distances. The mathematical solution will indicate $p_2 - p_1^{e1} > t_1\hat{u}_2$ in the dual and the flow from A to market 2 as equal to zero in the primal. The economic interpretation of this result is of course that no shipments move along routes where the profits involved in shipping are negative. Analogously we will have $p_1 - p_2^{e2} > t_2\hat{u}_1$ and zero flows from Firm B to market 1 as equilibrium conditions to our primal-dual problem.

The above location problem could be extended. We could introduce geographically fixed rivals in product markets and/or input markets as we did in Section 2. We could introduce input supplies or product demands or both with zero elasticities of supply and demand respectively. Alternatively we could develop a related class of multi-firm location problems with alternative input sources where the alternative sources are located in different geographic sites. Equilibrium would obtain which were analogous to those for the problem illustrated in Figures 5 and 6. The salient price equilibrium conditions in the primal-dual formulation would reflect the zero flows from certain alternative input sources to certain firms entering. Extensions could be made analogous to those discussed in Section 2. Locational problems with both alternative demand sites and alternative input sources could be solved in the same fashion we have seen for the case of alternative demand sites.

The treatment of many (more than two) firms simultaneously locating is not difficult. In some cases we might observe a redundancy in the number of entrants. That is, the number of optimal sites for entrants will be less than the specified number of entrants to be assigned. This special phenomenon will be indicated in the price equilibrium conditions in the solution to the primal-dual problem.[7]

4. Location of Firms Producing Different Products

The principle of rent maximization can be utilized to determine optimal locations for firms entering and producing different products to the same site or town, utilizing different inputs from the same geographic site or town, and/or requiring as inputs products of the other firm or firms entering. Let us consider for purposes of illustration the case of the entry of two firms A and B which produce different products to a town at a fixed site.

Let us assume that there are two inputs available from geographically distinct sites and the quantity of inputs supplied are an increasing function of price. Each product requires both inputs but in different fixed proportions. There are no rival producers supplying the markets or rival demanders requiring inputs.

The demand conditions are as follows:

$$q_i = D^i(p_1, p_2) \qquad (i=1,2)$$

where q_i is quantity demanded of the ith product in the town of delivery, and p_1, and p_2 are prices for the delivered products. We assume $\partial q_i / \partial p_i < 0$ and to be monotonic. An equilibrium is illustrated in Figure 7 below.

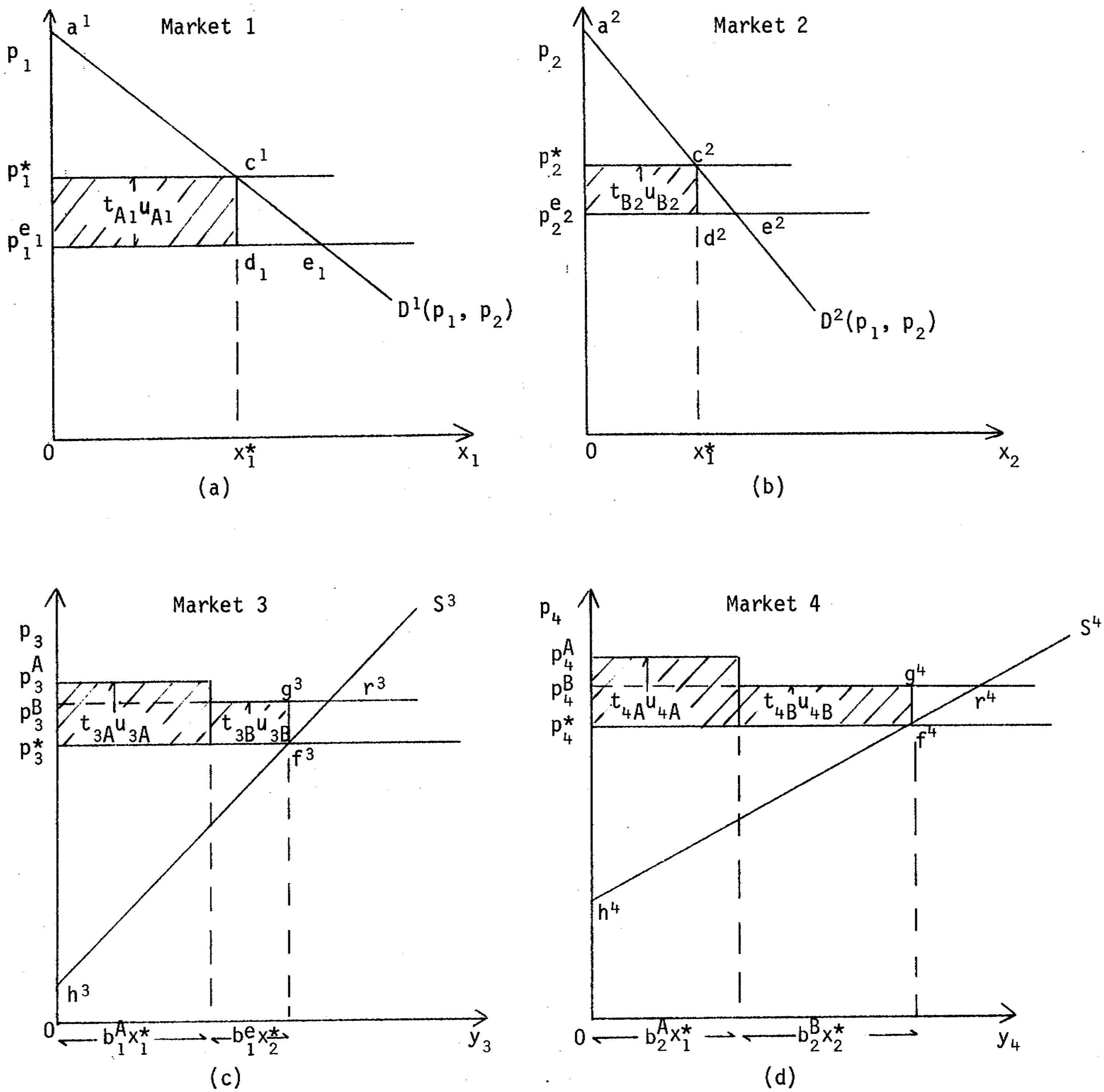


Figure 7.

In Figure 7, u_{Aj} is the distance from firm A to market j, and u_{iA} is the distance from market i to firm A. t_{Aj} and t_{iA} are the corresponding transport costs per unit distance. We have similar definitions for firm B. b_i^A is the amount of input i required per unit output of product at firm A. b_i^B is similarly defined. We assume the different products produced by A and B require different proportions of inputs.

The optimal locations for A and B will result from the maximization of economic rent or areas $a^1c^1p_1^*$ plus $a^2c^2p_2^*$ plus $p_3^*f^3h^3$ plus $p_4^*f^4h^4$ in Figure 7. We can then directly obtain prices at various points which sustain the equilibrium flows by solving the dual problem which is a constrained economic rent foregone minimization problem. The constraints are that along routes of potential shipment, the difference between the delivered price and the shipping price must be greater than or equal to the cost of transport along that route. p_1^*, p_2^*, p_3^* and p_4^* are the quilibrium prices for products and inputs in markets 1,2,3 and 4. p_1^{e1}, p_3^A, and p_4^A are the prices at firm A's site for the product and the two inputs respectively. p_1^{e2}, p_3^B, and p_4^B are analogously defined.

The case of interdependent demands illustrated in Figure 7 can be further developed to incorporate interdependent supplies of inputs from a point. We can also handle $n > 2$ interdependent demands and/or supplies in terms of rent maximization. The basic model with multiple interdependencies in demands and supplies is Mosak's [12] and this latter model has been altered to incorporate exogenously given transportation costs between points in [14],[15], and [16]. An additional interdependency can be introduced into the locational model with many different firms and that is by having various firms entering supply and/or demand inputs to and/or from other firms entering. No economic rents will be yielded from these transactions but these interdependencies will influence the over-all locational equilibrium through the effects arising from the minimization of transportation costs between firms.

5. General Equilibrium and Location of Firms

The most general model would depart from those we have examined above in two ways. First all firms or production units should be located simultaneously with optimal interpoint flows emerging along with the optimal dual price system. Secondly transportation should be a produced good or endogenous to the economy and its price and output level determined simultaneously with all other prices and outputs. In the absence of some assumption concerning the geographic immobility of certain plants, our general problem will be geographically anchored and transportation costs will be minimized by having all activities at one point. The anchoring assumptions we shall employ are twofold. First it will be assumed that certain sites have locational efficiency indices which induce certain activities or firms to locate at these sites under all circumstances. These peculiar efficiencies may arise from irregularities in the topography such as good potential port sites or some such related phenomenon. Secondly it will be assumed that agglomeration occurs at certain sites or nodes. We simply require that transportation costs be infinite on two commodities one of which is required as an input for production in the other.

If we postulate a finite number of goods required both as final products and as inputs[8] and a finite number of geographically fixed sites and a finite number of possible producers of each product, we can define a primal-dual rent maximization - rent foregone minimization problem which will locate all firms simultaneously and result in an equilibrium to obtain which has the same price structure and output flow structure as that observed in reality.[9]

The introduction of transportation costs as endogenous entails the increase in the mathematical complexity of exhibiting a general equilibrium with location. We must appeal to the mathematical literature on fixed points in point to set mappings. Let us outline the general problem with endogenous transport costs. Transportation goods will be assumed to be producible in all geographically fixed points. These goods will incur no cost of transportation in transit. At each geographically fixed market there will be a general supply function for transportation.

$$q_t = f_t(p_1,\ldots,p_n,p_t)$$

where $\dfrac{q_t}{p_t} > 0$ and is monotonic

We shall assume now that each unit of commodity (other than the transportation good) shipped a unit distance requires a fixed physical amount of the transportation good. Hence if $t_i^{kl}u^{kl}$ was cost of transporting a unit of commity i between k and l now we have $p_t r_i^{kl} u^{kl}$ as the same cost where r_i^{kl} is the physical amount of transportation good required to transport a unit of commodity i a unit distance.

In a general equilibrium, the amount of transportation good required must just equal the amount supplied. This is the transport equilibrium condition. If we solved a primal dual rent maximization-rent foregone minimization problem for all sections except the transport sector for all possible price of transport p_t, we will trace out a set of feasible price vectors satisfying the primal-dual problem. One point in this set will be a vector of prices which also satisfies the transportation equilibrium condition and corresponds to a certain p_t. This vector or fixed point will provide the solution prices to our general equilibrium location problem. The solution locations and inter-market flows will also obtain from the problem.

6. Conclusions

We have dealt throughout this paper with production units called firms. In certain cases it might be useful to consider these units as industries. In this latter case attention must be paid to the nature of aggregating firms to industries.

This analysis has abstracted from market areas and considered only points. Although generality is sacrificed by this abstraction I do not believe the essential nature of many locational problems is lost. Modern industrial nations have well over three quarters of their populations concentrated at points on maps or in cities and inter-urban flows are the crucial ones in industrial location analysis.

The abstraction from the most of the irregularities in topography in this analysis does not seriously detract from the relevance of this analysis for real locational inquiries. The Interstate Highways system in the United States has done much to knit the towns and cities together in a network in which the unit costs of transport from any point in one direction are the same as those in another direction. A homogeneous plain dotted with cities has been emerging in the U.S. Consider for example the decision of a major American electronics company to locate its color television set production in Bloomington, Indiana. The generalized Weberian analysis in this paper appears to satisfactorily explain this decision. A transportation cost minimization criterion appears to have dictated such a locational decision.

FOOTNOTES

1. See the modern treatment by Kuhn and Kuenne[9].

2. In [6] I demonstrated how the von Thünen model of agricultural land allocation could be extended to a model of multiregional general equilibrium through the application of the Cournot-Samuelson principle of economic rent maximization and generalizations of the principle in [14],[15], and [16]. Isard [8] utilized the principle of economic rent maximization in his pioneering work on the "optimum space-economy" but failed to develop the approach fully.

3. Chipman [1] developed a model in which changes in consumer surplus are isomorphic to changes in social welfare. His result turns on his particular choice of individual utility functions.

4. In this classic Weber case we are left with a problem with open-ended scale: that is we simply determine the point where a firm would locate so as to minimize transportation costs on a unit of the product supplied to the product market. We do not know how much will be produced in general unless additional information is introduced.

5. Throughout this paper we shall abstract from a consideration of costs of fabrication at a specific site. If there are constant returns to scale in fabrication and the necessary inputs (aside from those transported from supply points) are ubiguitous at fixed prices then fabrication costs will not qualitatively affect the analysis.

6. Gale [3, pp. 14-17], neglects to mention this same point in his exposition of the transportation problem in linear programming. By just satisfying demands, that is having no excess deliveries, transportation costs can be minimized. Thus we should always expect non-zero shadow prices on delivered product.

7. Important welfare considerations arise when one reflects on the possibility of an optimal number of firms. Such questions can only be answered when one considers a model which has an explicitly developed production and cost side where there are non-constant returns to scale. Losch [10, p. 113] and Mills and Lav [11, pp. 286-88] have commented on the welfare aspects of multi-firm location and

the optimum number of firms. Location problems can be viewed as orthodox non-spatial economic problems with a transportation cost constraint. The notion of welfare optima has been exhaustively analyzed for the unconstrained problems but has not been definitively treated for problems with transport or transaction cost constraints.

8. The formulation of the equilibrium of the solution of the primal-dual program breaks down if interdependencies feed back on themselves in loops. That is firm A requires an input from firm B which in turn requires an input from firm A. The objective functions then become not concave in all instances.

9. This is an interpretation devoid of welfare suggestions and is inspired by Samuelson's interpretation of the related Cournot-Enke problem as being "artificial" or a fiction which resulted in an observed equilibrium to obtain.

REFERENCES

[1] Chipman, John S., "External Economies of Scale and Competitive Equilibrium", Quarterly Journal of Economics, 84, August, 1970, pp. 347-385.

[2] Cournot, Augustin, The Mathematical Principles of the Theory of Wealth, (translated by N.T. Bacon from the French edition of 1838), New York: Macmillan, 1927.

[3] Gale, David, The Theory of Linear Economic Models, New York: McGraw-Hill, 1960.

[4] Hartwick, John M., "A Generalization of the Transportation Problem in Linear Programming and Spatial Price Equilibrium", Queen's University Economic Research Institute, Paper # 30, October, 1970.

[5] Hartwick, John M., "Effective Protection, Transportation Costs and the Location of Firms", Queen's University Economic Research Institute, Paper # 18, March, 1970.

[6] Hartwick, John M., "The Pricing of Goods and Agricultural Land in Multiregional General Equilibrium", Queen's University Economic Research Institute, November, 1970.

[7] Hotelling, Harold, "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates", Econometrica, 6, 1938, pp. 242-69.

[8] Isard, Walter, "A General Location Principle of an Optimum Space-Economy", Econometrica, 20, July 1952, pp. 406-30.

[9] Kuhn, Harold W. and Robert E. Kuenne, "An Efficient Algorithm for the Numerical Solution of the Generalized Weber Problem in Spatial Economics", Journal of Regional Science, 4, 1962, pp. 21-33.

[10] Losch, August, The Economics of Location, (translated by W.H. Woglom from the 1942 German edition), New York: John Wiley, 1967.

[11] Mills, E.S. and M. Lav, "A Model of Market Areas with Free Entry", Journal of Political Economy,

[12] Mosak, J., General Equilibrium Theory in International Trade, Bloomington, Indiana: Principia Press, 1944, Chapter 5.

[13] Samuelson, Paul A., "Spatial Price Equilibrium and Linear Programming", American Economic Review, 42, June, 1952, pp. 283-303.

[14] Takayama, T. and G.G. Judge, "Alternative Spatial Equilibrium Models", Journal of Regional Science, 10, April 1970, pp. 1-12.

[15] Takayama, T., and G.G. Judge, "Equilibrium Among Spatially Separated Markets: A Reformulation", Econometrica, 32, 1964, pp. 510-524.

[16] Takayama, T., and A. Woodland, "Equivalence of Price and Quantity Formulation of Spatial Equilibrium. Purified Quality in Quadratic and Concave Programming", Econometrica (forthcoming).

[17] Weber, A., Alfred Weber's Theory of the Location of Industries, (translated by C.J. Friedrich from the German edition of 1909), Chicago: University of Chicago Press, 1929.