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THE PRICING OF GOODS AND AGRICULTURAL LAND IN MULTIREGIONAL GENERAL EQUILIBRUIM

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Von Thünen taught us, with the aid of a simple model [15], how to allocate and price agricultural land around a town. Cournot [1], Enke [3], and Samuelson [10] demonstrated how to price commodities at different points in space so as to cause efficient interregional trade to result when transportation costs impede the flow of commodities. The conceptual framework which Cournot and Samuelson utilized can be shown to yield the equilibrium which von Thünen discovered. The first part of this paper is concerned with the elucidation of the von Thünen equilibrium and illustrates the power of the Cournot-Samuelson approach. In Sections 2 and 3, the von Thünen model is extended to incorporate more than one town or region when these towns are involved in trade, to incorporate many agricultural commodities with demand functions not exhibiting only infinite elasticities of demand, and to incorporate commodities produced solely in town rather than in fields around the town. In Sections 4 and 5, the trade model is extended to incorporate endogenously produced transportation goods. In Section 6, the general model is compared with other models of spatial general equilibrium.

The determination of and the nature of equilibria are the principal themes of this paper. We observe that the equilibria can be defined to result from a mathematical problem of maximizing economic rent subject to certain economic feasibility restrictions. This was the approach taken by Cournot[1] and Samuelson [10] and utilized by Takayama and Judge [12], [13], for the case of many commodities. Hartwick [4] indicated that the dual to the rent maximization problem was a problem of minimizing rent foregone because of the existence of positive transportation costs as contrasted with a situation in which there were no transportation costs. This dual appears naturally in the simple von Thünen model and the extensions in this paper. Hotelling [5] rigorously analyzed the welfare aspects of the economic rent minimization problem when excise taxes were the distortion rather than transportation costs.[2]

The minimization of economic rent foregone does serve as a welfare criterion.[3]

The mathematical foundation for this economic analysis is a primal-dual non-linear program. The proofs of the existence of solutions become difficult as the economic problem becomes more general. No attempt is made in this paper to demonstrate mathematically the existence of solutions to the programs; the diagrammatic presentation of the economic equilibria is straightforward and diagrams are frequently utilized. Proofs of the existence of solutions to allied problems have been successfully developed.[4]

SECTION I. SIMPLE VON THÜNEN MODEL

The von Thünen world consists of a homogeneous plain with a population center, say a town, located in the plain. Various types of crops are grown in the area around the town and these crops are distinguished from each other by different prices in consumption and also by the fact that each requires a different area of land per unit output.

It is assumed that the crops produced are shipped from the field to town at some transportation cost. It is well known that the different crops will be grown in different belts or rings around the town as in Figure I. The von Thünen problem[5] is to determine the frontiers of these rings and how these frontiers or demarcation lines depend on the nature of production, transportation costs, and values assigned to the various crops at the town. Figure 1 illustrates the allocation of land in equilibrium in a simple von Thünen model.

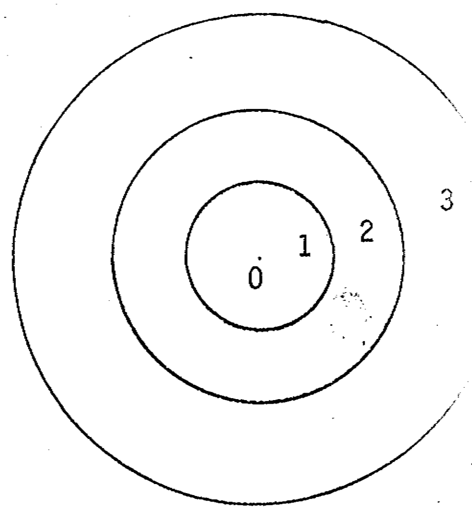


Figure 1.

The town is at 0. Crop 1 is grown in the first ring, crop 2 in the second and crop 3 in the third.

The model is of a partial equilibrium type. We assume fixed prices for agricultural commodities at the town. Transportation costs per unit distance are also exogenous. We will deal with the case in which there are three different crops which can be grown around the town. The n crop case has no essential differences.

We assume that each crop requires a fixed amount of land per unit of output, e.g. it might be 1 acre per bushel. The coefficients are as follows.

a_i is the area of land required to produce a unit of crop i.

$i = 1,2,3$

We wish to maximize the value of the crops delivered to the town less the transportation costs, i.e. determine frontiers u_1, u_2 and u_3 which

$$\text{maximize:} \quad V = p_1x_1 + p_2x_2 + p_3x_3 - T_1 - T_2 - T_3 \qquad (1)$$

where p_1, p_2, and p_3 are the prices in the town for units of crops 1, 2, and 3 respectively.

T_1, T_2, and T_3 are the total costs of transporting amounts of the three crops x_1, x_2, and x_3 respectively.

Let us assume that crop one is grown in the ring closest to the town, crop two in the next ring and crop three in the last ring. Below we will determine the conditions on our coefficients which assures that this will be so. The total amount of the first crop x_1 grown will be

$$x_1 = \int_o^{2\pi} \int_o^{u_1} a_1 u \, du \, d\,\theta = 2\pi \, a_1 \frac{u_1^2}{2}$$

That is the total area used for x_1 times the amount grown per acre used. Similarly

$$x_2 = \int_o^{2\pi} \int_{u_1}^{u_2} a_2 u \, du \, d\,\theta = 2\pi \, a_2 \left(\frac{u_2^2}{2} - \frac{u_1^2}{2}\right)$$

$$x_3 = \int_o^{2\pi} \int_{u_2}^{u_3} a_2 u \, du \, d\theta = 2\pi \, a_3 \left(\frac{u_3^2}{2} - \frac{u_2^2}{2}\right)$$

Transportation costs for a unit of crop 1 will be t_1 per unit distance. t_2 and t_3 are transportation cost for a unit of goods and 2 and 3 per unit distance.

$$T_1 = \int_o^{2\pi} \int_o^{u_1} a_1 u t_1 u \, du \, d\,\theta = 2\pi \, a_1 t_1 \frac{u_1^3}{3}$$

$$T_2 = \int_o^{2\pi} \int_o^{u_2} a_2 u t_2 u \, du \, d\,\theta - \int_o^{2\pi} \int_o^{u_1} a_2 u t_2 u \, du \, d\,\theta$$

$$= 2\pi a_2 t_2 \frac{u_2^3}{3} - 2\pi a_2 t_2 \frac{u_1^3}{3}$$

$$T_3 = \int_o^{2\pi} \int_o^{u_3} a_3 u t_3 u \, du \, d\,\theta - \int_o^{2\pi} \int_o^{u_2} a_3 u t_3 u \, du \, d\,\theta$$

$$= 2\pi a_3 t_3 \frac{u_3^3}{3} - 2 \, a_3 t_3 \frac{u_2^3}{3}$$

T_1 is the transportation cost of moving x_1 to the town.

Recall crop 1 was grown in the first ring.

T_2 is the transportation cost of moving x_2, the amount grown in the second ring to town. T_2 is determined by finding the cost of moving the amount of crop 2 grown in a circle of radius u_2 around the town minus the cost of moving the amount of crop 2 grown in a circle of smaller radius u_1 around the town.

T_3 is the transportation cost associated with x_3.

We can now substitute for x_1, x_2, x_3, T_1, T_2, and T_3 in (1) above and maximize the expression. We want to find what values u_1, u_2, and u_3, the frontiers between different crops assume when V or the value of the crops in town is at its maximum. The conditions are:

$$\frac{\partial V}{\partial u_1} = 2p_1\pi a_1u_1 - 2p_2\pi a_2t_1u_1^2 + 2\pi a_2t_2u_1^2 = 0$$

$$\text{or} \quad u_1 = \frac{p_2a_2 - p_1a_1}{t_2a_2 - t_1a_1}$$

$$\frac{\partial V}{\partial u_2} = 2p_2\pi a_2u_2 - 2\pi p_3a_3u_2 - 2\pi a_2t_2u_2^2 + 2\pi a_3t_3u_2^2 = 0$$

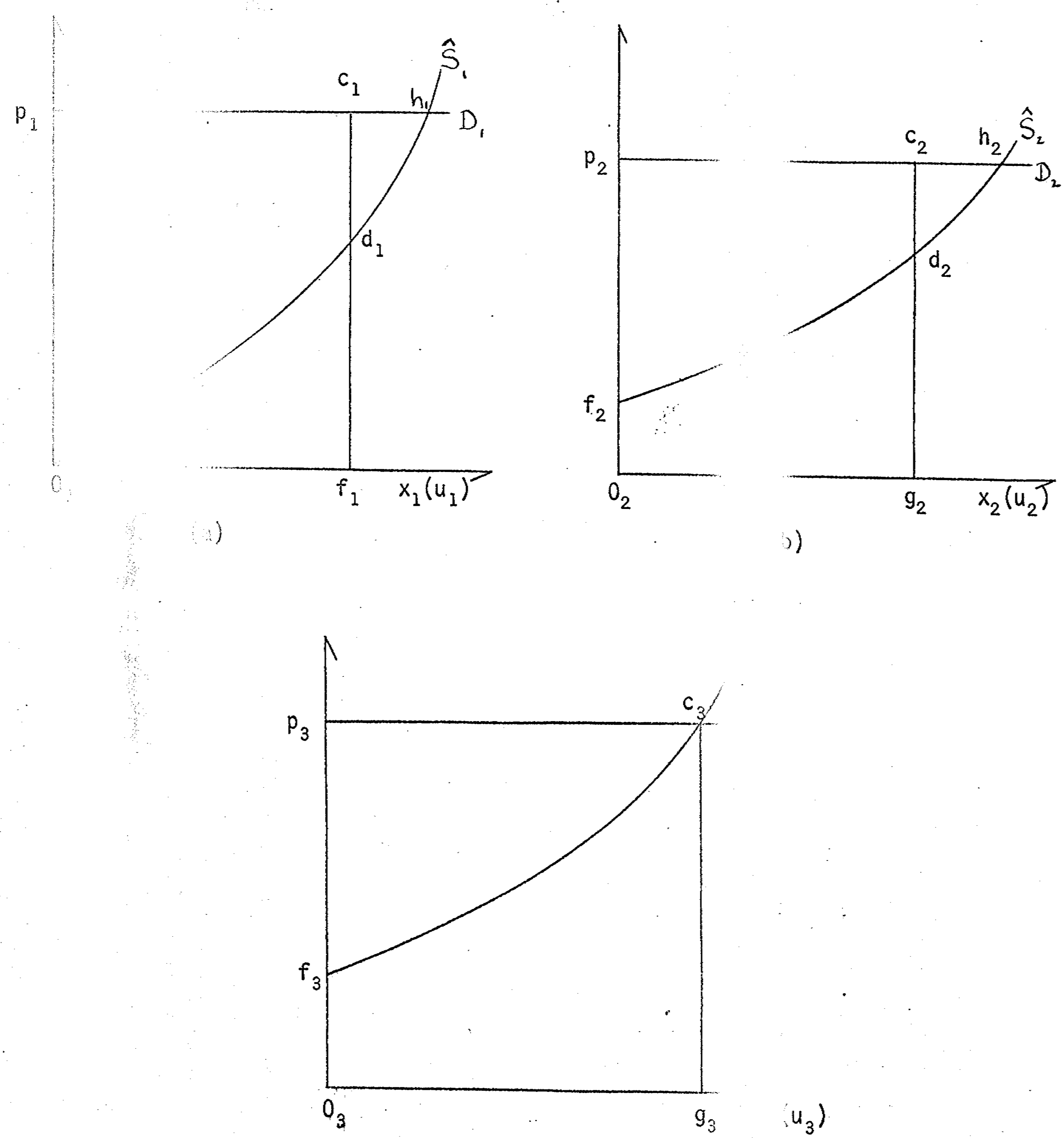
$$\text{or} \quad u_2 = \frac{p_3a_3 - p_2a_2}{t_3a_3 - t_2a_2}$$

$$\text{and} \quad \frac{\partial V}{\partial u_3} = 2p_3\pi a_3u_3 - 2\pi a_3t_3u_3^2 = 0$$

$$\text{or} \quad u_3 = \frac{p_3}{t_3}$$

The second order conditions assuring a maximum rather than a minimum obtains are that the determinants of the principal minors of the matrix of second partial derivatives of V with respect to u_1, u_2, and u_3 must alternate in sign starting with $\frac{\partial^2 V}{\partial u_1^2} > 0$.

Diagrammatically, our maximization problem can be expressed as follows: determine u_1, u_2, and u_3 so that area $p_1c_1d_1o_1 - d_1o_1f_1 + p_2c_2d_2f_2 - d_2f_2o_2g_2 + p_3c_3f_3 - c_3f_3o_3g_3$ is a maximum in Figure 2abc.


p_1
c_1
h_1
Ŝ_1
D_1
d_1
f_1
x_1(u_1)
p_2
c_2
h_2
Ŝ_2
D_2
d_2
f_2
0_2
g_2
x_2(u_2)
p_3
c_3
f_3
0_3
g_3
(u_3)

(a)

(b)

(c)

Figure 2.

Observe that in Figure 2a area $p_1c_1d_1o_1$ is the total land rent associated with the production of crop 1 on ring 1. Area $o_1d_1f_1$ is the total transportation cost associated with the output of crop 1 on ring 1. The corresponding areas in Figure 2b and 2c have a similar interpretation. The curvature of the $\hat{S}$ schedules in Figure 2 results from the fact that output increases more than proportionately as distance from town as we move out along a radius from town. Distances o_2f_2 and o_3f_3 are respectively the cost of transporting a unit of commodity 2 and 3 from the frontier of the field nearest town, to the town.

Our problem can thus be viewed as one of maximizing the sum of land rents less transportation costs. These land rents can also be viewed as producers' surpluses. Thus in the parlance of Samuelson we are maximizing "net social payoff". It will be apparent from Figure 2 that the von Thünen model could easily be amended to include demand schedules which are not infinitely elastic in the delivered price. Consumers' surpluses as well as producers' surpluses will then figure in our maximand. We shall develop this new model in Section 2.

In order to have the rings in the sequence suggested above, we require

$$0 < u_1 < u_2 < u_3$$

$$\text{or} \quad 0 < \frac{p_2a_2 - p_1a_1}{t_2a_2 - t_1a_1} < \frac{p_3a_3 - p_2a_2}{t_3a_3 - t_2a_2} < \frac{p_3}{t_3} \tag{2}$$

To assure the positivity of u_1 and u_2 (u_3 is assured since p and t are assumed to be positive) we require for u_1 for example that

$$p_2a_2 < p_1a_1 \quad \text{and} \quad t_2a_2 < t_1a_1$$

Similarly for u_2. We will see below that conditions for all of crops 1, 2, and 3 to be grown and in the first, second, and third rings consecutively, are

$$p_1a_1 > p_2a_2 \quad \text{and} \quad t_1a_1 > t_2a_2$$

$$\text{and} \quad p_2a_2 > p_3a_3 \quad \text{and} \quad t_2a_2 > t_3a_3$$

and the satisfaction of the inequalities in (2).[6]

There exists a problem dual to (1), the solution of which yields the price equilibrium conditions in a von Thünen model and in more general land rent models. We can illustrate the problem in Figure 2. We find frontiers which minimize rent foregone (areas $c_1h_1d_1$ plus $c_2d_2h_2$) subject to the condition that the difference between the price of a crop in the field and the price in town must be less than or equal to the cost of transporting a unit of the crop to town. This latter constraint is the familiar one inhering in any problem of spatial price equilibrium.[7] formally, the dual problem is determine non-negative u_1, u_2, and u_3 which

minimize

$$\int_{p_1(u_1)}^{p_1} x_1(u)dp + \int_{p_2(u_2)}^{p_2} x_2(u)dp + \int_{p_3(u_3)}^{p_3} x_3(u)dp - \left\{ \sum_{i=1}^{3} p_i x_i(u_i) - \sum_{i=1}^{3} p_i(u_i) x_i(u_i) \right\}$$

subject to:

$$f_i(u) - p_i + p_i(u) \geqq 0 \qquad (i = 1, \ldots, 3)$$

where $f_i(u)$ is the price of a unit of i in the field and $p_i(u_i)$ is the value on the vertical axes in Figure 2 equal to $t_i u_i$, the transport cost of a unit of commodity i from u_i to the town. $p_i(u_i) = t_i u_{i-1} + t_i u_i x_i(u_i)$, $(i = 1, \ldots, 3)$ and $u_0 = 0$. Since land has been assumed to be the only input into the production of a commodity[8] we can easily express the value of land in terms of the value of the commodity.

Define

$$R_i(u) = a_i(p_i - p_i(u_i))$$

or $R_i(u)$ as the rent on land devoted to crop i at distance u from the town.

We may deduce the rent per unit of land in a different manner. The rental value on a piece of land depends how far it is from the town and what crop, if any, is grown on it. Define

$$A_i = \int_0^{2\pi} \int_0^{u_i} u \, du \qquad i = 1,2,3$$

where A_i is an area of land devoted to growing crop i alone. Also $V_i = p_i x_i - T_i$. Now dV_i/dA_i is the increase in the value of crop i from increasing the amount of land devoted to i by one unit. It is in fact the rental value or implicit price of land when used to produce crop i.

$$\frac{dA_i}{du} = 2\pi u$$

Thus

$$\frac{dV_i}{dA_i} = a_i(p_i - t_i u_i) \qquad i = 1,2,3 \qquad (3)$$

$$= R_i(u)$$

Expression (3) shows that the rental value on land when cultivated with crop i is a linear function of the distance u one is from the town. We assume that the rental value is positive for some values of u and reaches zero at some u. u, of course, is necessarily non-negative, being a distance. We can also see that the value of u which satisfies $R_1(u) = R_2(u)$ is the same u as we derived in (3).

Similarly for $R_2(u) = R_3(u)$.

We can now produce the familiar companion diagram to Figure 1. Figure 3 shows the rent functions defined in (3) and the frontiers, u_1, u_2, and u_3 for the crops.

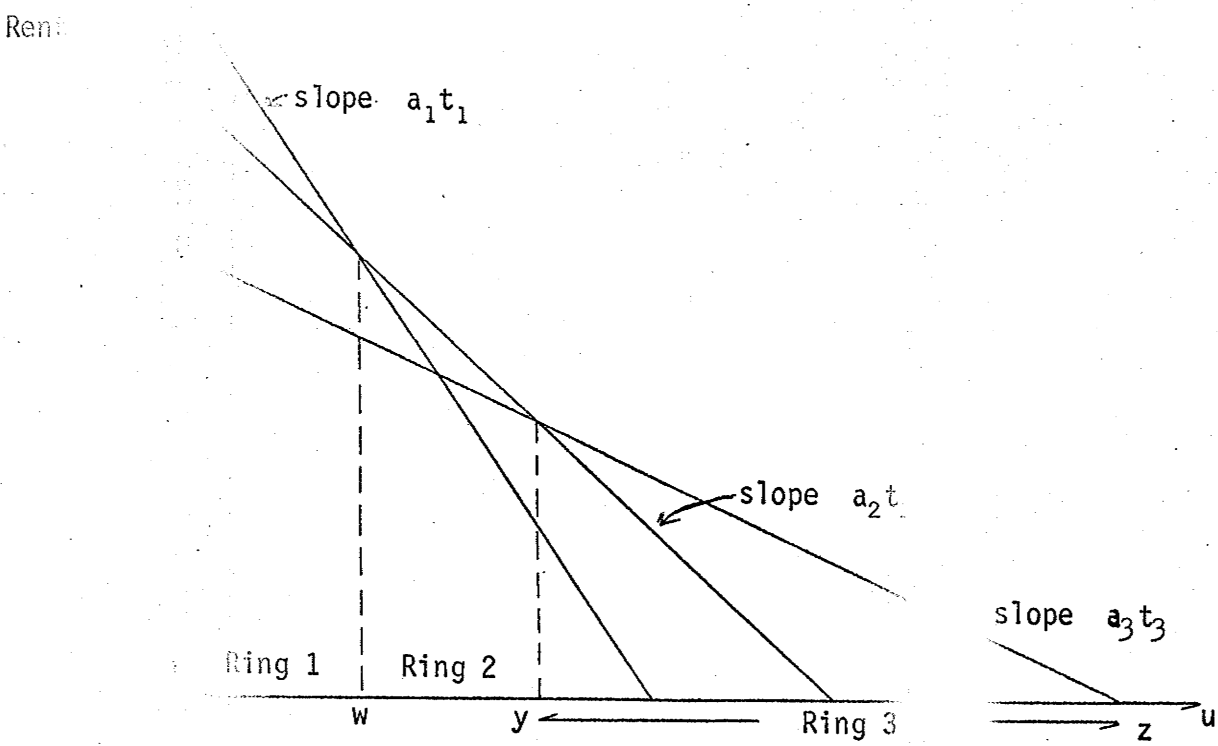


Figure 3.

Observe that as transportation costs rise, the rents on land fall to zero more rapidly. $p_1a_1 > p_2a_2$ indicates that the value of a unit of land in terms of the amount of crop it yields is higher for crop 1 than 2. Observe the relative magnitudes of the slopes a_it_i and intercepts p_ia_i required to assure that crops 1, 2 and 3 are grown and that they are grown in the rings 1, 2, and 3 consecutively. If p_3a_3 were very close to p_2a_2 in Figure 2 with the same slopes, then none of crop 2 would be grown and there would only be two rings around the town. If the intercepts in Figure 2 remained the same but the slope a_1t_1 became less in absolute value than the other two, then only crop 1 would be grown and there would be one ring around the town. The reader can see the myriad possibilities for different configurations, each depending on the relative values of the slopes and intercepts of the rent functions.

The basic equilibrium condition indicating a maximum for V, that is $R_i(u) = R_j(u)$ at a frontier point u, is not dependant on the fact that the world is shaped as a circular disc. The town might be located on a coast and the resulting agricultural land cultivated would be half a disc divided into various rings of crops.

We can deduce the conditions which define a frontier in a slightly different way. A necessary economic condition defining a frontier point where crop types are switched is that the values of the two crops at the frontier must be equal, i.e. where $\frac{dV_i}{du} = \frac{dV_j}{du}$:

$$\frac{dV_i}{du} = 2p_i \pi a_i u - 2\pi a_i t_i u^2$$

and the value of u when $\frac{dV_i}{du} = \frac{dV_j}{du}$

is $$u = \frac{p_j a_j - p_i a_i}{t_j a_j - t_i a_i}$$

which is the same condition we got above.

We can present these results in a diagram. Let us graph $\frac{dV_i}{du}$ as a function of u for i = 1,2,3. We observe $\frac{dV_i}{du} = 0$ when u = 0 and $u = \frac{p_i}{t_i}$. Also $\frac{d^3V_i}{du^3} < 0$ implying that the graph is convex upwards. The graph is symmetric about $u = \frac{p_i}{2t_i}$ which occurs when $\frac{d^2V_i}{du^2} = 0$. Graphs of $\frac{dV_i}{du}$, i = 1,2,3 are presented in Figure 4.

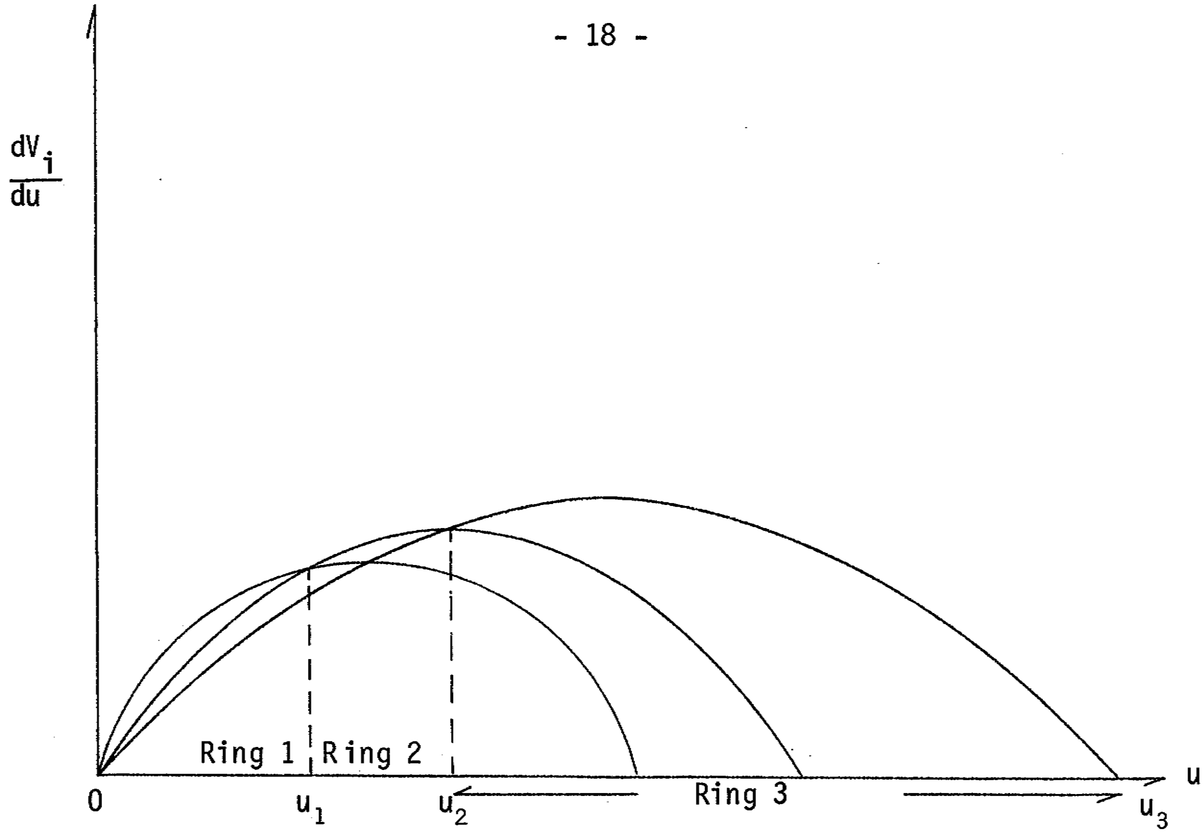


Figure 4.

Note that we obtain three distinct rings with frontiers u_1, u_2, u_3 because of the special assumptions made concerning the parameters p_i, t_i, and a_i, $i = 1,2,3$ above. Other choices of parameters would give different sequences and likely different numbers of rings.

When the rental value on land for crop 1 equals the value for crop 2, say, then

$$u = \frac{p_2 a_2 - p_1 a_1}{t_2 a_2 - t_1 a_1}$$

which is the same value we got for the ring frontier value u_1. Thus the equilibrium conditions to the maximum problem indicate that the rental value on land for any two crops must be equal where there is a switch-over from the cultivation of one crop to that of another.

SECTION 2. LAND RENT AND SPATIAL PRICE EQUILIBRIUM

Let us now consider situations when the demands for commodities are functions of delivered price rather than when demands are infinitely elastic at fixed prices. It is simplest to assume demand a function only of the price of the single commodity in question in the first instance. Thus we shall only deal with one agricultural commodity. However we shall deal with two towns and their environs rather than one as before. Our problem will be to determine the size of the ring around each town devoted to cultivation, the prices of the crop in each town assuming that we can ship the commodity between the towns, the amount shipped between the towns, and the rents on the lands around the towns.

In each town, demand for the commodity will be a function of price. In the diagrammatic presentation we shall assume the function to be linear.

$$x^i = a^i - b^i p^i \qquad (i = 1,2)$$

There is only one crop grown around the two towns. a^i is land required per unit output of the crop at town i. t_i is the cost of moving the crop from the field to the town. u^i is the radius of a circular agricultural area around town i. s^{ij} is the cost transporting a unit of the commodity from town i to town j. We assume that $s^{ij} = s^{ji}$.

The equilibrium is illustrated in Figure 5.

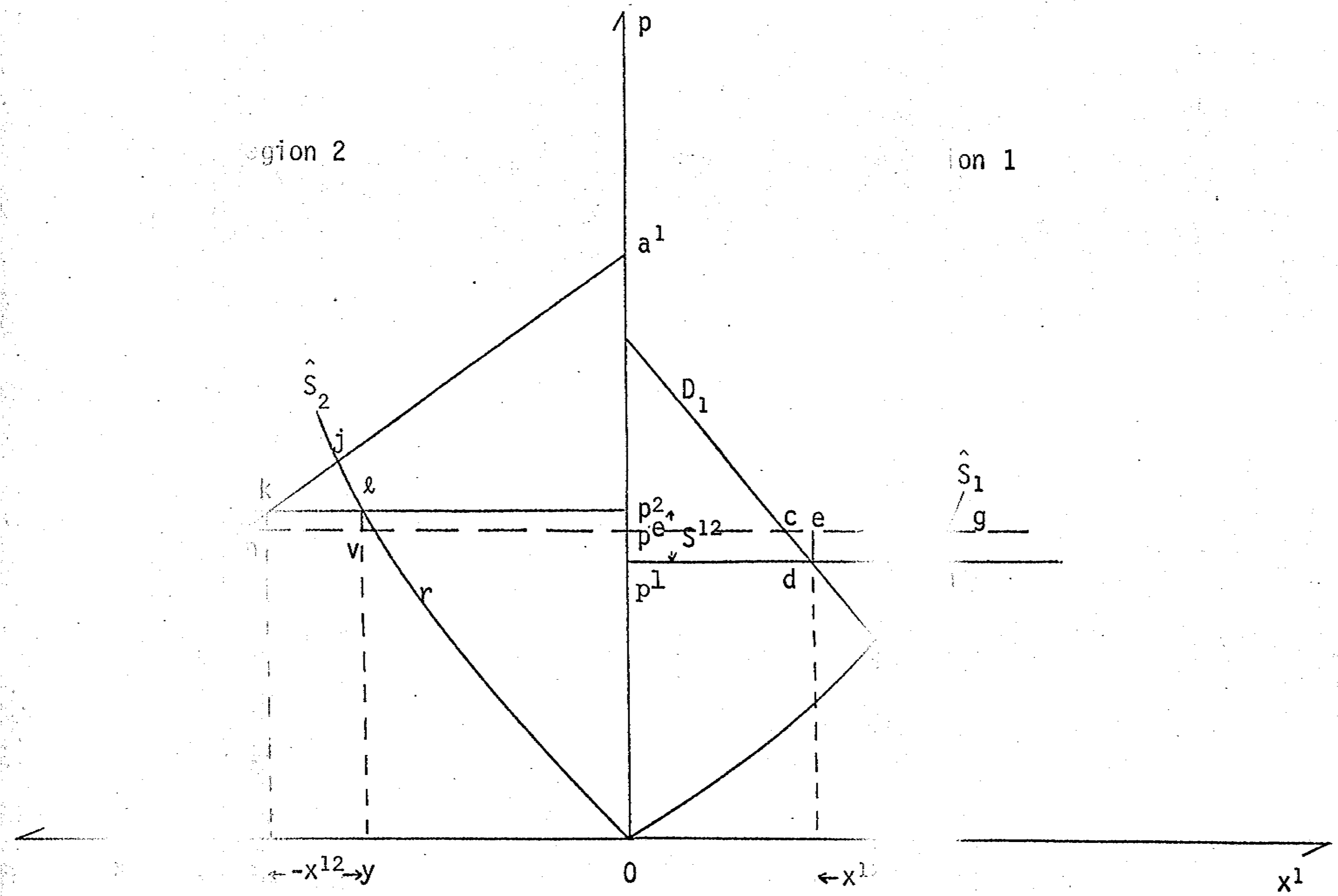


Figure 5.

In Figure 5 are the "supply" and demand schedules for regions 1 and 2 in a back-to-back trade diagram. D_1 and D_2 are the linear demand schedules for regions 1 and 2 respectively. $\hat{S}_1$ is a schedule indicating the amounts $x^1(u^1)$ of the commodity grown in various sized rings of radius u_1 about the town. Corresponding to each amount $x^1(u^1)$ produced in the fields is a unit transport price t^1u^1 indicating the cost of moving one unit of the commodity from the perimeter of the ring under cultivation to the town. For a town in isolation the sum of consumers' surplus plus land rents (producers' surplus) is maximized when the area under cultivation is such that the delivered price of a unit of the commodity equals the cost of transporting a unit of the commodity from the perimeter of the area under cultivation to the town, or at i and j for regions 1 and 2 respectively, i.e. $p^1 = t^1u^1$. With trade, there exists the possibility of expanding economic rents or sums of producers' and consumers' surpluses beyond levels attainable under conditions of no trade. Furthermore an equilibrium corresponding to a welfare maximum can be described by the conditions under which economic rents are maximized. We have an $\hat{S}_2$ in region 2 analogous to the $\hat{S}_1$ and region 1.

The economic conditions defining an equilibrium in the above model derive from the equilibrium conditions to the following two mathematical problems.

Primal: Determine non-negative u^1 and u^2 and hence x^{12} such that areas efhid + jknvl - {knvl + efhd} are maximized.

Dual: Determine non-negative u^1 and u^2 and hence p^1 and p^2 such that areas cdhd + kmrl - {efhd + klvn} are minimized subject to

$$p^1 - p^2 + s^{12} \geqq 0$$

Note that the primal is an economic rent maximization problem whereas the dual is a problem in which economic rent foregone because of the presence of transportation costs is minimized. Areas klvn plus efhd define the total transportation costs involved in shipping x^{12} from region 1 to region 2. p^e is the price for the commodity which would obtain in trade between the regions were permitted at zero transportation costs. In equilibrium, for region 1 area a^1dp^1 is the consumers' surplus; area p^1ho is the land rent or producers' "surplus"; and area ohw is the total transportation cost involved in getting the agricultural commodity to town. In equilibrium, for region 2 area a^2kp^2 is the consumers' surplus; area olp^2 is the land rent or producers' "surplus"; and area oly is the total transportation cost involved in getting the agricultural commodity to town.[9]

Consider the rent per unit radius function in this case when demand has some elasticity. From the fact that $\frac{dp}{dx} < 0$ or the demand curve has a negative slope for each commodity (x is quantity demanded) and $\frac{dx}{du} = ku$ where k is a constant, we note that $\frac{dp}{du} < 0$. From this fact we can see that $\frac{d^2W}{du^2} < 0$ where W is economic rent including consumers' surplus. So long as demand is monotonic with respect to price, then the rent function W(u) be shaped as below in Figure 6.

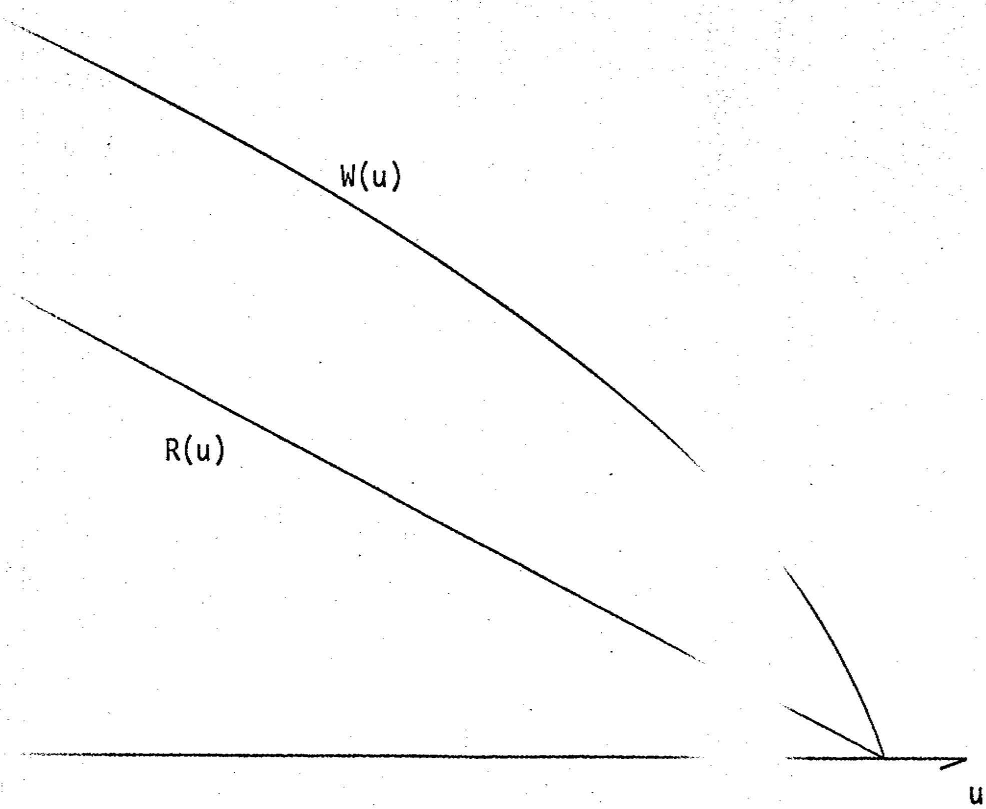


Figure 6.

Note that the land rent function is still a straight line. For the spatial price equilibrium and land rent problem illustrated in Figure 5, the regions' respective rent per unit radius functions will appear as illustrated below in Figure 7.

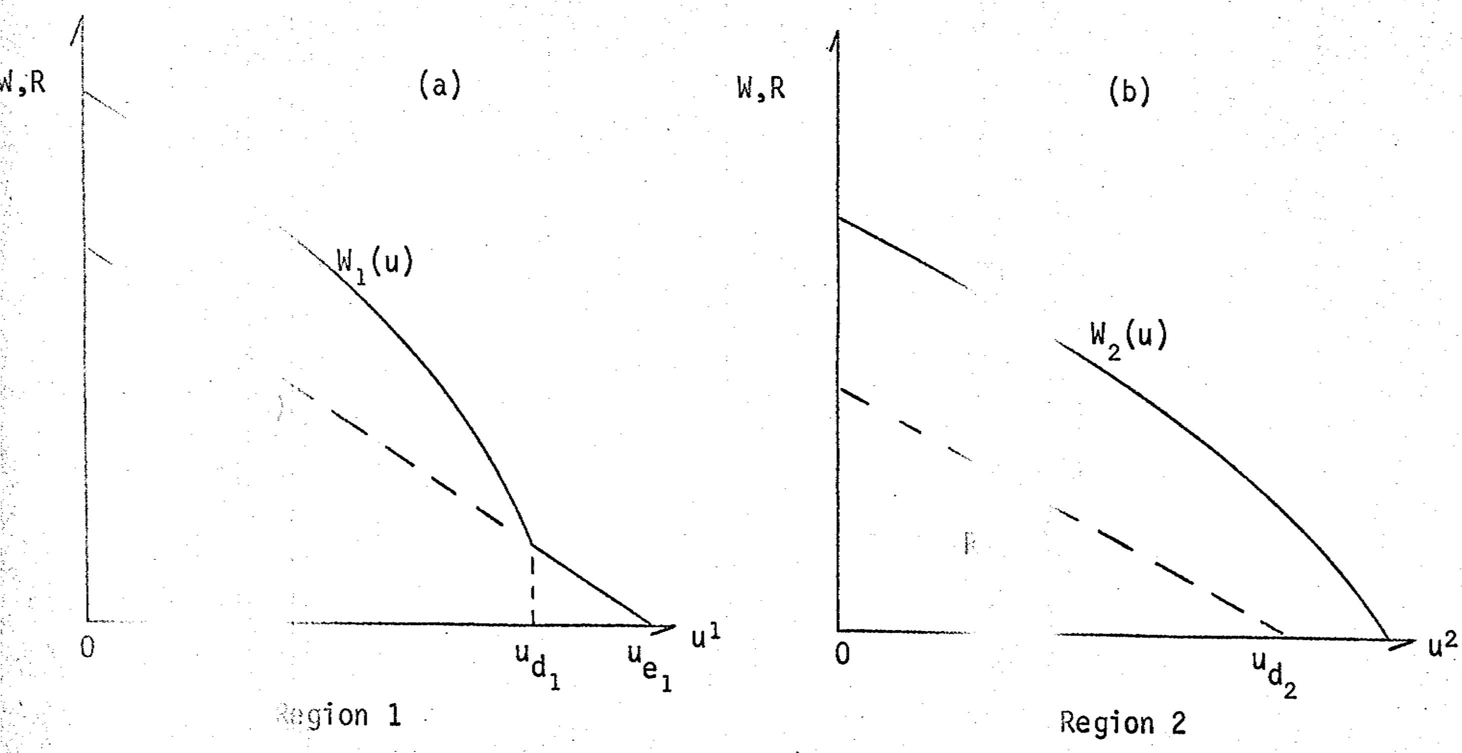


Figure 7.

In Figure 7a distance $u_{e_1} - u_{d_1}$ is the part of region 1's radius denoting how much land is devoted to growing the crop for export to region 2. Beyond u_{d_1} there is no more consumer surplus accruing in region 1, only land rent; $W_1(u)$ has a kink at u_{d_1} and becomes a straight line. In Figure 7b u_{d_2} is the size of the radius defining the amount of land devoted to agriculture. Land rent becomes zero beyond u_{d_2} but economic rent is still positive because consumers' surplus is still positive corresponding to the imports of the commodity from region 1. Thus the $W_2(u)$ function dominates the $R_2(u)$ function for region 2.

The analysis in this section has been restricted to the case of two regions or towns. The extension to n towns for n greater than 2 is straightforward and follows the path laid down in Samuelson [10] or Hartwick [4]. There will be the possibility of no trade occurring between regions.[10]

SECTION 3. MANY COMMODITIES IN A REGION WITH GENERAL DEMAND CONDITIONS

Let us consider a single region world with n commodities in which the demand for commodity i is a function of the prices of all other commodities. In section 1 we dealt with the special von Thünen case where demands were infinitely elastic at fixed prices.[11] We can extend our set of n commodities to include not only those which involve land directly in production but also those which are produced in town and use no land. The commodities produced in town will have demands and the demand functions will in general be functions of the prices of all commodities produced in the economy. Associated with these town commodities will be supply functions which will in general be functions of all prices of produced goods in the economy.[12] The demand function for good i will be assumed to be a continuously declining function of the price p_i for commodity i for all commodities. The supply function for town good j will be assumed to be a continuously rising function of the price p_j for commodity j. An economic equilibrium will result from the solution to the following primal-dual problems. Let us assume that there are k town goods subscripted $1,\ldots,k$ and n-k agricultural goods subscripted $k+1,\ldots,n$.

Primal: Determine n-k u's (i.e. $u_{k+1},\ldots,u_n$) and hence n outputs x_i $(i=1,\ldots,n)$ which result in the maximization of economic rent (the sums of producers' and consumers' surpluses for the town goods and the sums of land rents and consumers' surpluses for the agricultural goods) for the n commodities taken together.

Dual: Determine n-k u's (i.e. $u_{k+1},\ldots,u_n$) and hence n prices p_i $(i=1,\ldots,n)$ which result in the mini-

> mization of economic rent foregone in the production of the n-k agricultural commodities, where the rent is foregone because of the presence of transportation costs in getting the agricultural products to town. The minimization being constrained by the condition that the difference between price of a unit of an agricultural commodity in town and in the field must be less than or equal to the cost of transporting a unit of product to town.

Consider the case of three commodities, one produced in town 1, and two produced in the fields around the town, 2 and 3. Let us take demand and supply functions as linear. The equilibrium is illustrated in Figure 8 below.

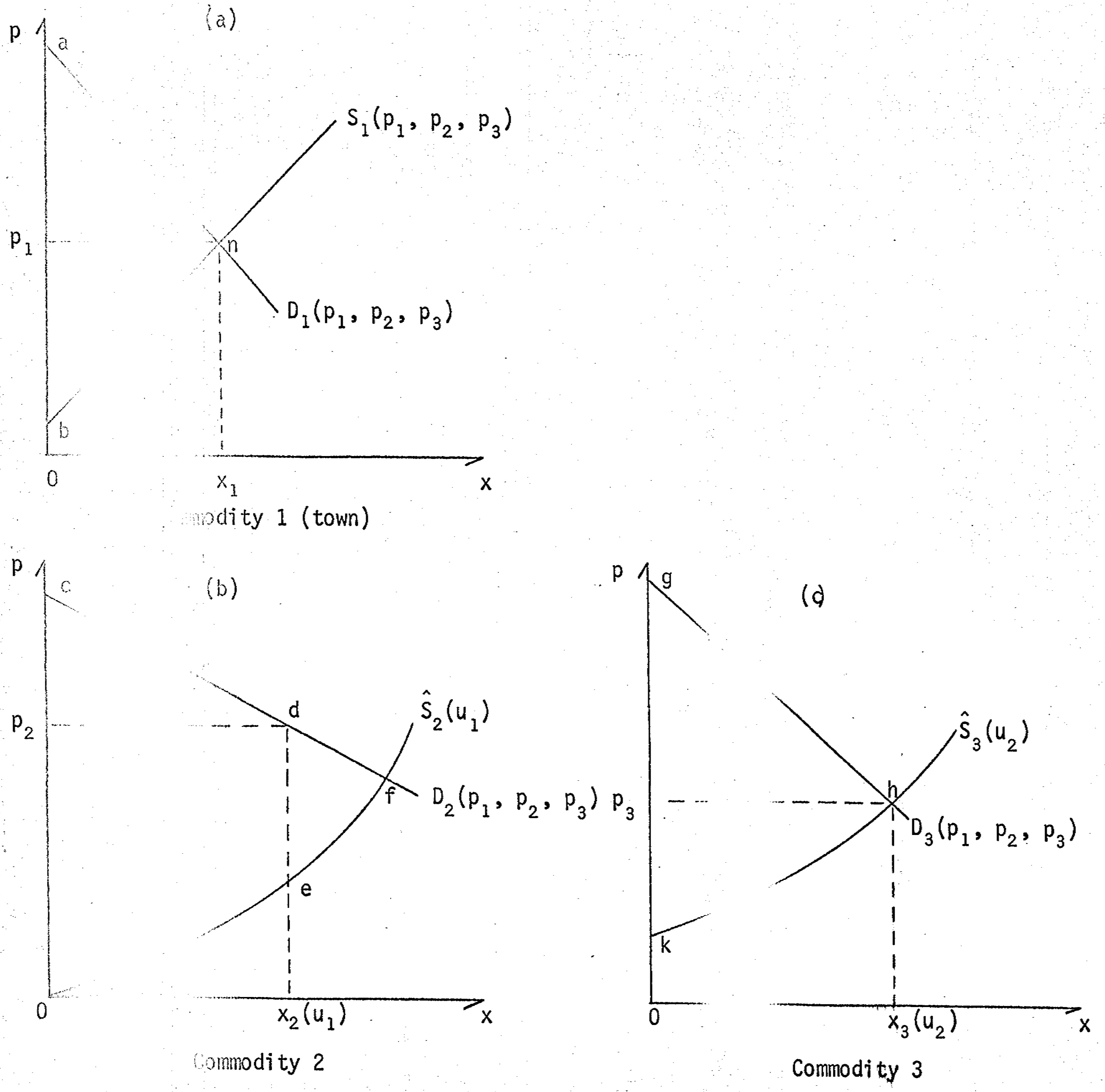

(a)
p
a
p1
b
0
S1(p1, p2, p3)
n
D1(p1, p2, p3)
x1
x
modity 1 (town)
(b)
p
c
p2
d
Ŝ2(u1)
f
D2(p1, p2, p3)
e
0
x2(u1)
x
Commodity 2
p
g
p3
k
0
Ŝ3(u2)
h
D3(p1, p2, p3)
x3(u2)
x
Commodity 3

Figure 8.

In Figure 8, the primal problem is to find u_1 and u_2, the radii of the circles defining the areas devoted to producing commodities 2 and 3, so as to maximize area anb + cdeo + ghk. In Figure 8, the dual problem is to find u_1 and u_2 so as to minimize area def (in Figure 6b). The quantities which define an equilibrium x_1, x_2, and x_3 as well as the prices p_1, p_2, and p_3 result from the solution to the maximum-minimum economic rent problem.

The rent functions for the three commodity problem presented in Figure 8 are diagrammatically presented in Figure 9 below.

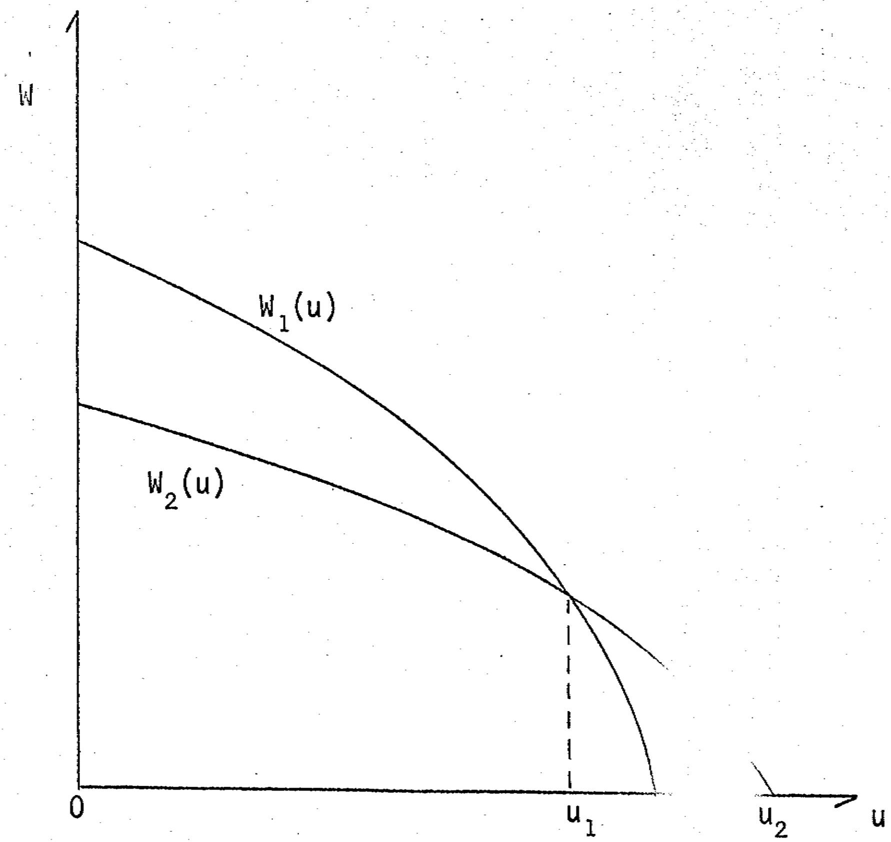


Figure 9.

The frontier for crop 1 closest to town will be at u_1 indicated in Figure 9 by the point where W_2 cuts W_1. At u_2 economic rent has fallen to zero and no crops will be produced beyond the frontier defined by u_2. It is apparent that the straight line economic rent function characteristic of the simple von Thünen model is a special limiting case when consumers' surplus zero or demands are infinitely elastic. Note that certain demand conditions will cause rings of crops to be repeated with a different crop separating two rings of the same crop. Consider a straight line rent function in Figure 9 cutting the other W function in two places corresponding to two different u's. In this latter case repeated rings will occur. The straight line rent function will occur if demand is infinitely elastic for one product.

SECTION 4. SPATIAL PRICE EQUILIBRIUM WITH MANY AGRICULTURAL COMMODITIES

We can combine the models of Sections 2 and 3 into one with n commodities and m regions or towns. There will be a non-empty set of commodities produced in the circular fields around each town and some produced only in town.[13] We do not require that every commodity be produced in or around every town. A necessary condition for the production of all n commodities in our system is that at least one region has a supply function for that commodity and at least one region has a demand function for that commodity.[14] Provided the equilibrium prices in antarky for one commodity in two regions differ by more than the cost of transporting a unit of the commodity between the regions trade will be induced to take place. We shall then take as given fixed transportation costs s_k^{ij} for moving a unit of commodity k ($k=1,\ldots,n$) between town i and j ($i,j=1,\ldots,m$). We desire to determine the nature of an equilibrium in our n commodity, m region world. That is given trade, we want to determine the prices in each town for the commodities, the amounts of the commodities produced in and around each town, to determine the areas of land devoted to the production of agricultural commodities around towns, and finally to determine the configuration of land rents around each town.

The economic equilibrium in this model will result from the solution to the economic rent maximization problem and from the constrained rent minimization problem.

Primal: Determine radii $\left\{u_1, u_2, \ldots, u_{r^k}\right\}^k$ ($k=1,\ldots,m$) where $r \leqq n$ and hence outputs $\left\{x_1, \ldots, x_{s^k}\right\}^k$ ($k=1, \ldots,m$) where $s \leqq n$ and $r \leqq s$ which maximize the sum of economic rent accruing to consumers and producers in all regions and in all markets in each region. r^k is the index of the final agricultural commodity in the k^{th} region and s^k is the index of the final commodity, including both agricultural and town commodities, produced in the k^{th} region.

Dual: Determine radii $\{u_1, u_2, \ldots, u_{rk}\}^k$ $(k=1,\ldots,m)$ where $u_i \geqq 0$ and $r \leqq n$ and hence prices $\{p_1, \ldots, p_{sk}\}^k$ $(k=1,\ldots,m)$ where $p_i \geqq 0$ and $s \leqq n$ and $r \leqq s$ which minimize economic rent foregone because of transportation costs, economic rent foregone in all regions and in all markets in each region. The minimization is to be conducted subject to the u's satisfying the following two general constraints.

1) In region k, the difference between price of an agricultural product in town and in the field must be less than or equal to the cost of transporting a unit of the product to town; $(k=1,\ldots,m)$.

2) For any commodity i, the difference between the price in town k and town l must be less than or equal to the cost of transporting a unit of the product from town k to town l; $(i=1,\ldots,n)$.

In Figure 10 below is an illustration of a two region, two agricultural commodity equilibrium with interregional trade.

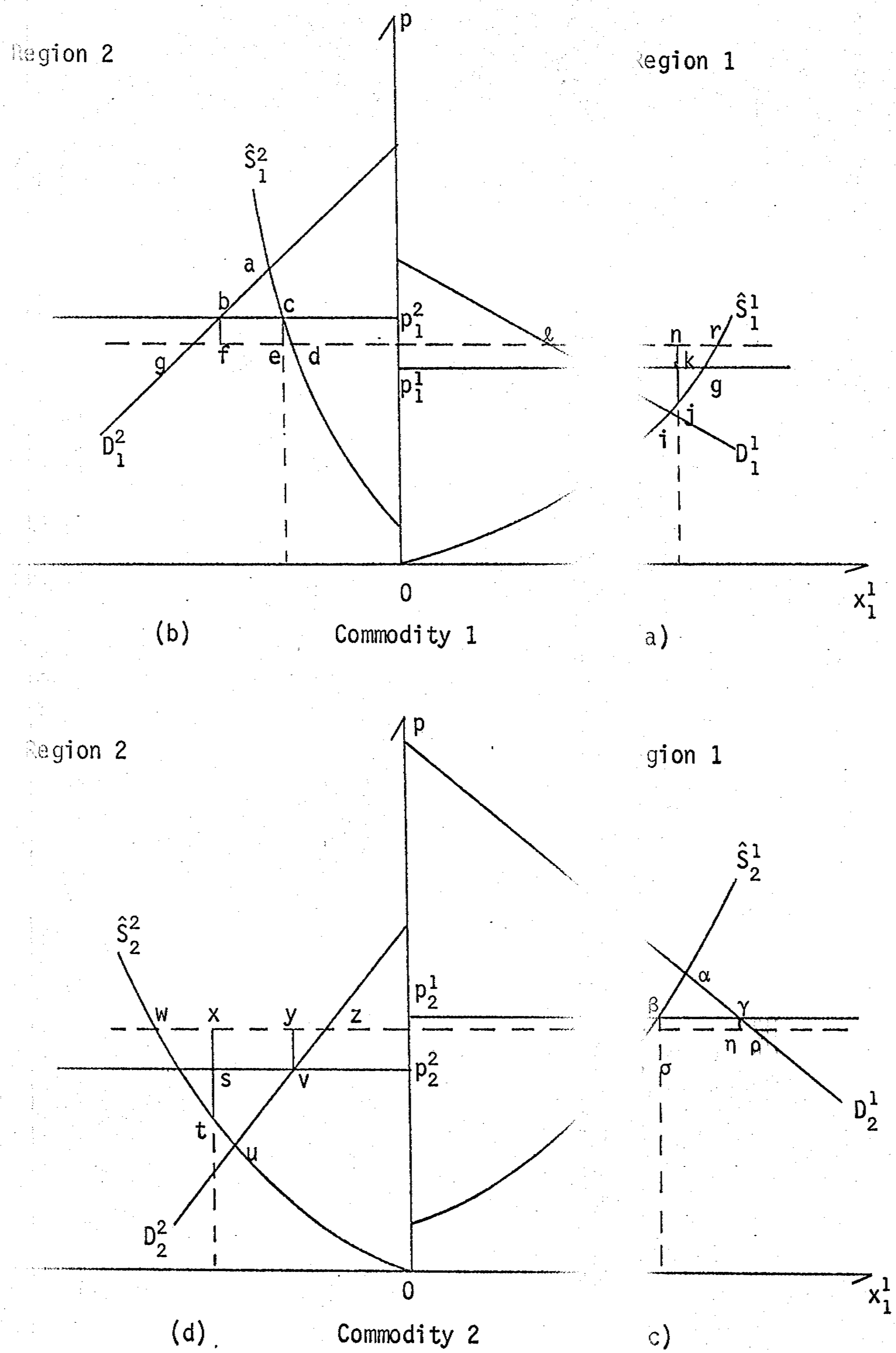

Region 2
p
Region 1
$\hat{S}_1^2$
a
b
c
p_1^2
ℓ
f
e
d
g
p_1^1
D_1^2
n
r
$\hat{S}_1^1$
k
g
j
i
D_1^1
0
x_1^1
(b)
Commodity 1
a)
Region 2
p
gion 1
$\hat{S}_2^1$
$\hat{S}_2^2$
α
p_2^1
w
x
y
z
β
γ
η
ρ
s
v
p_2^2
ρ
t
u
D_2^2
D_2^1
0
x_1^1
(d)
Commodity 2
c)

Figure 10.

In Figure 10(a) we have a "supply" schedule for commodity 1 in region 1, $\hat{S}_1^1$, and a demand schedule D_1^1. The super script indicates the region and the subscript the commodity. Schedules are similarly defined in Figure 10(b),(c), and (d). The primal problem is to determine ring sizes in regions 1 and 2 which maximize areas abc plus hijk, plus stuv plus $\alpha\beta\gamma$ in Figure 10. The export of commodity i from one region must equal the import of that commodity by the other. The dual problem is to determine ring sizes which minimize areas bgf plus cde plus lmh plus jnr plus wtx plus zyv plus $\beta\delta\sigma$ plus $\gamma\eta\rho$ subject to the price conditions stated above.

Observe that line gfedlmnr in Figure 10(a)(b) indicates what price would prevail in regions 1 and 2 if there were no transport costs in shipping commodity 1 between the two regions. The distance $p_1^2 - p_1^1$ is equal to the fixed cost of transporting a unit of commodity 1 between the town in region 1 and that in region 2. The equilibrium in Figure 10(c)(d) can be similarly interpreted. In equilibrium commodity 1 will be grown in the ring closest to town in region 1 and vice versa for region 2.

SECTION 5. BALANCE OF PAYMENTS EQUILIBRIUM AND AN ENDOGENOUS TRANSPORTATION SECTOR

The models we have examined to this point have been less than completely general because we have neglected to consider the long-run equilibrium given regional trade and we have chosen to consider trasportation as a non-produced commodity. In long-run equilibrium, we should expect the balance of payments in interregional trade to be in equilibrium, that is for each region k $(k=1,\ldots,m)$ the value of exports is just equal to the value of imports, that is

$$\sum_{i=1}^{n} \sum_{\substack{l=1 \\ k\neq l}}^{m} p_i^k x_i^{kl} = \sum_{j=1}^{n} \sum_{\substack{l=1 \\ k\neq l}}^{m} p_j^l x_j^{lk} \quad (k=1,\ldots,m) \tag{4}$$

where x_i^{kl} is the export of the i^{th} commodity from the k^{th} to the l^{th} region and p_i^k is the price of the i^{th} commodity in the k^{th} region.

Secondly by introducing a transportation sector into our model by making it one of the n commodities we will have an additional equilibrium condition to be satisfied. That is the value of the transportation good produced in all regions must equal the value of the transportation used in interregional trade. If we assume that the transportation good can be transported at zero transportation cost then the price of a unit of the good will be the same in all regions and the physical amount of the good produced will equal the physical amount of the good required for trade.

We shall introduce the transportation sector in this manner. Let $a_{t_j}^{kl}$ be the physical amount of transportation good required to transport a unit of j between regions or towns k and l. Now $p_t^k \, a_{i_j}^{kl}$ is the value of the transportation good in region k required to transport a unit of j between k and l. Thus our s_j^{kl} from Section 2 is equivalent to $p_t^k \, a_{t_j}^{kl}$ where j was the only good in the economy in Section 2. We also had transportation costs t_i^k indicating the cost of transporting a unit of agricultural commodity i a unit distance from the field to town in region k. With transportation costs endogenous t_i^k will equal $p_t^k \, a_{t_i}^k$ where $a_{t_i}^k$ is the physical amount of the transportation good required to transport a unit of good i a unit distance in region k. Finally in at least one town and perhaps all m, there will be a supply function for transportation goods.

$$S_t^k = S_t^k \; (p_1,\ldots,p_t,\ldots,p_n) \qquad (1 < k \leqq m)$$

We assume that the supply curve slopes upward with respect to the price of transport p_t. Once again, we might remark that all regions are not required to produce all commodities and so some p_i's can be zero in the supply function. The transportation use equilibrium condition is

$$\sum_{k=1}^{m} p_t x_t^k = p_t \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} x_i^{kl} a_i^{kl}$$

$$+ 2\pi p_t \sum_{k=1}^{m} \left[\int_{0}^{u_1^k} a_{t1}^k x_1^k (u_1^k) u\,du + \dots + \int_{u_{h-1}}^{u_h^k} a_{th}^k x_h^k (u_h^k) u\,du \right] \qquad (5)$$

where p_t is the price of a unit of the transportation good

x_t^k is the amount of transportation good produced in region k

x_i^{kl} is the flow of commodity i from town k to town l

u_h^k is the radius of outer frontier of a circle around town k and commodity h is grown in the corresponding ring in region k

$x_h^k(u_h^k)$ is the amount of crop h produced in a ring with an outer frontier with a radius u_h^k in region k.

To determine the existence of an equilibrium in our multiregional model with endogenous transportation costs requires the use of advanced mathematics, namely a fixed point theorem. Heuristically, however, we may illustrate the procedure as follows. Fix p_t the price of transportation in all regions at some positive value. Determine then t_j^{kl} and t_i^k and solve the primal and dual problems in Section 4 excluding the transportation sector from the n sectors. Then test to see whether the balance of payments condition (4) and the transportation condition (5) are satisfied. If the two conditions are not satisfied, choose a new price p_t for the transportation good and repeat the optimization problem set out in Section 4.[15] Provided our problem is mathematically well-behaved, there will be a p_t which will yield solutions to the optimizing problems and simultaneously satisfy the balance of payments and transportation requirements condition. Note that in general the absolute prices will be unique. This is because the level of economic activity feeds back on the volume of economic activity through the transportation sector in the same way the level of economic activity feeds back on the volume of economic activity through changes in the size of real balances in a monetary model.

SECTION 6. SPATIAL GENERAL EQUILIBRIUM

We have so far dealt with the case in which agricultural areas are circular in an equilibrium. However it is apparent that packing will occur in general - packing being the squeezing together of agricultural lands which are oriented toward different towns.[16] Frontiers will cease to be circular when pressed against each other. I use the expressions with a physical connotation such as packing, squeezing and pressing together because the process can validly be envisaged as a physical phenomenon such as two ameba pushing against one another on a slide under a microscope.

The nature of our equilibria with many regions requires only minor modification when packing occurs. The agricultural commodity "supply" curve associated with the commodity whose field is distorted will be less curved when the field is distorted than when circular.[17] The distortion will cause the size of other fields to change in general equilibrium but in general not the shape. Any point on a frontier separating the fields of two different regions will be one of equal rent with respect to the value of the respective agricultural products in the respective towns. Interstices in which no agriculture is undertaken will not be unusual. In fact with or without some packing, the economic topography will display no pattern; the pattern being a predominant feature of a Lösch or Christaller spatial general equilibrium.

The irregular economic topography can result from differences in tastes among dwellers in different towns, (demand functions will differ), from differences in factor endowments among regions (supply functions will differ) and from differences in efficiency in various regions (transport costs may differ for the same agricultural product within two different regions and/or agricultural output per unit land may differ for the same product between two regions).[18] Of course with special assumptions a Lösch or Christaller economic landscape could be constructed from the basic elements in the general equilibrium model of section 5. In the general model in this paper towns are taken as fixed in location whereas in the Lösch and Christaller models we seek to explain the location of towns in a system.

FOOTNOTES

1. Cournot summarizes "the highest development of communications between fractions of the same territory does...necessarily raise the real value of the national income (Samuelson's "Net Social Payoff") to a maximum, and brings about the most advantageous working. So far as I know, this fundamental principle of political economy, though always vaguely understood, has never been demonstrated by strict reasoning or deduced from its real premises". [1, p. 135]

2. Hotelling concludes: "It is remarkable and may appear paradoxical that without assuming any particular measure of utility or any means of comparison of one person's utility with another's, we have been able to arrive at (rent minimized) as a valid approximation measuring in money a total loss of satisfactions to many persons that this result depends only on the conception of ranking, without measurement, of satisfactions by each person is readily apparent from the foregoing demonstration." [5; p. 295] In this paper we are contrasting situations in which transportation costs are positive with situations in which there are no transportation costs. Incomes will be different in the alternate equilibria and thus Hotelling's welfare analysis will not be strictly valid.

3. We can of course take Samuelson's, what might be termed agnostic approach and call economic rent, Net Social Payoff. "This magnitude is artificial in the sense that no competitor in the market will be aware of or concerned with it. It is artificial in the sense that after the invisible Hand has led up to its maximization, we need not necessarily attach any social welfare significance to the result." [10, p. 288]

4. Takayama and Judge [12], [13], and Takayama and Woodland [14] have proved the existence of solutions to a class of mathematical problems similar to those posed here. By assuming demand and supply functions to be linear and with the usual slopes, they converted the general programs to quadratic programs and appealed to the well-developed mathematical literature in that area. The mathematical problem posed in Section 5 which yields an economic equilibrium, that is the one involving balance of payments equilibrium

and endogenous transportation costs can be solved by appealing to Kakutani's fixed point theorem. In another paper I will be rigorously proving existence for a problem related to those posed here, namely the Mosak international trade model with endogenous transportation costs. Isard and Ostroff [8] have proved existence is a Mosak-like model but they did not use programming techniques; these latter techniques make the proof simpler.

5. Stevens [11] has analyzed the von Thünen model in some detail and his bibliography indicates other contributions to the analysis of the basic model.

6. We know that the distance u_i must be positive. We also know from the second order conditions, indicating a maximum rather than a minimum has obtained, that $t_j a_j - t_i a_i < 0$. Hence $p_j a_j - p_i a_i < 0$ also. These are necessary conditions for the existence of a frontier between bands devoted to crops i and j. However sufficient conditions involve a whole chain of inequalities being also justified as in (2).

7. This concept of a dual, that is minimization of rent foregone, was first utilized in [4] a contribution generalizing the transportation problem. Note that in equilibrium, a strict inequality obtaining in the dual price problem implies a zero output in the primal quantity problem. This is a familiar programming equilibrium condition and in particular for a problem involving transportation costs indicates that if the difference between the price (cif) and price (fob) is less than the costs of shipment then no shipment will be induced to take place. In our case, not only no shipment will occur for a case of strict inequality but that agricultural commodity will not be produced where the transportation costs make it unprofitable to produce it.

8. The maximization could be carried out over total economic rent, that is areas a^2klo plus a^1dho to yield the same economic equilibrium. But it suffices to just consider the net increase in economic rents for the two regions.

9. We require that the distance between towns is sufficient so that agricultural land areas in equilibrium do not overlap. If they did overlap we could still define an equilibrium but the analytics would be more complicated. For an economic equilibrium the

agricultural areas could not overlap but must press against one another in a fashion which distorts the circular shape. The frontier dividing the two agricultural areas would be a contour defining points of equal land rents for the two areas. The resulting distortion would be reflected in Figure 5 in the respect that $\hat{S}_1$ and $\hat{S}_2$ are still always increasing but with less curvature.

The analysis in Hyson and Hyson [6] can be readily applied here to land rent frontiers separating areas under cultivation rather than to price frontiers separating market areas. In fact the extension of Fetter's Law by Hyson and Hyson is unsatisfactory when applied to market areas since they must assume transportation costs per unit distance are different for two producers of an identical product on a <u>homogeneous</u> plain. It is difficult to interpret such an economic situation. However, for the case of rent frontiers, the Hyson and Hyson extension is perfectly appropriate. For two simple von Thünen models overlapping, the frontier will be defined by the u satisfying $a^1(p^1-t^1u) = a^2(p^2-t^2u)$ or when $R^1(u) = R^2(u)$ for regions 1 and 2 respectively.

10. The caveats in the previous footnote then apply with equal force in the n region case. We return to these problems in Section 6.

11. The case where demands for commodity i is a function only of the price for commodity i in one region with many commodities is a special case which we shall not consider because it can be developed as a simple special case of the model in this section. This particular special case is also less satisfactory from an economic theoretical point of view.

12. The qualification produced goods is inserted because transportation goods are still considered exogenous. We shall consider transportation goods as produced commodities in Section 5.

13. If all are produced in town, our model will be the same as that of Takayama and Judge [13]. We shall assume then that a non-zero number of commodities are produced in the fields.

14. The existence of an upward sloping supply function and a downward sloping demand function in at least one region (the schedules need not be in the same region) for all commodities, and the presence of transportation costs between the regions whose value is less than the difference between the price intercepts of the supply and demand functions are necessary and sufficient conditions for observing all n commodities produced in the multiregional system.

15. The town supply and demand functions will be functions of p_t as will the agricultural demand functions. Thus when we say that the transportation sector is excluded from the group of a commodity in the otpimization, it is only the quantity that does not figure. The fixed transportation good price figures prominantly.

16. Packing can result because we must assume our total area finite. The total area could be a continent or the globe.

17. In the limiting case when a field becomes a straight line radius, the "supply" curve will be a straight line. Otherwise there will be some curvature as illustrated.

18. We could of course generalize the model by considering commodities produced in processes requiring variable doses of factors say land and labor. In particular we could consider the possibility of varying the intensity of cultivation of various tracts of land. Samuelson [9] has examined models incorporating these considerations.

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