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A GENERALIZATION OF THE TRANSPORTATION PROBLEM IN LINEAR PROGRAMMING AND SPATIAL PRICE EQUILIBRIUM

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1. Introduction

The Hitchcock-Koopmans (HK) transportation problem in linear programming [4], [5] and the Cournot-Enke-Samuelson (KES) pricing problem in spatial price equilibrium [2], [3], [6] are two classic analyses in the theory of transportation costs. In this paper I generalize the HK analysis to take account of variable supplies and demands for a product at the diverse geographically separated locations. The solution to the new problem is an optimizing problem of a form of great similarity to the Samuelson optimization problem for spatial price equilibrium. Primal and dual formulations are compared. I then show that the generalized HK problem is a special case of the pricing problem in spatial equilibrium and that there exists a principle unifying the analysis of spatial price equilibrium by optimizing techniques and there is a single optimizing problem that can solve a pricing problem in spatial equilibrium that incorporates the HK problem, the generalized HK problem and the KES problem.

2. The Generalized Transportation Problem

The Hitchcock-Koopmans problem can be described as follows. There are m supply points geographically separated from each other each with a fixed positive endowment of a commodity. There are n demand points geographically separated from each other and from the supply points and these demand points have fixed positive demands for the commodity at the supply points. The sum of supply endowments is as least as great as the sum of the demand requirements. There exist fixed transportation costs for moving a unit of the commodity between any pair of points, one of supply and one of demand. The problem is to transport the supply endowments to the demand requirements so as to minimize total transportation costs. That is determine x_{ij} for all i and j so as to

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} \tag{1}$$

$$\text{subject to:} \quad \sum_{i=1}^{m} x_{ij} \geqq \delta_j \qquad j=1,\ldots,n$$

and

$$\sum_{j=1}^{n} x_{ij} \leqq \sigma_i \qquad i=1,\ldots,m \tag{2}$$

$$x_{ij} \geqq 0 \quad \text{for all } i \text{ and } j.$$

where

t_{ij} is the cost of transporting a unit of the commodity from supply point i to demand point j.

x_{ij} is the flow of the commodity from i to j.

$_j$ is the fixed demand for the commodity at point j.

$_i$ is the fixed supply of the commodity at point i.

The dual to problem (1),(2) is to determine supply point prices p'_j $j=1,\ldots,n$ so as to

$$\text{maximize} \quad \sum_{j=1}^{n} p_j \delta_j - \sum_{i=1}^{m} p_i' \sigma_i \tag{3}$$

subject to:

$$p_j - p_i \leqq t_{ij} \quad \begin{Bmatrix} i=1,\dots,m \\ j=1,\dots,n \end{Bmatrix} \tag{4}$$

$$p_j \geqq 0 \text{ and } p_i' \geqq 0 \quad \text{for all } i \text{ and } j.$$

The equilibrium conditions indicating that the problems have been solved are of course

$$\text{if} \quad p_j - p_i < t_{ij} \quad \text{then} \quad x_{ij} = 0 \quad \begin{Bmatrix} i=1,\dots,m \\ j=1,\dots,n \end{Bmatrix}$$

$$\text{and if} \quad \sum_{i=1}^{m} x_{ij} > \delta_j \text{ then } p_j = 0 \quad (j=1,\dots,n)$$

$$\text{and if} \quad \sum_{j=1}^{n} x_{ij} < \sigma_i \text{ then } p_i' = 0 \quad (i=1,|||,m)$$

and the economic interpretation of these conditions is now well known.

We generalize the HK problem by assuming that there is a supply function at each supply point i, in which the amount supplied is a function of p_i' fob, that is quantity supplied $x_i = \hat{S}_i(p_i')$ $i=1,\dots,m$; and there is a demand function at each demand point j, in which the amount demanded is a function of p_j cif, that is quantity demanded $x_{ij} = \hat{D}_j(p_j)$. The demand and supply curves are assumed to have the usual slopes as indicated in Figure 1. We assume unit transport costs between i and j are again given. This new problem is to determine flows between points

$x_{ij} \begin{Bmatrix} i=1,\dots,m \\ j=1,\dots,n \end{Bmatrix}$ that obtain in spatial price equilibrium. This problem differs from the KES one in that we have only one schedule, either demand or supply at every point whereas the KES problem has both demand and supply schedules at every point.

We can see straightaway that if there is only one supply point i and one demand point j the equilibrium will be as indicated in Figure 1.

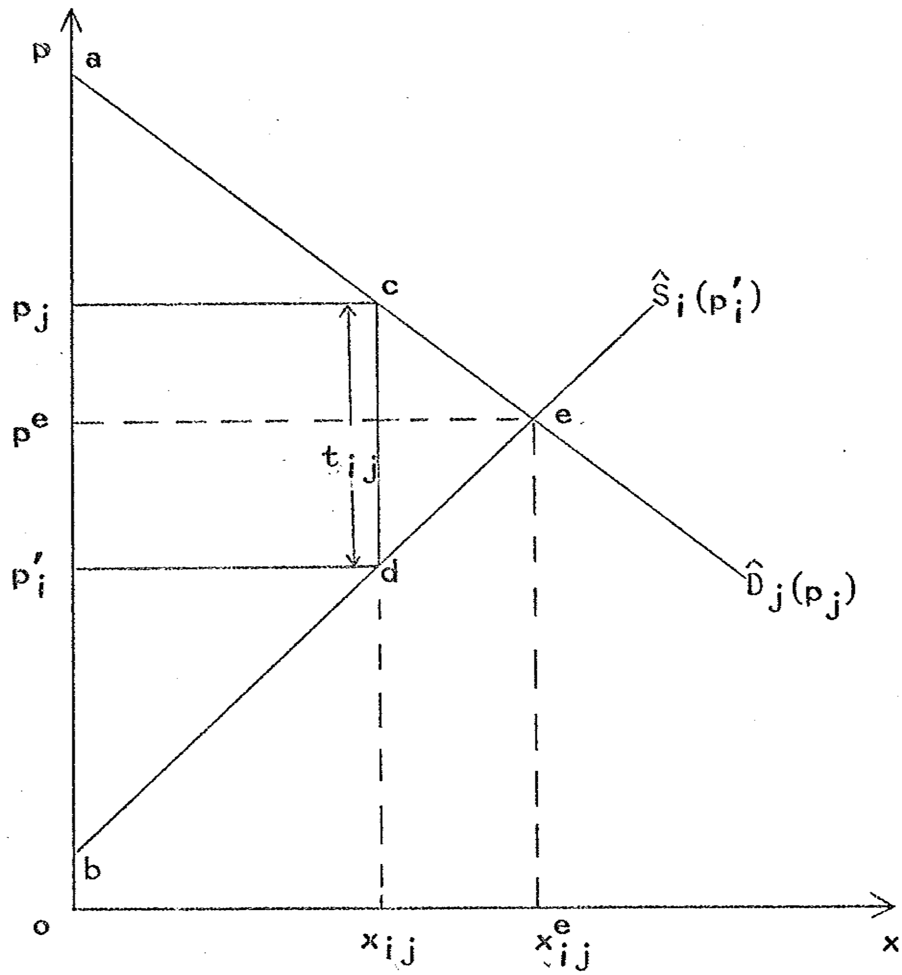

Figure 1

In Figure I, amount x_{ij} will flow from i to j. The equilibrium price at j will be p_j and at i will be p_i'. Also $p_j - p_i' = t_{ij}$. We observe further that in the absence of transportation costs the equilibrium price would be p^e and equilibrium flow x_{ij}^e. It would appear then that demanders always act to drive the price down or to maximize their consumers' surplus $ap^e e$ and suppliers act to drive the price up or to maximize their producers' surplus $bp^e e$ subject to the flow never exceeding x_{ij}^e. With transportation costs, it appears that consumers and producers continue to maximize their respective surpluses but are constrained from reaching p^e by the total transport cost bill $p_j c d p_i'$. Formally the equilibrium illustrated in Figure I is the solution to the following problem:

$$\text{maximize} \quad \int_o^{x_{ij}} D(u)du - \int_o^{x_{ij}} S(u)du - t_{ij}x_{ij} \qquad (5)$$

with respect to x_{ij}. D(u) is the function indicating price as a function of quantity or the inverted demand function. S(u) is the function indicating supply price as a function of quantity. The generalized HK problem where there are $m > 1$ suppliers and $n > 1$ demanders can be formulated as an extention of the above single supplier, single demander case. We simply maximize the sum of all consumers' and producers' surpluses at all geographic points and net out the total transportation cost bill. Thus the generalized HK problem is to determine the $x_{ij} \begin{Bmatrix} i=1,\ldots,m \\ j=1,\ldots,n \end{Bmatrix}$

which maximize

$$\sum_{j=1}^{n}\int_{0}^{\sum_{i=1}^{m} x_{ij}} D_j(u_j)du_j - \sum_{i=1}^{m}\int_{0}^{\sum_{j=1}^{n} x_{ij}} S_i(u_i)du_i - \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} \tag{6}$$

where $x_{ij} \geqq 0$ for all i and j.

The equilibrium conditions indicating that a maximum has been attained are the same as in the simple HK transportation problem and in the KES problem.

$$\text{if} \quad D_j(u_j^*) - S_i(u_i^*) < t_{ij} \tag{7}$$

$$\text{then} \quad x_{ij}^* = 0$$

where the stars indicate the values of the variables when a maximum has been attained. $D_j(u_j^*) = p_j^*$ and $S_i(u_i^*) = p_i^{*\prime}$. This condition indicates that if the difference between the price at a supply point and the price at a demand point is less than the transport cost for shipping a unit of the commodity between those points, then no shipment will take place.

In Figure 2, we have an illustration of the equilibrium situation for demanders at point j receiving shipments from supply points 3,6, and 7.

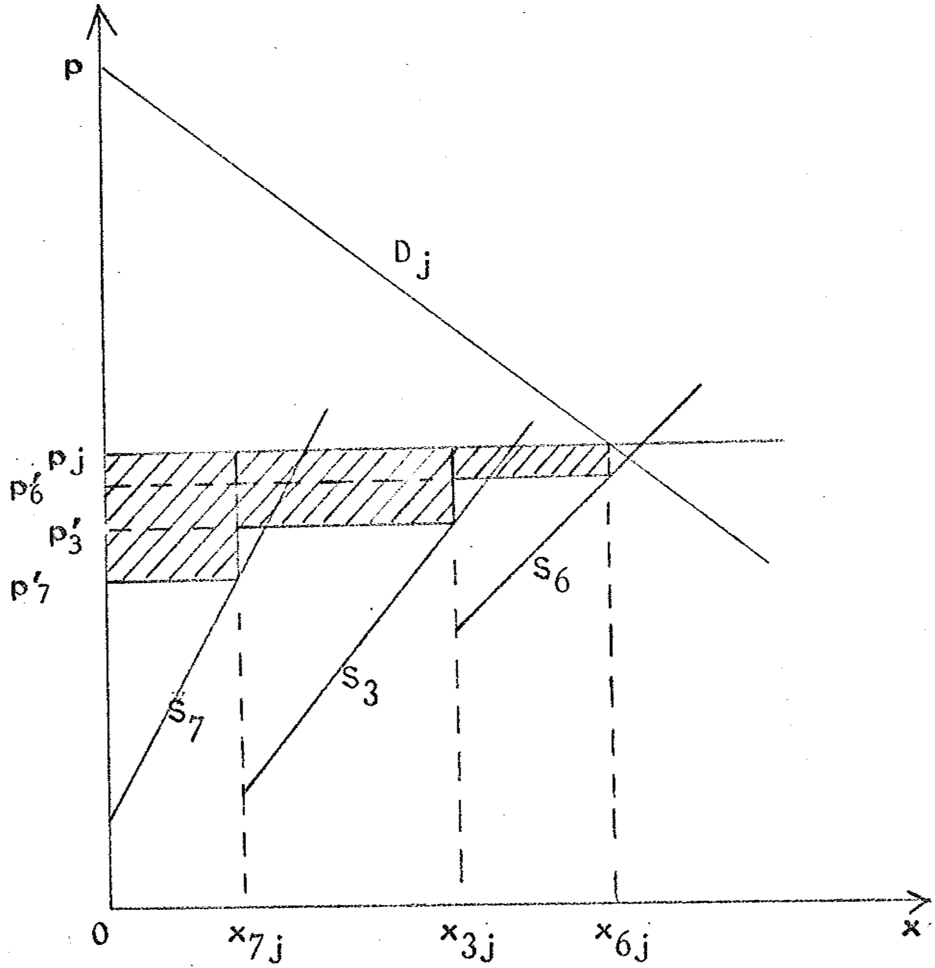

Figure 2

In Figure 2, the lined areas indicated total transportation costs. Also $p_j - p_7' = t_{7j}$, $p_j - p_3' = t_{3j}$ and $p_j - p_6' = t_{6j}$. An analogous diagram could be drawn for equilibrium at a supply point when numerous demand points are being supplied from that particular point.

Note that if supplies and demands are reduced to fixed positive numbers rather than schedules for all points then we are back in the HK world and the expressions with integrals in (6) indicating areas under demand and supply curves respectively become zero. We then have a problem of minimizing transportation costs alone and the supply and demand inequality constraints must be reintroduced in order to have a well-defined optimizing problem. The implicit constraints to (6) are that the solution must satisfy

$$\sum_{i=1}^{m} x_{ij} = D_j(u_j^*) \qquad (j=1,\ldots,n)$$

or that the sum of all flows from the m supply points must equal the demand at point j for all j, and

$$\sum_{j=1}^{n} x_{ij} = S_i(u_i^*) \qquad (i=1,\ldots,m)$$

or that the sum of all flows received at all demand points j from point i must exactly equal the supply at i for all i. In the HK problem flows to a demand point j can exceed the fixed demand at j and flows from supply point i can be less than the fixed supply at i. In addition, the solution to the generalized HK problem must satisfy

$$\sum_{j=1}^{n} D_j(u_j^*) = \sum_{i=1}^{m} S_i(u_i^*)$$

or total demands must equal total supplies. It is well known that the HK transport cost problem has a solution if and only if

$$\sum_{j=1}^{n} \delta_j = \sum_{i=1}^{m} \sigma_i \quad ,$$

or that the total of fixed supplies must at least equal the total of fixed demands.[1]

The similarity of the formal enunciation of the generalized HK problem in (6) to the formal statement of Samuelson's approach to solving the KES problem is striking. In Samuelson's treatment, the functions under the integrals in (6) would be excess demand functions in the receiving regions or points and sending regions respectively. The upper limits of integration would have a negative sign. Samuelson's maximand has "social surpluses" beneath the integrals whereas we have the sums of consumers' and producers' surpluses.

3. Duality and an Optimality Principle

For the two point case, the dual[2] to (5) is simply to determine two non-negative prices, one at the supply point and one at the demand point which minimize the area ced in Figure 1 subject to the condition that the difference between the price at the demand point and that at the supply point is less than or equal to the unit transportation costs between the two points. We are minimizing the sum of all consumers' and/or producers' surpluses foregone. For the

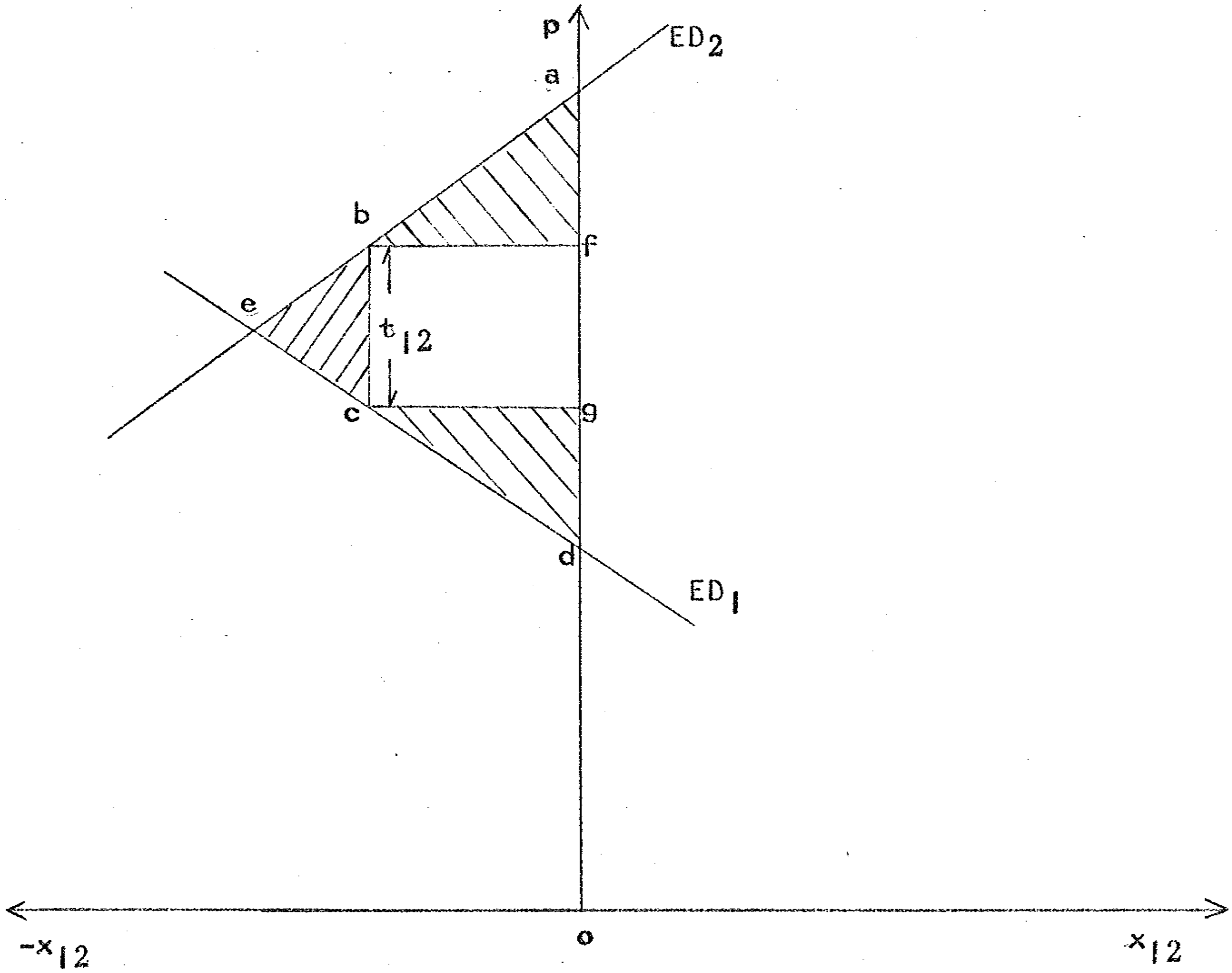

Figure 3

many supply point, many demand point problem, we can formally present the dual to (6) or the dual to the generalized HK problem as: determine non-negative prices p_i' $(i=1,\ldots,m)$ and p_j $(j=1,\ldots,n)$ so as to

$$\text{minimize} \quad \sum_{i=1}^{m} \int_{p_i'}^{p_i^e} \hat{S}_i(w)\,dw + \sum_{j=1}^{n} \int_{p_j^e}^{p_j} \hat{D}_j(w)\,dw \tag{8}$$

$$-\left(\sum_{j=1}^{m} p_j \hat{D}(p_j) - \sum_{i=1}^{n} p_i' \hat{S}(p_i') \right)$$

subject to:

$$p_i' - p_j + t_{ij} \geqq 0 \qquad \begin{Bmatrix} i=1,\ldots,m \\ j=1,\ldots,n \end{Bmatrix} \tag{9}$$

If we replace $\hat{S}_i(w)$ by $\widehat{ED}_i(w)$, excess demand at i, and $\hat{D}_j(w)$ by $\widehat{ED}_j(w)$, excess demand at j then we have the dual to Samuelson's optimizing problem developed to solve the KES problem.

For a two-region example, we can illustrate Samuelson's problem and its dual in Samuelson's excess demand diagram, Figure 3 below.

Samuelson's problem was to maximize the area abcd minus the area bcgf, where bcgf is the total cost of transport and t_{12}=bc is the cost of transport per unit flow. The dual to Samuelson's problem is to minimize area bec subject to the condition that distance bc(=fg) must be at least equal to distance t_{12}. Recall equations (8) and (9). The, at first blush, remarkable similarity between the form of the generalized HK problem and Samuelson's approach to the KES problem is brought out by comparing the structure of the equilibrium for the KES problem illustrated in Figure 3 with that for the generalized HK problem illustrated in Figure 1.

There exists a unifying principle underlying the solution to the KES problem and the generalized HK problem. We can illustrate this principle by returning to the diagram with supply and demand curves underlying Samuelson's excess demand schedule diagram above. In Figure 4 below we have a back-to-back two-region supply and demand model with transportation costs included.

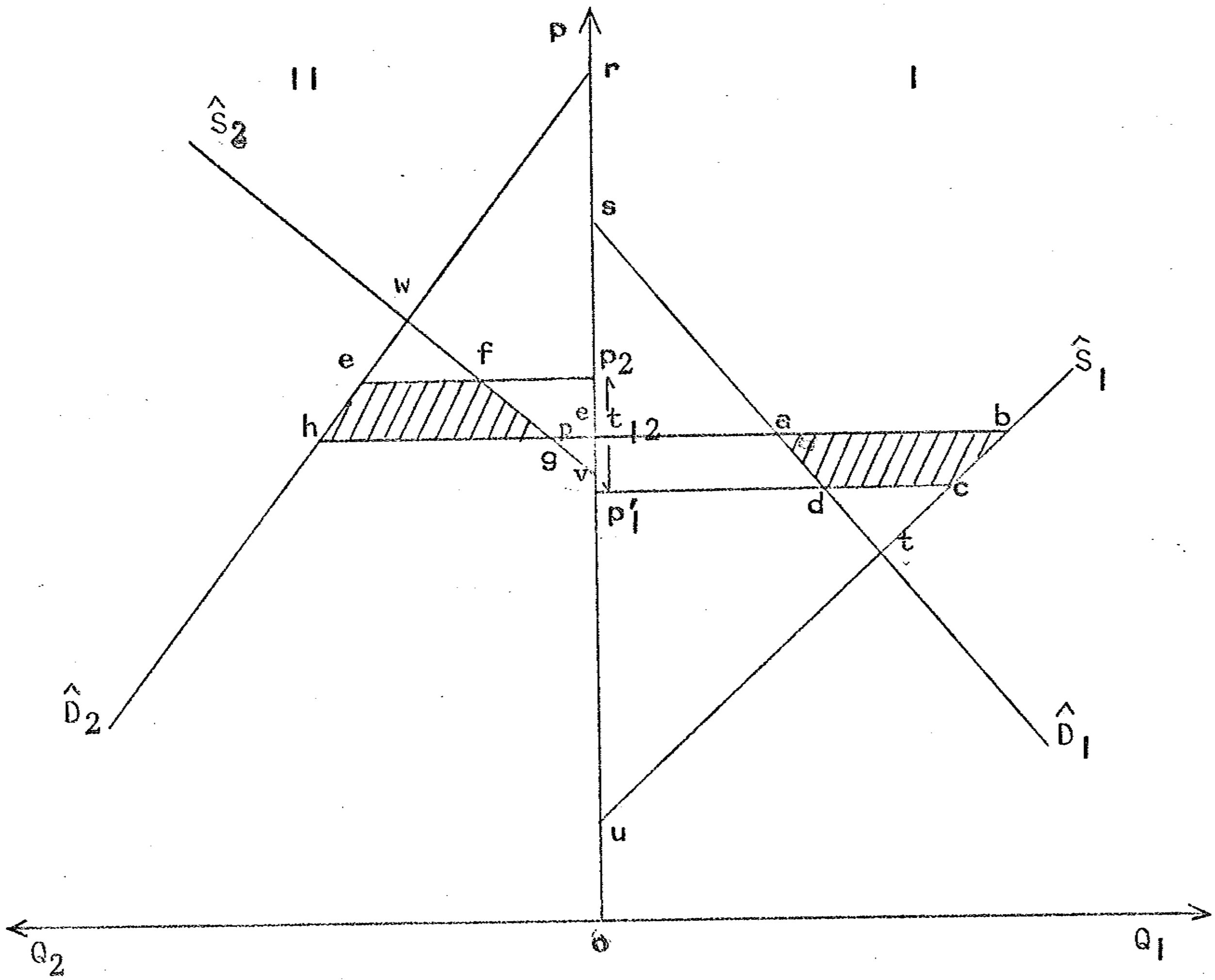
p
II
I
r
$\hat{S}_2$
s
w
e
f
p_2
$\hat{S}_1$
h
p^e
t_{12}
a
b
g
v
c
d
p'_1
t
$\hat{D}_2$
$\hat{D}_1$
u
Q_2
o
Q_1

Figure 4

If there were no interregional flows, the price and quantity in region I would be at t and in region 2 at w. If there were interregional flows but no transportation costs the equilibrium price, p^e, would be the same in each region lying between those associated with t and w and being fixed where flows from region I equalled flows received in region 2. If transportation costs were fixed at t_{12} then the equilibrium price would be at p_2 in region two and at p_1' in region I. p_2-p_1' would equal t_{12} and x_{12} would equal $-x_{12}$.

Consider the following two problems. First, maximize area dct plus area wef subject to the condition that ef equals dc. The resulting equilibrium will be the solution to the problem of determining interregional flows and prices when there are no transportation costs. The same equilibrium would be obtained if we minimized area abcd plus efgh subject to the condition that the price was the same in both regions. In other words the equilibrium is achieved by maximizing the rent in both regions which is gained from permitting interregional transfers subject to flows from one region equalling flows received by the other in a two region world. The same equilibrium is achieved by minimizing the deviations of rents from their equilibrium values.

The second problem which is the KES problem as developed by Samuelson is: Maximize areas ctd plus wef minus total transportation costs subject to the condition that flows from region I equal flows received in region 2. This is also a rent maximization problem as was the maximization problem illustrated in Figure 1. An equilibrium will also be obtained for the problem illustrated in Figure 4 if we minimize area abcd plus efgh subject to the condition that the price in region 2 minus that in region 1 will equal transportation costs t_{12}. Recall that the minimization problem associated with Figure 1 and the generalized HK problem was one of minimizing rent foregone owing to the existence of transportation costs. The minimization problem for the KES problem is also the minimizing of rent foregone owing to the existence of transportation costs, that is rent foregone from that which would accrue to both regions if there were transfers at zero transportation costs. The maximization approaches for the generalized HK problem and the KES problem are the universal ones of maximizing rent or the sum of producers' and consumers' surpluses in one or more markets at one time.[3] The limit of the rent maximization for the generalized HK and KES problems obtains where there are no transportation costs or no impediments to flow of any kind. In other words the solutions to the generalized HK and KES problems are special cases of equilibria in any market. It will be clear, now, that the sum of areas dct plus wfe in Figure 4 are equal to areas abf plus cgd in Figure 3 and that area ebc in Figure 3 is equal to the sum of the areas of rent deviations abcd and efgh from the rents at the zero transport cost equilibrium in Figure 4.

We have noted that the primal problems for approaching the generalized HK and the KES are both problems of maximizing economic rent. The dual problems are minimizing the rent foregone because of the presence of transportation costs, rent foregone from that which would obtain in equilibrium if transportation costs were zero. In view of the nature of the minimum problem, we can deduce the general optimality principle for spatial economic system. <u>If transportation costs are introduced into a system of markets in a spatial economic equilibrium, the system will reach a new equilibrium such that the economic loss resulting from the introduction of the cost of transportation is minimized.</u>

4. Spatial Price Equilibrium Synthesized

A general spatial equilibrium problem for one commodity has elements of the HK problem, the generalized HK problem and the KES problem. Consider a two region example incorporating a generalized HK problem and a KES problem.

Assume that we have two regions or points - region 1 has a demand and supply schedule for a commodity and region 2 has only a demand schedule. We may associate region 1 with the KES problem and region 2 with the generalized HK problem. We assume of course that transport costs per unit flow of the commodity are known and fixed at t_{12}. If the three schedules are such that there is a flow from region 1 to region 2 then an equilibrium will obtain as in Figure 5 below.

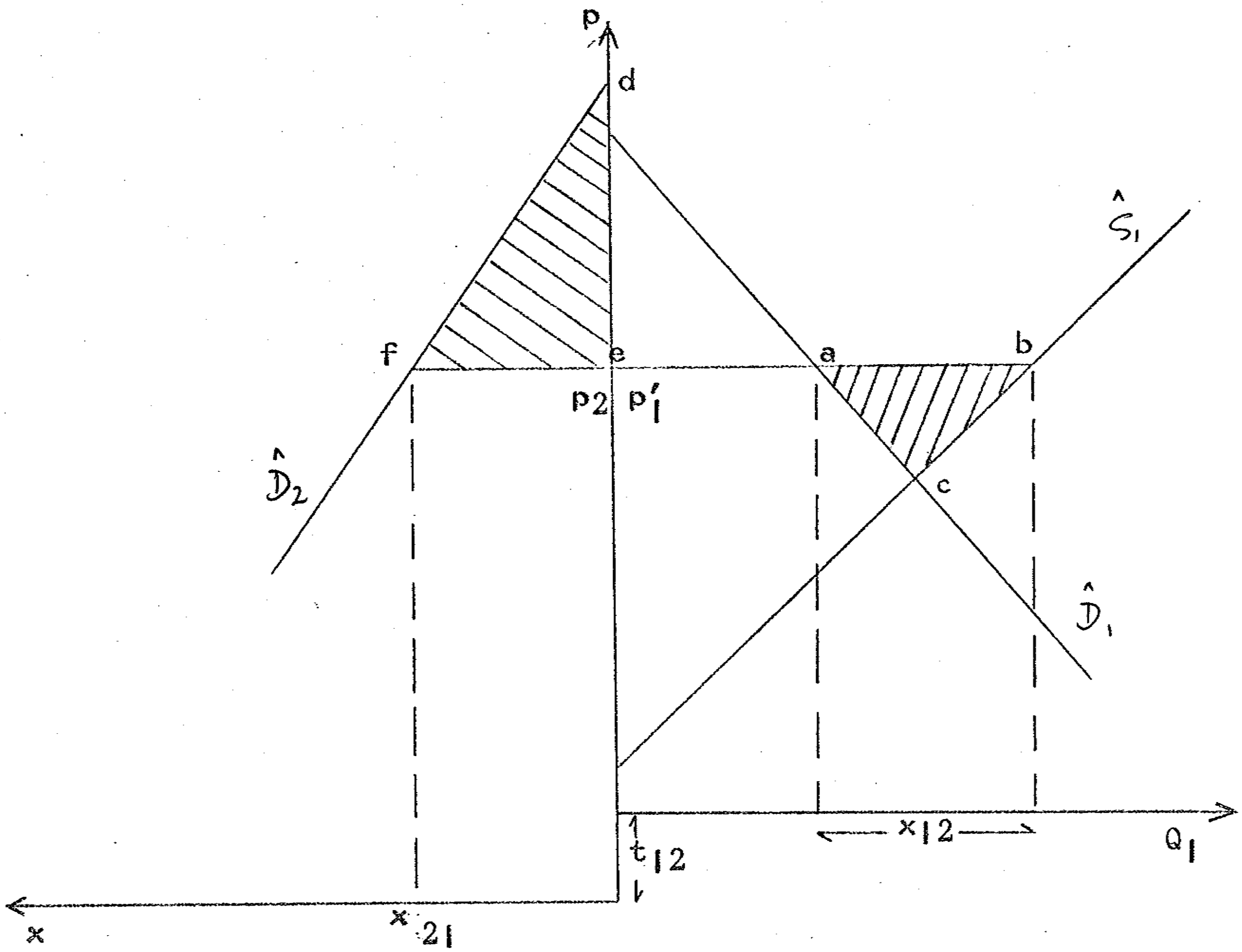

Figure 5

In the KES problem there are economic forces causing the area of triangle abc to be maximized in region 1 in the right-hand part of Figure 5 and in region 2 the generalized HK problem indicates that the area of triangle def will be maximized. In equilibrium $p_2-p_1'=t_{12}$ and $x_{12}=x_{21}$ or flows from region 1 equal flows to region 2. Thus an optimizing problem which consists of maximizing area abc plus area def, that is a combination of a KES problem and a generalized HK problem, will cause a spatial price and flow equilibrium to obtain.

Let us now assume that the demand is fixed in region 2 or of zero elasticity, that is D_2 is a vertical line in the left-hand side of Figure 4. This is the HK assumption. If D_2 intersects the horizontal x axis at x_{21} then there exist prices p_2 and p_1' such that $p_2-p_1'=t_{12}$ and $x_{12}=x_{21}$. The maximization of the area of triangle abc yields the equilibrium. If D_2 cuts to the right of x_{21} then the equilibrium will tend to settle where flows from region 1 equal flows to region 2 and the $p_1'-p_2+t_{12} \geqq 0$ constraint in the dual will be violated. Thus no shipment will actually take place. If D_2 cuts the x axis to the left of x_{21} then the maximum flow from region 1 namely ab or x_{12} will not be sufficient to satisfy the demand in region 2 and one of the HK constraints in the primal, (1)(2), is not satisfied and there will be no solution to the problem. It is thus not nonsensical to consider a problem incorporating elements of the HK problem and of the KES problem. All three problems are formally and economically related.

Let us partition suppliers into three groups, k' suppliers with fixed supplies, l' with elastic supplies but where no demanders are present, and h' with elastic supplies where demanders also are present. $k'+l'+h'=n$. We will partition demanders similarly into three groups, k demanders with fixed demands, l with elastic demands but where no suppliers are present, and h with elastic demands where suppliers also are present. $k+l+h=m$. Note that h' and h are disjoint sets since one group is located at points which are net suppliers and the other net demanders. A spatial equilibrium will consist of nm non-negative flows x_{ij} which

maximize

$$\sum_{j=k+1}^{l} \int_{o}^{\sum_{i=1}^{m} x_{ij}} D_j(u_j)\,du_j - \sum_{i=k'+1}^{l'} \int_{o}^{\sum_{j=1}^{n} x_{ij}} S_i(u_i)\,du_i$$

$$+ \sum_{j=l+1}^{h} \int_{o}^{\sum_{i=1}^{m} x_{ij}} ED_j(y_j)\,dy_j - \sum_{i=l'+1}^{h'} \int_{o}^{\sum_{j=1}^{n} x_{ij}} ED_i(y_i)\,dy_i$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} t_{ij}$$

subject to:

$$\sum_{i=1}^{m} x_{ij} \geqq \delta_j \qquad (j=1,\ldots,k)$$

$$\sum_{j=1}^{n} x_{ij} \leqq \sigma_i \qquad (i=1,\ldots,k')$$

(where ED_i is the price as a function of interregional flow x or excess demand) and nm non-negative prices p_i' $(i=1,\ldots,m)$, p_j $(j=1,\ldots,n)$ which

minimize

$$\sum_{i=k'+1}^{l'} \int_{p_i'}^{p_i'^{e}} \hat{S}_i(w)dw + \sum_{j=k+1}^{l} \int_{p_j^{e}}^{p_j} \hat{D}_j(w)dw$$

$$- \left\{ \sum_{j=k+1}^{l} p_j \hat{D}(p_j) - \sum_{i=k'+1}^{l'} p_i' \hat{S}(p_i') \right\}$$

$$+ \sum_{i=l'+1}^{h'} \int_{p_i'}^{p_i'^{e}} \hat{ED}_i(z)dz + \sum_{j=l+1}^{h} \int_{p_j^{e}}^{p_j} \hat{ED}_j(z)dz$$

$$- \left\{ \sum_{j=l+1}^{h} p_j \hat{ED}_j(p_j) - \sum_{i=l'+1}^{h'} p_i' \hat{ED}_i(p_i') \right\}$$

$$- \left\{ \sum_{j=1}^{k} p_j \delta_j - \sum_{i=1}^{k'} p_i' \sigma_i \right\}$$

subject to:

$$p_i' - p_j + t_{ij} \geqq 0 \qquad \left\{ \begin{array}{l} i=1,\ldots,m \\ j=1,\ldots,n \end{array} \right\}$$

The equilibrium conditions defining an optimum for the HK problem and its dual reported following equations (3) and (4) continue to apply to this more general problem. The essentially spatial price equilibrium condition is $p_j - p_i' < t_{ij}$ implies $x_{ij}=0$.

5. Conclusion

The maximization of economic rent and the minimization economic rent foregone because of the existence of transportation costs have been found to be two principles which unify the theory of interregional flows at non-zero transportation costs. We have developed the analysis in terms of one commodity and many spatially separated markets. The problem of introducing many commodities to a KES one commodity model has been analyzed by Takayama and Judge [8]. Our discovery of the optimality principles underlying spatial economic equilibrium could assist in reinterpreting Takayama and Judges results and assist in generalizing their model to incorporate elements of the HK and generalized HK models. The development of numerical techniques for solving complex spatial equilibrium models remains a difficult problem,

FOOTNOTES

1. Hitchcock [4] in fact defined the original HK programming problem with equalities in all constraints.

2. The word dual is not used to denote what Wolfe has defined as the dual in non-linear programming. See Balinski and Baumol [1] for the Wolfe definition and its interpretation. Our notion of dual is borrowed from Smith's analysis [7] of Samuelson's formulation of the KES problem. It is simply another optimization problem, in this case a minimization one where our primal was a maximization one, whose equilibrium conditions define a spatial price equilibrium in the original sense of HK. Our primal and dual are of course related by a saddle-point structure.

3. Smith [7] defined a problem which was to be dual to Samuelson's and the economic rationale underlying the new problem was found to be more plausible than that underlying Samuelson's "artificial" problem of maximizing "net social surplus". Smith's problem was the minimization of areas sdctu plus rwefgv subject to $p_1' - p_2 + t_{12} \geqq 0$. The areas stu and rwv do not affect the solution in any way. By minimizing area dct plus wef subject to $p_1' - p_2 + t_{12} \geqq 0$ Smith is <u>inverting</u> Samuelson's problem since Samuelson was maximizing the areas of these triangles subject to $x_{12} = -x_{12}$. The problem illustrated in Smith's Figure 2 has an interior of non-corner solution. For a minimum, we require that

$$\begin{vmatrix} \hat{S}_1' - \hat{D}_1' & 0 & -1 \\ 0 & -(\hat{S}_2' - \hat{D}_2') & +1 \\ -1 & +1 & 0 \end{vmatrix} < 0$$

or $(\hat{S}_2' - \hat{D}_2') - (S_1' - D_1') < 0$. However the above exposition makes clear that Smith intended to be maximizing rents. Hence $(\hat{S}_2' - \hat{D}_2') - (\hat{S}_1' - \hat{D}_1')$ must be less than zero. Note Smith has a sign error in his term $\frac{\partial \phi}{\partial p_2}$ following the Lagrangian (5). It should read

$$\frac{\partial \phi}{\partial p_2} = D_2(p_2) - S_2(p_2) + \lambda_{12} - \lambda_{21} \geqq 0.$$

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