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DUOPOLY IN SPACE

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1. INTRODUCTION

Hotelling's 1929 article concerning the behavior of duopolists in a spatial setting has had a lasting influence in economics and political science. With a simple model he was able to elucide why "our cities become uneconomically large and business districts within them too concentrated"; and why "Methodist and Presbyterian churches are too much alike; cider is too homogeneous" [8 ; p. 484]. In American politics, he showed "why each party strives to make its platform as much like the other's as possible" [8 ; p. 482].

Hotelling was aware that the zero elasticity of demand ascribed to consumers, though it contributed to the simplicity of his formal model, made his predictions rather extreme. "The elasticity of demand of particular groups does mitigate the tendency to excessive similarity of competing commodities, but not enough" [8 ; p. 484]. Smithies [16] proved this result in a subsequent paper although the mathematical details were not published. Devletoglou [3] demonstrated the same result in a quite different model.

In this paper we develop the constant non-zero elasticity of demand case which Hotelling referred to.[1] Hotelling's zero elasticity model emerges as a special case. The analytical details of Smithies linear demand function are also presented. We show that the final equilibrium position can lie

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beyond the bounds which Hotelling mooted and Smithies asserted. A new model is not required to show polarization as an equilibrium condition. Comparative static exercises concerning the effects of different elasticities of demand and different rates of transportation cost on the final equilibrium position are reported. These issues were commented on by Hotelling and Smithies. The second major part of this paper deals with competitive reactions between duopolists and the convergence to a stable equilibrium for both rivals. The conclusion we reach is that what Samuelson has termed, in the context of location,"the random walk of history" [15; p. 126] is beautifully illustrated by the spatial duopoly model; the final equilibrium is shown to depend on initial prices and locations. Hotelling's theorem on clustering is shown to require restrictive assumptions on the process of competitive reaction to equilibrium. Finally the limitations of the Hotelling-Smithies model in explaining the behavior of political parties are analyzed. A combination of mathematical analysis and numerical simulations were the principal techniques used to examine the above issues.

2. THE MODEL AND EQUILIBRIUM

Our economy consists of a market along a line of length ℓ with two sellers. This line might be a beach or a transcontinental railroad. Families dwell in this market forming a regular pattern or in the constant density D per unit distance. In Figure 1 we have the market with two rivals located. Competitor 1 is u_1 units from the left end and competitor 2 is u_2 units from the same end. The total length is ℓ, a fixed value.

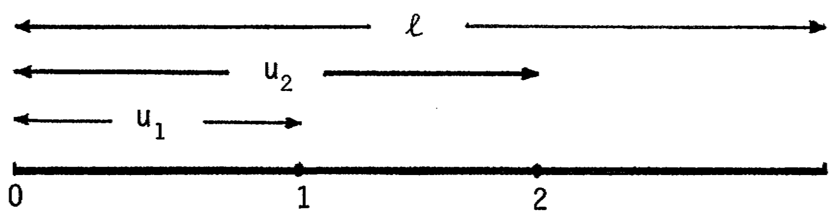

Figure 1.

Each family has a demand function per unit time which will in general be a function of delivered price.

$$x_F = f(p + tu) \tag{1}$$

where x_F is quantity demanded per unit time by a family, p is the price at the factory f.o.b., t is the constant transport cost per unit distance and u is the distance from the factory or firm.

We consider the case of two firms supplying the market at some cost. In general the total cost for each firm per unit time will be assumed to be linear in quantity produced.

$$C_i = A_i + k_i y_i \qquad (i = 1, 2) \tag{2}$$

where A_i is the fixed cost per unit time, k_i is the marginal cost per unit time and y_i is the quantity produced per unit time.

An equilibrium with two sellers is depicted in Figure 2.

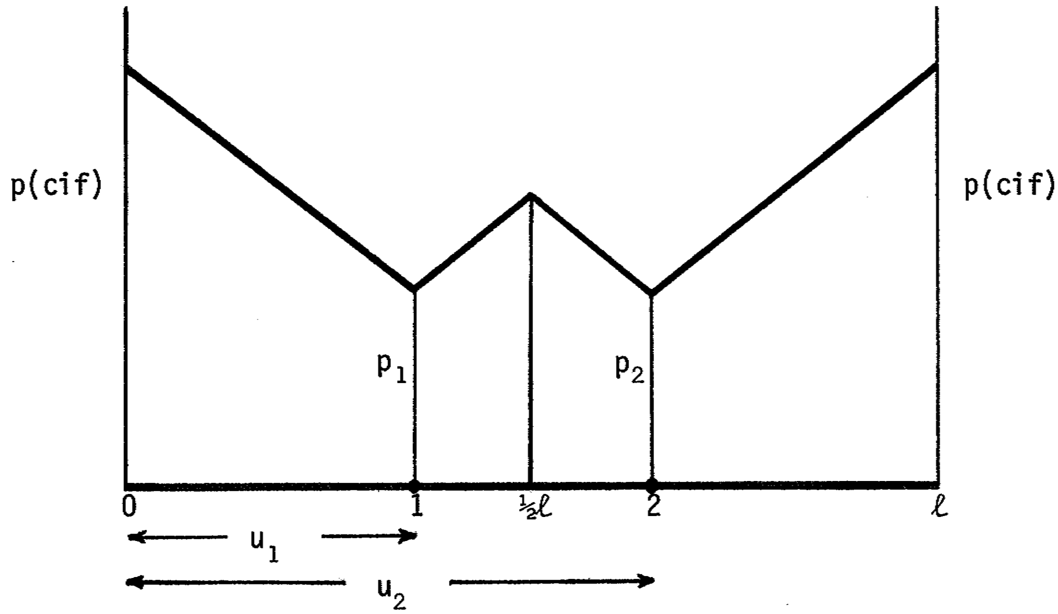

Figure 2.

In Figure 2 are two sellers in equilibrium when the elasticity of demand is non-zero. Note $\ell - u_2 = u_1$ and $u_2 - \frac{1}{2}\ell = \frac{1}{2}\ell - u_1$. The price (cif) is the same at the center of the market and $p_1 = p_2$. The regions u_1 and $\ell - u_2$ will be referred to as the hinterlands of sellers 1 and 2 respectively, and the regions $\frac{1}{2}\ell - u_1$ and $u_2 - \frac{1}{2}\ell$ will be referred to as the local markets of sellers 1 and 2 respectively. For the sake of variety, sellers will also be referred to as firms, plants, competitors, duopolists, and parties.

In this section we will investigate the nature of the equilibrium for the two sellers. The process of competitive reaction is analyzed in Section 3. We will appeal to only two types of behavior by sellers. For the most part, we assume atomistic competition, that is each seller reacts by changing his location and price (f.o.b.) in order to maximize his profits given the price (f.o.b.) and location of his rival. The other assumption made is that a central authority locates the sellers in order to maximize their combined profits or in order to make zero profits.

<u>Demand has a constant elasticity.</u>

Each individual demand function will be of the form

$$x_F = a(p + tu)^{\lambda} \qquad \lambda < 0 \tag{3}$$

We first assume that each competitor competes with his rival in an area consisting of exactly half the market. One seller remains in his half and the other in his half. Competition will still take place resulting in an equilibrium for each rival indicating its equilibrium location in half the market and its price given the location and price of his rival. The reason for analyzing this case is only to illustrate the analytics of an equilibrium.

We do not assume that competition actually takes place in this way. In the general problem each competitor competes in the whole market with his rival. Our economic analysis and numerical simulations are based on this latter model but fewer analytical mathematical details can be presented for this case owing to the complexity of the various mathematical conditions required to be satisfied in the model. Competition in half the market has the same qualitative aspects as atomistic competition to be discussed below in the whole market. In equilibrium where each duopolist faces the same cost and demand conditions each seller's delivered price will be the same at the mid point of the market. A plant's total sales will be obtained by integrating family sales over half the market.

$$y_1 = D\{ a \int_0^{u_1} (p_1 + t(u_1 - u))^\lambda du + a \int_{u_1}^{\frac{1}{2}\ell} (p_1 + t(u - u_1))^\lambda du \}$$

$$= \frac{D.a.t}{(\lambda+1)} \{(p_1 + tu_1)^{\lambda+1} + (p_1 + t(\tfrac{1}{2}\ell - u_1))^{\lambda+1} - 2p_1^{\lambda+1}\} \quad \text{for } \lambda \neq -1 \qquad (4)$$

where p_1 is seller 1's price f.o.b., and u_1 is seller 1's position in half (the left half) the market of length ℓ.

Seller 1's profit will be

$$\Pi_1 = p_1 y_1 - (A_1 + k_1 y_1)$$

$$= \frac{(p_1 - k_1)\, D.a.t}{\lambda + 1} \{(p_1 + tu_1)^{\lambda+1} + (p_1 + t(\tfrac{1}{2}\ell - u_1))^{\lambda+1} - 2p_1^{\lambda+1}\} - A_1 \quad \text{for } \lambda \neq -1 \qquad (5)$$

Seller 2's quantity sold can be determined by integrating over its half of the market in a similar way and its profit function obtained.

$$y_2 = \frac{D.a.t}{(\lambda+1)}\{(p_2 + t(u_2 - \tfrac{1}{2}\ell))^{\lambda+1} + (p_2 + t(\ell - u_2))^{\lambda+1} - 2p_2^{\lambda+1}\}$$
$$\text{for } \lambda \neq -1 \qquad (6)$$

and
$$\Pi_2 = \frac{(p_2-k_2)\ D.a.t}{\lambda + 1}\{(p_2 + t(u_2 + \tfrac{1}{2}\ell))^{\lambda+1} + (p_2 + t(\ell - u_2))^{\lambda+1} - 2p_2^{\lambda+1}\} - A_2$$
$$\text{for } \lambda \neq -1 \qquad (7)$$

The condition for equal prices at the mid-point is:

$$p_1 + t(\tfrac{1}{2}\ell - u_1) = p_2 + t(u_2 - \tfrac{1}{2}\ell)$$

$$\text{or} \qquad p_1 = p_2 + t(u_1 + u_2 - \ell) \qquad (8)$$

Substitute for p_1 in (5) from p_1 in (8) and we can express seller 1's profit as a function of his position in the market and seller 2's price and position.

$$\Pi_1 = \frac{(p_2 + t(u_1 + u_2 - \ell) - k_1)\ D.a.t}{\lambda + 1}\{(p_2 + t(2u_1 + u_2 - \ell))^{\lambda+1} + (p_2 + t(u_2 - \tfrac{1}{2}\ell))^{\lambda+1} - 2(p_2 + t(u_1 + u_2 - \ell))^{\lambda+1}\} - A_1 \qquad (9)$$

If we place firm 1 at a point in the market symmetric to firm 2's position, that is $u_1 = \ell - u_2$ (and $p_1 = p_2$) then the profits of the two firms are of course equal. However, it is of interest to examine the condition in which each firm maximizes its profits by varying its position subject to the condition that the competitor's price and position are fixed. In the following section it will be demonstrated that, following a convergent process of competitive readjustment, in equilibrium, each firm is maximizing its profits subject to the condition that the other competitor's price and position are fixed and that each firm is symmetrically stationed about the mid-point of the market. Consider now that final equilibrium position.

We require first the position u_1 such that profits Π_1 are maximized, given p_2 and u_2 fixed. Hence at any time solving $\frac{d\Pi_1}{du_1} = 0$ gives

$$(p_2 + t(u_1 + u_2 - \ell) - k_1)\{2t(\lambda+1)(p_2 + t(2u_1 + u_2 - \ell))^{\lambda}$$
$$- 2t(\lambda+1)(p_2 + t(u_1 + u_2 - \ell))^{\lambda}\} + t\{(p_2 + t(2u_1 + u_2 - \ell))^{\lambda+1}$$
$$+ (p_2 + t(u_2 - \tfrac{1}{2}\ell))^{\lambda+1} - 2(p_2 + t(u_1 + u_2 - \ell))^{\lambda+1}\} = 0 \qquad (10)$$

This implicit equation (10) contains a solution u_1 which indicates that at one point in the process of competitive reaction, seller 1's profits are maximized given the price and position of seller 2. Note that $u_1 + u_2 - \ell$ is never negative. Otherwise it would imply that p_1 was negative by equation (8). For $\lambda < -1$, then the value in the first curly bracket is positive and the second negative. Also p_2 is always greater than $k_1 = k_2$. For $-1 < \lambda < 0$ the values in the first curly bracket will be negative and in the second curly bracket positive. Thus we have weak evidence that equation (10) does have the desired non-negative solution.

The second order condition indicating a maximum is

$$\frac{d^2\Pi_1}{du_1^{\,2}} = 2Dat^3\{\lambda(p_2 + t(u_1 + u_2 - \ell) - k_1)[2(p_2 + t(2u_1 + u_2 - \ell))^{\lambda-1}$$
$$- (p_2 + t(u_1 + u_2 - \ell))^{\lambda-1}] + 2[(p_2 + t(2u_1 + u_2 - \ell))^{\lambda}$$
$$- (p_2 + t(u_1 + u_2 - \ell))^{\lambda}]\} < 0$$

This required negative sign will hold if

$$0 < u_1 < \frac{(1-\alpha)(p_2 - t(\ell - u_2))}{(2\alpha - 1)t}$$

where $\alpha = 2^{\left(\frac{1}{\lambda-1}\right)}$ $\lambda < 0$. Of course $p_2 > k_1 = k_2$.

Now $t(\ell - u_2)$ is cost of transporting the good to the right-hand end of the market from the position of 2. p_2 must exceed this cost in order for u_1 to be positive. Thus we have a constraint condition relating one competitor's location to the price and location of his rival. A similar result could be determined for competitor 2, and in the equilibrium defined by its exhibiting the property that neither firm wishes to change its price or position, symmetry in positions and equality in prices obtains or $u_1 = \ell - u_2$. In this situation:

$$\frac{d\Pi_1}{du_1} = 2t(p_2 - k_1)\{(\lambda+1)(p_2 + t(\ell - u_2))^\lambda - (\lambda+1)p_2{}^\lambda\}$$
$$+ t\{(p_2 + t(\ell - u_2))^{\lambda+1} + (p_2 + t(u_2 - \tfrac{1}{2}\ell))^{\lambda+1} - 2p_2{}^{\lambda+1}\} = 0 \qquad (11)$$

Note that the equilibrium position for firm 1 is a function of the position and price for firm 2 and that there is a whole family of (u_2 , p_2) which could be exhibited. Thus the equilibrium positions (and prices) for the two competitors are not unique. We will demonstrate below that the final equilibrium positions are, given the mode of competition, are functions of the initial, and in general arbitrary, position and price of the first entrant to the market. Not only does the final equilibrium depend on initial values but a process of atomistic competition will be shown to cause the equilibrium to obtain.

In the process of competition, each firm takes the whole market as its potential sales domain rather than simply half the market as we assumed above. Half the market will be shown to be an equilibrium condition signalling the end of competition. Let us assume one competitor fixed at u_2 and that another enters and locates at u_1 where his profits are maximized given the other locator. They will divide the market at u_k, not necessarily $\frac{1}{2}\ell$ as before. Thus firm 1's profits Π_1 will be obtained in an expression like (5) above, but where y_1 is determined by integrating sales over the market from 0 to u_k rather than from 0 to $\frac{1}{2}\ell$. For firm 2, y_2 is determined by integrating from u_k to ℓ.

At the market division point u_k, the delivered price from the two competitors must be equal. That is:

$$u_k = \frac{1}{2t}(p_2 - p_1 + t(u_1 + u_2))$$

Given the position of firm 2, u_2 and the price (f.o.b.) for firm 2, p_2, firm 1's profits are functions of u_1 and p_1 in the following expression.

$$\Pi_1 = \frac{(p_1 - k_1).D.a}{t(\lambda+1)} \{(p_1 + tu_1)^{\lambda+1} + \left(\frac{p_1 + p_2 + t(u_2 - u_1)}{2}\right)^{\lambda+1} - 2p_1^{\lambda+1}\} - A_1 \qquad (12)$$

Π_2 can be determined similarly.

Consider now how the nature of the equilibrium is affected by varying certain parameters in the model.

<u>Case 1 (Hotelling, 1929)</u>: Make costs zero, that is $A_1 = k_1 = 0$. Set the elasticity equal to 0, that is $\lambda = 0$. Make $D = a = 1$. Then

$$\Pi_1 = \frac{p_1}{t}\{p_1 + tu_1 + \left(\frac{p_1 + p_2 + t(u_2 - u_1)}{2}\right) - 2p_1\}$$

$$= \frac{p_1}{2}\{\frac{p_2 - p_1}{t} + u_1 + u_2\} \qquad (13)$$

This is Hotelling's profit function which was to be maximized with respect to p_1 and u_1 given p_2 and u_2. By inspection, we see that Π_1 can be maximized by making u_1 large or having firm 1 cluster close to firm 2 and vice versa for firm 2. Hence the celebrated clustering theorem.

Note also that the model is open-ended with respect to prices. ".... since demand is inelastic, we may imagine the two alleged competitors to be amicably exploiting the consumers without limit by raising their prices". Hotelling [8 ; p. 475]. We shall return to this point in Section 3.

Case 2CE (Constant Elasticity) Two Plant Monopoly. We presuppose a solution in which each plant is symmetrically located about the mid-point of the market. Total profits are

$$\begin{aligned}
\Pi &= \Pi_1 + \Pi_2 \\
&= \frac{D.a.t}{\lambda+1}\{(p_2 + t(u_1 + u_2 - \ell) - k_1)[(p_2 + t(2u_1 + u_2 - \ell))^{\lambda+1} \\
&\quad + (p_2 + t(u_2 - \tfrac{1}{2}\ell))^{\lambda+1} - 2(p_2 + t(u_1 + u_2 - \ell))^{\lambda+1}] + (p_2 - k_2) \\
&\quad \cdot[(p_2 + t(u_2 + \tfrac{1}{2}\ell))^{\lambda+1} + (p_2 + t(\ell - u_2))^{\lambda+1} - 2p_2{}^{\lambda+1}]\} - (A_1 + A_2) \qquad (14)
\end{aligned}$$

The necessary condition for total profits in (14) to have a maximum is that

$$\frac{\partial\Pi}{\partial u_1} = \frac{\partial\Pi}{\partial u_2} = \frac{\partial\Pi}{\partial p_1} = \frac{\partial\Pi}{\partial p_2} = 0$$

Rather than compute these derivatives, we might observe simply that profits are unbounded above with respect to prices and that a two-plant monopolist will simply raise prices indefinitely in this model. This is similar to the phenomenon that Hotelling observed for his case with price competition. This unboundedness is true for all positions including the quartiles which we shall see satisfy a maximum profit position for the two plant monopoly in the Smithies model below.

Case 3CE Equilibria with Different Elasticities of Demand. This case was investigated by means of numerical simulations. The conclusions which emerge are that given the same parameters and same initial conditions for the first competitor, higher elasticities result in final equilibria farther from the center of the market than with lower elasticities. An equilibrium for a firm occurs when there is no tendency to change price or location given the price and location of the other competitor. These results were suggested by Hotelling [8 ; p. 484]. The following numerical illustration in Table 1 depicts one set of equilibria varying with the values of the elasticity of demand.

Table 1

EFFECTS OF VARYING VALUE OF ELASTICITY OF DEMAND

$a = 1$, $k_1 = k_2 = 0$, $\ell = 1.0$, initial conditions $u_2 = 0.5$, $p_2 = 0.5$

Values of Variables in Equilibrium

λ	t	u_1	p_1	Π_1	u_2	p_2	Π_2
-2.0	1.5	0.4480	0.0020	1.3417	0.5520	0.0020	1.3417
	4.0	0.1440	0.0360	0.4791	0.8560	0.0360	0.4791
	4.5	0.2000	0.0500	0.4248	0.8000	0.0500	0.4258
-0.01	1.5	0.4500	0.8810	0.4398	0.5500	0.8810	0.4398
	4.0	0.4907	2.1970	1.0861	0.5093	2.1970	1.0861
	4.5	0.4500	2.6330	1.3001	0.5500	2.6330	1.3001

One feature of the results in Table 10 is that one pair of equilibria for the two competitors obtains beyond the quartiles. Hotelling did not anticipate this. He stated that higher elasticities would cause competitors to be separated "but he (B) will not go as far from A as the public welfare would require". Hotelling [8 ; pp. 117-118].

Case 4CE Equilibria with Different Rates of the Costs of Transportation. Examples were computed to investigate this situation. The same initial conditions and parameter values were set. A process of competitive adjustment was run and equilibria were obtained as transportation costs were varied. No pattern of location equilibria changes emerged in response to transportation cost changes. The example is presented in Table 2 below.

Table 2

EFFECTS OF VARYING VALUE OF TRANSPORT COSTS

$a = 1$, $k_1 = k_2 = 0$, $\lambda = -0.5$, initial values $u_2 = 0.65$, $p_2 = 0.5$

Values of Variables in Equilibrium

t	u_1	p_1	Π_1	u_2	p_2	Π_2
0.5	0.4500	0.4650	0.3103	0.5500	0.4650	0.3103
1.0	0.4500	0.4300	0.2747	0.5500	0.4300	0.2747
1.5	0.4680	0.6170	0.3233	0.5320	0.6170	0.3233
2.0	0.4783	0.8108	0.3671	0.5217	0.8108	0.3671
2.5	0.4500	1.0259	0.4216	0.5500	1.0259	0.4216
3.0	0.4500	1.2356	0.4629	0.5500	1.2356	0.4629
3.5	0.4673	1.5461	0.5173	0.5327	1.5461	0.5173
4.0	0.4673	1.6955	0.5385	0.5327	1.6955	0.5385
4.5	0.4673	1.8450	0.5590	0.5327	1.8450	0.5590
5.0	0.4500	2.1000	0.6051	0.5500	2.1000	0.6151

Note the regular increase in prices and profits for each firm when transportation costs rise above 1.0.

Smithies' Approach: Linear Demand Functions

Smithies conjectured that the clustering so prominent in the Hotelling model was a result of the peculiar demand assumptions - namely the zero elasticity of demand for each consumer. Hotelling in fact stated this point of view in his concluding comments to his original presentation.[2]

Smithies redeveloped the Hotelling model with demand functions linear in delivered price. Elasticity will vary along such a demand curve but will always be non-zero. Hotelling chided in his concluding comments that mathematical complexities increase with a non-zero elasticity of demand.[3] Smithies undaunted let

$$x_F = b + a(p + tu) \qquad a < 0$$

be his demand function. All variables have been defined above. Let u_k be the point where firm 1's market ends. Then his total demand will be

$$y_1 = D\left[\int_0^{u_1} [a(p_1 + t(u_1 - u)) + b]du + \int_{u_1}^{u_k} [a + t(u - u_1)) + b]du\right] \tag{15}$$

We shall first consider competition in half the market as we did with the constant elasticity of demand case. $u_k = \frac{1}{2}\ell$.

$$y_1 = D\left\{[(a(p_1 + tu_1) + b)u]_0^{u_1} + [-\frac{at}{2}u^2]_0^{u_1} + \left[[a(p_1 - tu_1) + b]u\right]_{u_1}^{\frac{1}{2}\ell} + [\frac{at}{2}u^2]_{u_1}^{\frac{1}{2}\ell}\right\}$$

$$= D\left\{(ap_1 + b)\frac{\ell}{2} - \frac{at\ell}{2}(u_1 - \frac{\ell}{4}) + atu_1^2\right\} \tag{16}$$

and

$$y_2 = D\left\{(ap_2 + b)\frac{\ell}{2} - \frac{at\ell}{2}(3u_2 - \frac{5\ell}{4}) + atu_2^2\right\} \tag{17}$$

At the mid-point of the market, the delivered prices of the two competitors must be equal. That is once again

$$p_1 + t(\tfrac{1}{2}\ell - u_1) = p_2 + t(u_2 - \tfrac{1}{2}\ell)$$

or

$$p_1 = p_2 + t(u_1 + u_2 - \ell) \tag{18}$$

We also, following Smithies, explicitly introduce costs into the model in the simple form

$$C_i = k_i y_i + A_i \qquad i = 1, 2.$$

Now $\Pi_1 = (p_1 - k_1)y_1 - A_1$

$$= (p_1 - k_1)\, D\{(ap_1 + b)\frac{\ell}{2} - \frac{at\ell}{2}(u_1 - \frac{\ell}{4}) + atu_1^2\} - A_1$$

and substituting for p_1 from (18)

$$\Pi_1 = D(p_2 + t(u_1 + u_2 - \ell) - k_1)\, \{\frac{\ell}{2}\, (a(p_2 + t(u_1 + u_2 - \ell)) + b)$$
$$- \frac{at\ell}{2}(u_1 - \frac{\ell}{4}) + atu_1^2\} - A_1 \tag{19}$$

and $$\Pi_2 = D(p_2 - k_2)\, \{\frac{\ell}{2}(ap_2 + b) - \frac{at\ell}{2}(3u_2 - \frac{5\ell}{4}) + atu_2^2\} - A_2 \tag{20}$$

Now setting $\dfrac{d\Pi_1}{du_1} = 0$ gives

$$D\{t[\frac{\ell}{2}(a(p_2 + t(u_1 + u_2 - \ell)) + b) - \frac{at\ell}{2}(u_1 - \frac{\ell}{4}) + atu_1^2]$$
$$+ (p_2 + t(u_1 + u_2 - \ell) - k_1)[\frac{at\ell}{2} - \frac{at\ell}{2} + 2atu_1]\} = 0 \tag{21}$$

Since $D \neq 0$, $t \neq 0$, then this condition becomes

$$\frac{d\Pi_1}{du_1} = 2au_1(p_2 + t(u_1 + u_2 - \ell) - k_1) + \frac{\ell}{2}(a(p_2 + t(u_1 + u_2 - \ell)) + b)$$
$$- \frac{at\ell}{2}(u_1 - \frac{\ell}{4}) + atu_1^2 = 0 \tag{22}$$

Now

$$\frac{d^2\Pi_1}{du_1{}^2} = D\{2at^2u_1 + 2at^2u_1 + 2at(p_2 + t(u_1 + u_2 - \ell) - k_1)\}$$

$$= 2atD\{p_2 + t(3u_1 + u_2 - \ell) - k_1\} \qquad (23)$$

and $\frac{d^2\Pi_1}{du_1^2} < 0$ if $p_2 > k_1$ and $u_1 > \frac{\ell - u_2}{3}$

Recall $a < 0$. A similar result can be determined for 2.

<u>Case 1L (linear) Hotelling 1929</u>: We examine firm 1's profit function when it locates with the market division point at some arbitrary spot u_k. That is integrate equation (15) and substitute y_1 in the profit function.

$$\Pi_1 = (p_1 - k_1)\, D\left\{\frac{p_2 - p_1 + tx_2}{8t}[a(3p_1 + p_2 + tu_2) + 4b] + \frac{u_1}{4}[a(3p_1 - p_2 - tu_2) + 2b + \frac{5atu_1}{2}]\right\} - A_1 \qquad (24)$$

To obtain Hotelling's case, set costs at zero $A_1 = k_1 = 0$, and $D = 1$, and elasticity of demand at zero $a = 0$, $b = 1$. Equation (24) becomes

$$\Pi_1 = p_1\left\{\frac{p_2 - p_1 + tu_2}{8t}(4) + \frac{u_1}{4}(2)\right\}$$

$$= \frac{p_1}{2}\left\{\frac{p_2 - p_1}{t} + (u_2 + u_1)\right\} \qquad (25)$$

and equation (25) is again Hotelling's profit function which we discussed above as Case 1CE.

<u>Case 2L: Two Plant Monopoly</u>: We consider now that the plants will split the market at the mid-point. The monopolist seeks to maximize total profits

$\Pi = \Pi_1 + \Pi_2$. Π_1 and Π_2 are expressed in equations (19) and (20) respectively. We require the solution to the equilibrium conditions $\frac{\partial \Pi}{\partial u_1} = \frac{\partial \Pi}{\partial u_2} = \frac{\partial \Pi}{\partial p_2} = 0$ in order to find the quality of the model when maximum profits are being obtained for the two plant monopoly. Of course the second order conditions must be satisfied in order to assure that we have in fact what is a maximum rather than a minimum.

$$\begin{aligned}
\frac{\partial \Pi}{\partial p_2} &= D\,\{\frac{\ell}{2}\,[a(p_2 + t(u_1 + u_2 - \ell)) + b] - \frac{at\ell}{2}\,(u_1 - \frac{\ell}{4}) \\
&\quad + atu_1{}^2 + \frac{a\ell}{2}(p_1 + t(u_1 + u_2 - \ell) - k_1) + \frac{\ell}{2}(ap_2 + b) \\
&\quad - \frac{at\ell}{2}(3u_2 - \frac{5\ell}{4}) + atu_2{}^2 + \frac{a\ell}{2}(p_2 - k_2) \\
&= D\,\{a\ell(p_2 + t(u_1 + u_2 - \ell)) + b\ell - a\ell k_1 - \frac{at\ell}{2}(3u_2 + u_1 - \frac{3\ell}{2}) \\
&\quad + atu_1{}^2 + atu_2{}^2 + a\ell p_2\} = 0
\end{aligned} \tag{26}$$

We observe that if $k_1 = k_2$, then $u_1 = \frac{\ell}{4}$, $u_2 = \frac{3\ell}{4}$, and $p_1 = p_2 = \frac{1}{2}(k_1 - \frac{t\ell}{8} - \frac{b}{a})$ is a solution of (24) and (26). Rival 2's profits will also be maximized, i.e. $\frac{d\Pi}{du_2} = 0$.

In other words, as Smithies asserted, location of plants at the quartiles yields a profit maximizing solution for the two-plant monopolist. This is also the total transportation cost minimizing solution for the economy and thus is a social optimum in this limited sense.

Case 3L: Government Monopoly: If production and distribution were controlled by a government functioning in the interests of the public, we might suppose that the plants would be located at the quartiles in order to minimize transportation costs of distributing the product and zero profits would be set to obtain. It is of interest to determine how the government monopoly price compares with the two plant monopolist's (private monopolist) price. To investigate this question, we substitute the quartile positions $u_1 = \frac{1}{4}\ell$ and $u_2 = \frac{3}{4}\ell$ in (19) and (20), set the sum $\Pi = \Pi_1 + \Pi_2$ equal to zero and solve for p_2. It is easy to see that $\Pi_1 = \Pi_2$ and both have to be zero, so we can solve for p_2 after setting $\Pi_1 = 0$ alone. (19) becomes

$$\Pi_1 = (p_2 - k_1)\left\{\frac{a\ell p_2}{2} + \frac{\ell b}{2} + \frac{at}{16}\ell^2\right\} - \frac{A_1}{D} = 0$$

$$\Pi_1 = p_2^2 + (-k_1 + \frac{t\ell}{8} + \frac{b}{a})p_2 + (-k_1 - \frac{k_1 t\ell}{8} - \frac{2A}{a\ell D}) = 0 \qquad (27)$$

Note that the coefficient of p_2 in (27) is $-2\hat{p}$ where $\hat{p}$ is the two plant monopolist's f.o.b. price. Thus the government monopoly price p_2 is

$$p_2 = \hat{p} \pm \sqrt{\hat{p}^2 - c} \qquad (28)$$

$$\text{where} \quad c = (-k_1 - \frac{k_1 t\ell}{8} - \frac{2A}{a\ell D})$$

In (28) $(\hat{p}^2 - c)$ will be positive since c is always negative. For $|c| \neq 0$, p_2 will be negative since we assume only the negative sign before the square root in (28) to be operative in order to obtain economically meaningful solutions. Thus for $|c| \neq 0$, $p_2 < \hat{p}$. Similarly for p_1. Hence the government price will always be lower than the private monopoly price.

Case 4L. Equilibrium with Different Rates of Transportation Cost: For this linear demand case we obtain results similar to those for the constant elasticity of demand. The final equilibrium for the two firms displays a higher final price and profit as transportation costs rise. This result is not surprising since at each adjustment in competition profit maximization always obtains at a higher price f.o.b. with higher transportation costs than with lower. The cumulative effect of these results yields the higher final equilibrium price with the higher transportation costs. Smithies noted this [16; p. 497]. We also observe that as transportation costs rise there is a tendency at first for the equilibrium location values to approach the socially desirable values (the quartiles). However as transportation costs continue to rise, there is a tendency for competitors to move from near the quartiles toward the center. These observations are illustrated with the numerical examples reported in Table 3.

Table 3

EFFECTS OF VARYING VALUE OF TRANSPORT COSTS

$a = -0.1$, $b = 1.0$, $k_1 = k_2 = 0$, initial values $u_2 = 0.5100$, $p_2 = 0.500$

Values of Variables in Equilibrium

t	u_1	p_1	Π_1	u_2	p_2	Π_2
0.5	0.4900	0.5000	0.2345	0.5100	0.5000	0.2345
1.0	0.4900	0.5000	0.2315	0.5100	0.5000	0.2315
1.5	0.4900	0.5000	0.2285	0.5100	0.5000	0.2285
2.0	0.4637	1.0582	0.4502	0.5363	1.0582	0.4502
2.5	0.4639	1.1986	0.4950	0.5361	1.1986	0.4950
3.0	0.4565	1.5202	0.5966	0.5435	1.5202	0.5966
3.5	0.4565	1.6902	0.6401	0.5435	1.6902	0.6401
4.0	0.4565	1.8602	0.6789	0.5435	1.8602	0.6789
4.5	0.4570	2.0327	0.7134	0.5430	2.0327	0.7134
5.0	0.4580	2.2091	0.7437	0.5420	2.2091	0.7437

3. COMPETITIVE REACTIONS AND EQUILIBRIUM

The process of competition among two sellers in space has received considerable attention but the contributions still lack a synthesis.[4] Two reasons why this might be are that the problem of analyzing the reactions in a general case are mathematically difficult[5] and secondly that Hotelling and perhaps also Smithies appeared to have laid bare the technical aspects of the problem. The persistence with which economists returned to Hotelling's reaction scheme seems to be evidence that the Hotelling-Smithies competitive reaction model was not simply a special case of Cournot-Bertrand-Stackelberg analysis of duopoly or that if it was, it required subtle analysis to clarify it.

We will examine the approach to equilibrium in the various models under the assumption that each competitor takes the position and price of his rival as fixed. We assume that each competitor treats both his position and price as a policy instrument which can be adjusted to maximize his profits given the price and position of his competitor. We assume zero costs of relocation and price adjustment. We assume shortsightedness on the part of each competitor or that profits are maximized in the present period given the price and position of his rival in only the previous period. Demand and cost functions are identical for each competitor.

Proposition 1: [Hotelling Clustering (stable)]. Given atomistic competition according to the above rules and demand functions with zero elasticity, an equilibrium will establish where competitors 1 and 2 are back-to-back in the center of the market and maintain stable prices and positions only if

a) each competitor agrees not to shift his position and price at any time in such a way that the difference in prices (f.o.b.) exceeds the cost of transportation from 1 to 2. Hotelling [8 ; footnote 8]. We shall refer to this latter condition as the Hotelling Stability Condition (HSC).

b) Price increases by either competitor are not permitted after each competitor is located back-to-back in the center of the market.

We can distinguish two plausible approaches to equilibrium depending on the initial conditions. First there could be a single seller located in the center of the market selling at an arbitrary price. An entrant would locate next door and sell at the same price and each competitor would serve half the market. Any shift in price down is constrained by the HSC and the requirement of profit maximization. Any shift in price upwards is ruled out by condition b).[6]

Neither condition a) nor b) both necessary for the stability of an equilibrium in the center of the market have plausible behavioral foundations, at least under a regime of atomistic competition. They both can be justified in terms of some collusion axiom and it is no doubt for this reason that Hotelling frequently refers to various collusive measures that might arise in reality. In the absence of condition a), one competitor would find that by lowering his price beyond the point where the difference in prices (f.o.b.) equals the cost of transportation from 1 to 2, he could capture the entire market and increase his profits. The other competitor would follow suit and so on until price just equalled average cost and zero profits were being made by both competitors. Hotelling recognized this contingency and introduced condition a) to rule out its occurrence. Smithies [16; p. 496] was also well aware of this possible instability. Chamberlin [1 ; pp. 226-29] made it the chief subject of his critical review of Hotelling. Fellner [5 ; p. 88] overlooked this point.[7]

Rather than assuming one seller is located in the center of the market when a rival enters we can assume otherwise. The entrant, 2, will locate near 1 and on the side closest the center. His price will be below 1's but not low enough to violate the HSC. 2's sales and profits will exceed 1's, causing 1 to leapfrog over 2 and establish a new position on the side of 2 closest the center of the market. 1's price will be slightly below that for 2 but not low enough to violate the HSC. Eventually both sellers will end up back to back in the center of the market selling at the same price to half the market each. Observe that though the final locations are the same for the competitors as in the case when 1 was assumed to be first located in the centre of the market, the final prices will necessarily be different. We reach the important conclusion that the final equilibrium values of the prices depend on the initial values of the price and location of the single first seller.

Proposition 2: Symmetric Equilibrium . Given atomistic competition according to the above rules, and demand functions with non-zero elasticity, an equilibrium will be established where competitors 1 and 2 are separated by a positive distance. The prices (f.o.b.) will be the same in equilibrium and each competitor will be located an equal distance from the center of the market on opposite sides of the center.

This proposition was stated by Hotelling [8 ; p. 484] and by Smithies [16; p. 493-4], though neither demonstrated either numerically or analytically that it was true. Smithies in fact asserted a property of the competitive reaction to equilibrium which we have found to be false. Smithies asserted that "the equilibrium position is independent of the starting points"

[16; p. 493] whereas this result has never obtained in any of our numerical simulation nor is it a property of the Hotelling zero elasticity of demand case analyzed immediately above.

The analytical expressions denoting the mathematical qualities of each step in a path of competitive reaction are obtained for the case of a constant elasticity of demand by solving the equations for (p_1, u_1) and (p_2, u_2) resulting from setting the derivatives of (5) and (7) with respect to p_1, u_1 and p_2, u_2 equal to zero when the market is divided at some point u_k ; and for the case of linear demand by solving the equations for (p_1, u_1) and (p_2, u_2) resulting from setting the derivatives of (24) and its companion for firm 2 with respect to p_1, u_1 and p_2, u_2 equal to zero when the market is divided at some point u_k.

We chose to investigate the process of competition by numerically simulating the process with the aid of the mathematical expressions describing the process. We ran over one hundred simulations some of which were reported in Section 2. In all cases we obtained an equilibrium for each duopolist exhibiting the same price (f.o.b.) for each and a location for each symmetric about the mid point of the market as in Figure 2. Costs were always taken to be zero. The results will be unaffected by this assumption. In Figure 3, one can observe the convergence to equilibrium as both competitors react step-wise to the other's previous position and price.[8]

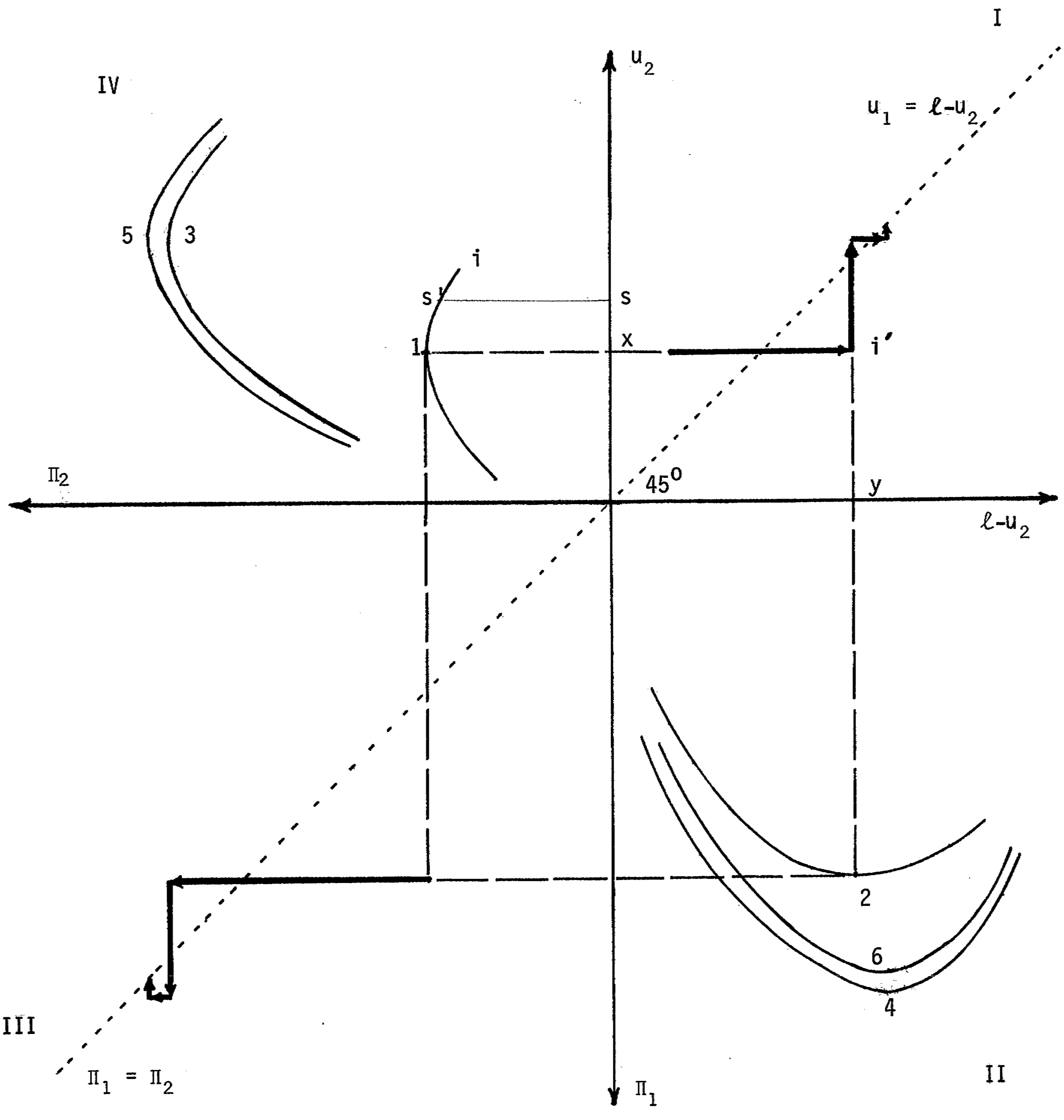

Figure 3.

Each schedule in either of quadrants II or IV indicates the maximum attainable profits for competitor 1 or 2 respectively. For example in quadrant IV schedule i shows the maximum profit attainable by competitor 2 as he varies his position (and price) when competitor 1's position and price are fixed. Competitor 2 really only varies his position because his price will be determined by his competitor's price and the point where the two competitors choose to divide the market. Let competitor 2 test point s. He then lets the frontier between his market and his competitor vary until he finds the point for the frontier of maximum profit. This maximum profit level will be at s'. Similarly all possible locations are tested and schedule i is mapped out. Point x is the one of maximum profit for that step in the competitive reaction. That position corresponds to competitor 1's position y. Thus we get point i' in quadrant I. Quadrant II contains maximum feasible profit schedules for competitor 1. Quadrant III shows the level of profits being made by competitors 1 and 2 at two consecutive reactions to one another's price and location. In equilibrium neither competitor is induced to move by the incentive of potentially higher profits since none can be attained. Profits will be equal as is indicated by their final points being on the 45^o line in quadrant III. Each competitor will be an equal distance from opposite ends of the market as indicated by the final positions being on the 45^o line in quadrant I.

The numerical example related to Figure 3 is presented below in Table 4.

Table 4

SIMULATION OF COMPETITIVE REACTIONS AND EQUILIBRIUM
(Linear Demand Functions)

$\ell = 1.0, \quad b = 1.0, \quad k_1 = k_2 = 0, \quad t = 1.5.$

	Step	u_1	p_1	Π_1	Step	u_2	p_2	Π_2
Initial values						0.6500	0.5000	---
	1	0.4036	0.7350	0.3263	2	0.5545	0.8128	0.3978
	3	0.4492	0.8211	0.4052	4	0.5489	0.8203	0.4065
	5	0.4511	0.8203	0.4055	6	0.5489	0.8203	0.4055

Note that the equilibrium positions for each competitor satisfy the condition which Samuelson[9] required of a satisfactory model, that is:

$$\max_{\{u_1,\, p_1\}} \Pi_1(u_1, p_1, \overline{u}_2, \overline{p}_2) = \max_{\{u_2,\, p_2\}} \Pi_2(\overline{u}_1, \overline{p}_1, u_2, p_2)$$

where u_i is position or location and p_i price (f.o.b.). A number of other important remarks are as follows.

(a) The final prices and positions are dependent on the initial price and position of the first competitor. For the linear demand function case, we can assume that the first competitor would initially locate in the center of the market and sell at a price which maximized his profits. Thus the initial position would always be the same and the initial price would depend on the parameters of the cost and demand functions. However profits are unbounded above for a single seller in the constant elasticity of demand case. Hence the initial price and position are always arbitrary and thus the final equilibrium positions for the duopolists cannot be predicted. The equilibria will vary with the initial position and we observe in location what Samuelson has aptly termed, "the random walk of history" [15; p. 126]. Table 5 contains numerical simulations.

Table 5

SIMULATION OF "RANDOM WALK"
(Linear Demand Functions)

$\ell = 1.0$, $b = 1.0$, $k_1 = k_2 = 0$, $t = 1.5$

Initial Values		Final Equilibrium Values					
u_2	p_2	u_1	p_1	Π_1	u_2	p_2	Π_2
0.65	0.75	0.4738	0.8563	0.4230	0.5262	0.8563	0.4230
0.65	0.50	0.4511	0.8203	0.4055	0.5489	0.8203	0.4055
0.75	0.50	0.4512	0.8157	0.4033	0.5488	0.8157	0.4033
0.80	0.50	0.4680	0.8420	0.4161	0.5320	0.8420	0.4161

(b) The final equilibrium prices and locations display a stability not inherent in the Hotelling zero-elasticity case. So long as there is some elasticity of demand, it will require a larger price decrease from the equilibrium price in order for a competitor to grab the whole market than in the zero elasticity case where an epsilon (small) decrease would suffice. The stability of the final equilibrium increases as the elasticity of demand increases for the constant elasticity of demand case or as the transportation costs increase for the linear demand case. In other words as long as there is some elasticity to demand the final equilibrium is more stable than in the Hotelling zero elasticity case. A non-zero elasticity of demand implies that each duopolist has an additional market he can rely on to buy his commodity that he did not have in the Hotelling case. A duopolist will not only have his hinterland but also his local market between himself and his competitor's local market, region $u_2 - u_1$ in Figure 2. These local markets act as buffers between competitors; they were not present in the Hotelling zero-elasticity of demand case. These local markets are larger in equilibrium for higher elasticities of demand.

We might also remark that the process of competitive reaction displayed no instability characteristic of the zero elasticity case. We required for each step in the zero elasticity of demand case that at each step in the reaction, no competitor lowered his price beyond the point where that new price differed from the competitor's price by more than the cost of transporting the commodity from location 1 to location 2. With a non-zero elasticity of demand (including the linear demand case) we find that at each step in the reaction, no competitor lowers his price and shifts his location in such a way that he grabs the whole market. The local markets act as a buffer even in disequilibrium.

(c) The final equilibrium prices and locations display a stability which Samuelson has defined as stability of the second kind [13; p. 226] or what is also known as stability in the sense of Lyapunov.[9] This stability implies that variables remain close to equilibrium without necessarily converging to it when perturbed slightly. If the competitors react to reach an equilibrium and then one shifts his position or price, a new equilibrium for both will be established after a competitive adjustment process. This adjustment will not be orderly if the price or location perturbation is large enough to cause one competitor to grab the whole of the market. Orderly means that the process of competitive reaction does not have to start again with only one competitor in the market at the outset. A numerical simulation is presented below in Table 6.

Table 6

SIMULATION OF PERTURBED EQUILIBRIUM

$\ell = 1.0$, $b = 1.0$, $k_1 = k_2 = 0$, $t = 1.0$

(Linear Demand Functions)

Initial Values		Final Equilibrium Values					
u_2	p_2	u_1	p_1	Π_1	u_2	p_2	Π_2
0.65	0.75	0.4993	0.7674	0.3798	0.5007	0.7674	0.3798
0.5007	1.50	0.4993	1.5000	0.7369	0.5007	1.5000	0.7369

We can relate the process of competitive reaction with non-zero elasticity of demand to the classical duopoly reaction theory.[10] Hotelling traversed this course in the first part of his article.[11]

$$\Pi_1 = f_1(u_1, p_1, u_2, p_2) \quad \text{and} \quad \Pi_2 = f_2(u_1, p_1, u_2, p_2)$$

as in equations (5) and (7).

At the point separating the two markets u_k , the delivered prices must be equal or

$$p_1 + t(u_k - u_1) = p_2 + t(u_2 - u_k)$$

In order to get a classical reaction function, we must fix u_k at some value, perhaps at the center of the market, $\frac{1}{2}\ell$, and eliminate p_1 from f_1. In competition, a firm selects u_k in order to maximize profits given the price and location of his rival. We now determine

$$\frac{\partial \Pi_1}{\partial u_1} = g_1(u_1, u_2, p_2) = 0 \tag{29}$$

and

$$\frac{\partial \Pi_2}{\partial u_2} = g_2(u_1, u_2, p_2) = 0 \tag{30}$$

where $\dfrac{\partial^2 \Pi_1}{\partial u_1{}^2}$ and $\dfrac{\partial^2 \Pi_2}{\partial u_2{}^2}$ are negative.

Now if p_2 is fixed initially we can solve (29) and (30) for the equilibrium positions u_1 and u_2 as in the classical reaction function theory. However this relationship between competition in a spatial duopoly model and a classical model is purely formal since at each reaction in the spatial model p_2 and u_k change resulting in two new reaction functions being defined. Thus the classical duopoly model has a unique pair of reaction functions whereas the spatial duopoly model with atomistic competition has implicit a family of reaction functions recursively linked to each other.

4. SPACE AS A QUALITY DIMENSION

Hotelling's article on spatial duopoly is well known for its ability to describe the behavior of two parties vying for a nation's votes, and the behavior of sellers of a slightly differentiable commodity vying for a market, as well as for its analysis of the location decision. The analogy between location in geographic space and location in quality space is imperfect especially when quality is the left to right political spectrum. Smithies introduction of some elasticity of demand improves the ability of Hotelling's model to explain the competition of political parties [8 ; p. 485]. Slight changes in price (party policy) no longer result in mass swings of political allegiance. Downs [4], Stokes [18] and Garvey [7] have elaborated on the shortcomings of the spatial model when used in political theory.

The major drawback of the spatial model, aside from the need for a non-zero elasticity of demand, is the fact that political parties vie for a market or constituency <u>per se</u> rather than for profits. The competition between political parties is a classic zero sum game and the rules of political competition reflect this fact; competition between sellers for profits is not a zero sum game. The warfare of political parties for a constituency takes place in Western countries with the weapons of persuasion as opposed to physical coercion and there are few rules of fair play. Either party will strive to force his rival to occupy as small a share of the political market as possible. The Lerner and Singer axioms of competitive behavior are relevant in this context. If we associate degree of preference with price, that is the lower the price the more the preference, we see that Hotelling's zero elasticity of demand formulation leaves his model open to the criticism of its extreme instability. A party can change its policy (lower its price f.o.b.) and capture the whole constituency. Even with a non-zero elasticity of demand this same extreme instability could arise with a sufficiently large change in price -- the 1964 presidential election? But swings of voter support are not in the extreme form of zero-one. The non-zero elasticity of demand case also denotes that people residing at the extreme ends of the political spectrum have a lower preference for the political parties than those near the center. Under normal circumstances, this group should be a very small group, i.e. the price should become infinite near the extremes but only near the extremes. Between the center of the political spectrum and the extremes should be the party faithful who vote for their party almost irrespective of the policies. This group should have the zero elasticity of demand. Between the two groups of party faithful should be the uncommitted or "swing" voters. This group is similar to the one Devletoglou introduced in his analysis of duopoly in space [3]. There should also be a relative price or preference level or set of policies which would induce a faithful party X voter to switch to party Y. The price party Y must pay to win a party X follower should be an increasing function of the distance the party X follower is from the center of the market. We shall assume equal density of political "consumers" along the length of the market. The principal innovation in this alternative spatial model of political party behavior is the assumption the voters distinguish between parties at all times. They have one preference function for Democrats and another for Republicans. The model is illustrated in Figure 4.

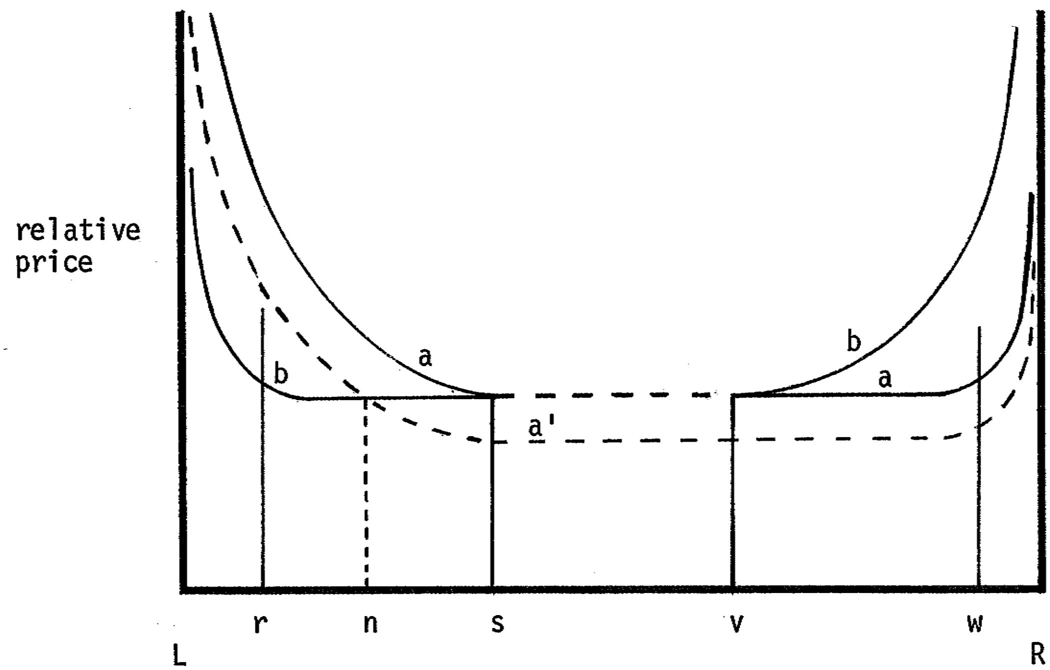

Figure 4.

In Figure 4, the extremists are in both ends of the political spectrum left to right. They cannot be persuaded to vote for any party under any circumstances. Their elasticities of demand tend to infinity. In the next regions rs and vw are the left wing and right wing party faithful. Their elasticity of demand is zero indicated by the schedules being horizontal to the horizontal axis. In the region sv are the uncommitted voters. Points s and v can be shifted by a change in the level of advertising on the part of the left wing and right wing parties respectively.

The major instrument open to a party is changing the relative price of itself vis-a-vis the other party. That is by espousing policies significantly different from its rival, it can markedly affect the relative attractiveness or price of itself in the eyes of the opposition faithful. A policy change shifts the relative price of the party in unknown directions. Policy innovations tend to be risky devices for winning elections. Suggesting a policy change as a shift in the party platform "to the right" or "to the left" is not particularly meaningful. A shift in a party to the right should mean that more voters on the right supported the party after the change than before. In Figure 4, the broken line schedule below the right wing party's schedule indicates a major change in the party platform or policy. In this case a shift to the right results. Voters nv are added to the right wing party's rolls of supporters. We can imagine a new group of uncommitted existing around point n.

The important point to recognize is that the consequences of significant policy departures from the line advocated by the rival are unpredictable. For example in 1964, the Republicans attempted to bolster their voting strength by advocating a very different party platform. Their relative price became higher in the eyes of the Democratic Party faithful and the swing was to the Democrats. In this two party system, the safest strategy is to copy the rival's platform and anticipate gaining on average about the same support the rival gets. Election strategy is usually focussed on wooing the uncommitted voters.

The relative price spreads viewed by "consumers" can vary depending on elasticities of demand. For items like toothpaste or detergent, both schedules may well be inelastic and a small change in price (policy) would cause a mass swing in support. This appears to capture the quality of fads and fashions. In this case advertising does indeed serve the purpose of informing the consumers what are the differences in toothpaste ingredients and such they should be aware of.

FOOTNOTES

1. "If one tries a particular demand functions the mathematical complications will now be considerable, but for the most general problems elasticity must be assumed". Hotelling [8; p. 484]. We switch to analysis by numerical simulations when the mathematical analysis gets excessively complex.

2. "With elastic demand the observations we have made on the solution will still for the most part be qualitatively true; but the tendency for B to establish his business excessively close to A will be less marked. The increment in B's sales to his more remote customers when he moves nearer them may be more than compensation to him for abandoning some of his nearer business to A. In this case B will definitely and apart from extraneous circumstances choose a location at some distance from A. But he will not go as far from A as the public welfare would require. The tempting intermediate market will still have an influence." Hotelling [8; p. 484].

3. Recall footnote 1.

4. For example Losch [10; p.13-15] discussed the issue in the German edition of his opus on location theory but the section was excised from later editions including the well-known Wolgom-Stolper English translation. Palander [12; pp. 370-94] devoted considerable attention to duopoly in space and invoked a variety of axioms of competitive behavior. Strangely though, he fails to acknowledge the existence of Hotelling's semial 1929 paper. Lerner and Singer [9] also analyzed duopoly in space selecting a variety of possible assumptions concerning competitive reactions as the focus of their analysis. Samuelson makes the salient point that the process of competitive reaction is of special interest to the theorist only if the assumptions concerning the competitive behavior of each duopolist are sufficiently general. The Cournot quantity adjustment reaction or Bertrand price adjustment reaction were suggested as satisfactory assumptions. "From this viewpoint, the Hotelling equilibrium point is the same in principle as the Cournot point, the Bertrand point, or the modern-day Nash equilibrium point for a non-constant sum game; namely each of two or more rivals is supposed to pick his x_1 strategy so as to end up with the system's achieving the following simultaneous maximum relationships

$$\max_{\{x_1\}} \Pi_1(x_1, \overline{x}_2) \quad \text{and} \quad \max_{\{x_2\}} \Pi_2(\overline{x}_1, x_2) \tag{F1}$$

where the barred magnitudes are subjectively taken as unalterably given.

Unless we have the special Neumann case where Π_1 and Π_2 add to a constant (or can be made to do so by scale changes) there need not be a nice saddle point solution and all such proposed Nash solutions are open to numerous economic objections." Samuelson [14; p. 1590]. We shall indicate below that our numerical simulations demonstrate that

(F1) is always satisfied for economically meaningful values of the coefficients.

5. There are two recent contributions to the theoretical analysis of competitive reactions which make use of game theoretic concepts and backward induction. They are by Friedman [6] and Cyert and De Groot [2]. They both consider a sequence of n periods over which two competitors react to each other in a non-spatial setting. Each competitor is required to design a sequence of outputs for the periods which maximizes profits over the sequence. "It is therefore natural to search for two sequences of reaction functions . . . with the following property: the sequence of outputs . . . for firm 1 that will be generated by these reaction functions will maximize Q (total profits for firm 1) against (firm 2's reaction functions) and, simultaneously, the sequence outputs for firm 2 that will be generated by these reaction functions will maximize R (total profits for firm 2) against (firm 1's reaction functions). Such reaction functions can be said to be optimal against each other". Cyert and De Groot [2; p. 414] (my inserts in brackets). These sequential models of competitive reaction presuppose that a competitor knows that competition will ensue for at least n periods. It is for this reason that they are considered as non-myopic models as contrasted with the Cournot-Bertrand models where each competitor maximizes profits in alternate periods given the price and output of his rival. In the Cournot-Bertrand framework competitors do not plan ahead to the equilibrium n periods hence. Non-myopic models assume that competitive reaction is a multiperiod process and that competitors realize when a reaction step is transitory and when it is to be a final equilibrium position. This assumption requires ascribing more information to competitors than Cournot-Bertrand envisaged. In the Cournot-Bertrand framework duopoly equilibrium is approached asymptotically in an infinite sequence of adjustments and the length of the horizon in reality is immaterial. Myopia is really a consequence of the horizon being of indeterminate length and should not be considered a deficiency of the Cournot-Bertrand approach or of the analogous approach taken in this section of the paper. In fact Cyert and De Groot [2; p. 416-17] prove that the Cournot myopic strategy is optimal for their n-period model under fairly general assumptions. However they suggest that it is unstable because the horizon is of known and finite length. Cyert and De Groot acknowledge that their analysis should be extended to the infinite horizon case [2; p. 420]. Their approach would then of course be closely allied to the myopic Cournot-Bertrand one.

6. Recall the quotation from Hotelling at the end of Case 1CE in Section 2.

7. Fellner [5; p. 88] "Hotelling obtains what essentially are price reaction functions. These intersect at a level securing profits for both firms. The difference between Hotelling's problem and Bertrand's is that in Hotelling's problem an infinitesimal price reduction does not give the undercutting firm the whole market but results merely in a slight increase of his market." Fellner was no doubt referring to Hotelling's Figure 2 and the related discussion. However Figure 2 is drawn under the assumption that the positions of the two competitors are both fixed and at a positive

distance from each other. The form of competition implicit in this part of Hotelling is one where the costs of relocation are high enough to preclude location adjustment. Hotelling introduced this case to illustrate how orthodox Cournot-Bertrand duopoly theory was related to his model. His point was: "If the purveyor of an article gradually increases his price while his rivals keep their fixed, the diminution in volume of his sales will in general take place continuously rather than in the abrupt way which has facitly been assumed." Hotelling [8; p. 467]. The point which Chamberlin rightly emphasized is that in the situation where location was a variable as well as price (f.o.b.) (we might call it Hotelling (1929) II) then the discrete diminution in volume of sales in response to a competitor's price reduction is the rule.

8. Some license was taken in drawing the schedules in Figure 3 in order to make the process clear. A diagram based on the actual numerical simulation reported below would have the final two profit points in each of quadrants 2 and 4 almost coincident with one another.

9. See Negishi [11; p. 461].

10. See Fellner [5; p. 71-91] for an exposition of classical reaction function theory and Cyert and De Groot [2] and Friedman [6] for recent extensions.

11. Recall footnote 7.

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