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LOSCH'S THEOREM ON HEXAGONAL MARKET AREAS

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Corrections to

"LOSCH'S THEOREM ON HEXAGONAL MARKET AREAS"

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Page 1, line 1, comma after: classic work.

Page 3, first complete paragraph, line 4: equilibrium not equikibrium.

Page 56, vertical axes of Figure 1a and 1b should be labelled: p, AC

also labels to Figures 1a and 1b should read $(p, U)_H$ =

and $(p,U)_C$ =

Page 10 Figure 2. Schedules $Y_C = 0$ and $Y_H = 0$

should display negative slopes for all p and U considered in the diagram. The schedules should not "bend up" at the right hand ends.

Page 12 line 2: or $\frac{dp}{dU} < 0$ not $\frac{dp}{dU} > 0$.

line 3: For Y_C the same not Y_t

Page 14 Section 3, 7th line: continually declining, not declined.

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LÖSCH'S THEOREM ON HEXAGONAL MARKET AREAS

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In his classic work. The Economics of Location, August Lösch claimed to have proved that the free entry of firms to a market in two dimensional geographic space characterized by uniform quality throughout its area would result in space-filling, hexagon-shaped market areas. Isard [2] demonstrated that his general location principle implied Losch's result.[1] Isard's treatment relied on two fundamental assumptions: 1) "Competition compresses irregular shapes into regular ones" and 2) "Of the various regular geometric shapes, only the equilateral triangle, square and regular hexagon can exhaust any given domain, as required by Lösch" [2; p. 14]. Both these assumptions should be treated as propositions either proveable or not. Mills and Lav [4] proceeded to set up a model and to examine this second proposition in detail.[2] They contended: "Lösch's mistake (repeated by Valavanis and by Kuenne)[3] is to *assume* that a necessary condition for industry equilibrium with profit maximization and free entry is that market areas be space-filling. This ought instead to be presented as a *theorem* and as such, it turns out to be false". [4; p. 278].

I will demonstrate that Mills and Lav did not faithfully invoke Lösch's assumptions explicitly set out in his *opus* and that Lösch's theorem is in fact true. Mills and Lav's analysis will be shown to suffer a significant technical error. Space-filling hexagonally-shaped market areas will obtain

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* I am indebted to Philip Hartwick for research assistance. Financial support was kindly afforded me by the Institute for Economic Research, and the Interim Research Committee, both of Queen's University.

in equilibrium with free entry. In addition I will show that if individual demand functions for the good are of the constant elasticity form rather than of the linear form assumed by Lösch and employed by Mills and Lav, then in general, with decreasing costs a single firm will monopolize the whole bounded market and no pattern will obtain. With a U-shaped average cost curve other market equilibria will result.

1. The Model

1) Each firm has decreasing costs (average costs) in production.[4]

$$C = A + k\,x$$

where C is total cost, x output and $A, k > 0$.

2) There is uniformity in transportation costs per unit distance in any direction,

$$T = tu$$

where T is total transportation cost, t is cost per unit distance for a unit of the commodity, and u is distance.

3) Potential customers are spread with a uniform density of D per square mile over a bounded plain of uniform quality.

4) Each customer has the same demand curve linear in delivered price.

$$x_c = a - b(p + tu)$$

where x_c is demand per customer and p is the f.o.b. price. $a,b > 0$.

We now solve for the total sales of a single firm for different various regular shaped market areas. We follow Mills and Lav.

a) For an regular s-sided polygon,

$$x = 2sD \int_0^{\pi/s} \int_0^{U/\cos\theta} [a - b(p + tu)]u\,du\,d\theta \qquad (1)$$

where U is the distance from the firm to the closest point on the perimeter of the market area.

b) For a circle

$$x = \int_0^{2\pi} \int_0^U [a - b(p + tu)] \, u \, du \, d\theta \qquad (2)$$

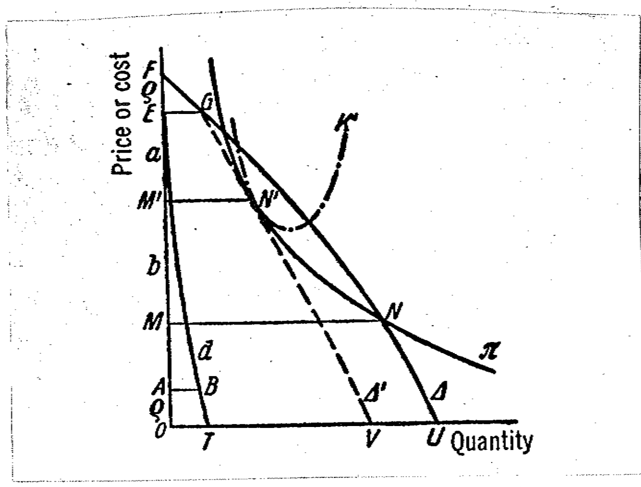
The firm's total profit will be

$$Y = px - A - kx \qquad (3)$$

Lösch determined that Y would be zero for all firms in a spatial equilibrium with free entry. This differs from the Mills-Lav deduction that profits would be non-negative and usually positive. Mills and Lav demonstrated that there could exist situations of spatial equikibrium in which firms could be making positive profits with circular market areas and not with hexagonal market areas and hence an equilibrium would obtain with all firms displaying circular market area.[5] However, if we derive a zero profit condition as a necessary consequence of Lösch's assumptions, we will find that only hexagonal market areas will obtain in equilibrium. We will examine the precise nature of an equilibrium as defined by Losch and employed by Mills and Lav below. The following quotations excerpted from the beginning of Lösch's chapter dealing with the hexagon conjecture makes clear why zero profits are an equilibrium condition in Lösch's model.

> "The deduction so far (an equilibrium for a firm) would be relevant if economic regions were circular in form. But they are not. Even if our district were full of breweries lying so closely together that their sales areas touched, one or another farmer would still be tempted to start a brewery for himself. And he could do so. First, because all the corners between the circles would not yet have been fully turned to account; and second, because the size of the individual brewery could be reduced from MN, in Figure 22, to M'N' without making the plant unprofitable."

"As long as the demand curve intersects the cost curve, surplus profits that attract competitors are possible. These will turn out differentiated products or, which is of special interest in the present context, will choose the location of their establishments in such a way that they are particularly convenient for some of the buyers. As a consequence of this loss of purchasers, the demand curves of the earlier enterprises will shift to the left until they are tangent to the cost curve and all surplus profits disappear. The tendency to the maximization of independent enterprises that underlies the process just described now reaches its limits. Small surplus profits may still remain, however, if an area is larger than necessary for n producers, but not large enough for n+1. If n=1, there is a monopoly which, of course, is restricted by latent competition that may become actual if the monopoly is exploited to the full. Then comes a struggle between the earlier and later firms, one of which must finally succumb since there is not room for both." [3 ;p. 109]



Equilibrium for a Firm with Free Entry

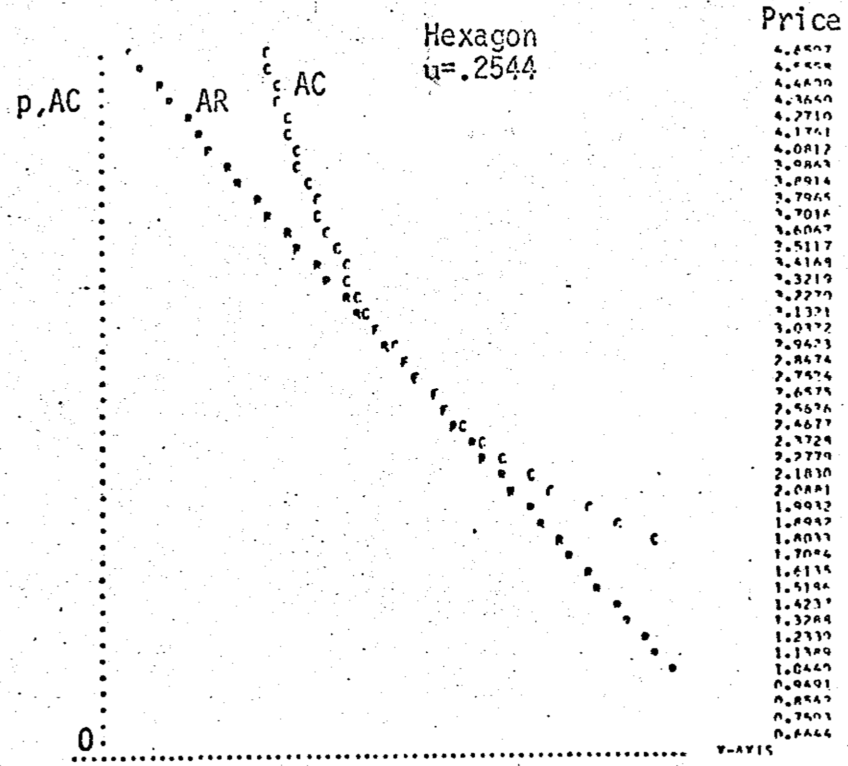
Δ is the market demand or average revenue curve,
π is the average cost curve.

FIGURE 22 (Lösch)

There are three salient features in the quotation which indicate how Lösch's assumptions are significantly different from the Mills-Lav assumptions. 1) Zero profit is a condition of equilibrium for each firm in a market with free entry. These zero profits can obtain from competition because 2) the homogenous market plain does not extend infinitely in all directions and 3) firms are assumed to compete with each other for markets. Mills and Lav assumed that the homogenous plain extended infinitely in space[6] and this assumption implies that each firm can act independently of others, or, in other words, not compete for market shares, and hence maximize profits as if in isolation. The multi-firm equilibrium is defined by Mills and Lav to be "some pattern of market areas that will permit more firms per square mile with all firms making at least zero profits". [4 ; p. 283]. This is obviously a quite different sort of equilibrium from the one which Lösch envisaged, that is one of Darwinian competition in which the zero profit condition signalled the end of further entry by new firms.

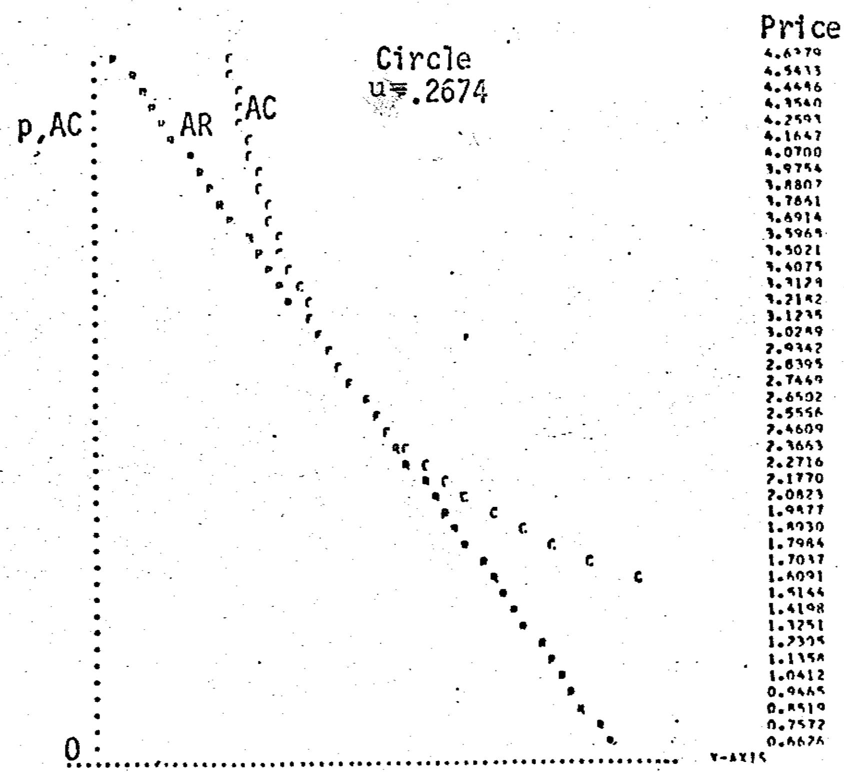
Lösch's Figure 22 reproduced above is compatible with the model outlined above except for two trivial modifications. The average cost function π will asymptotically approach the vertical axis as quantity decreases and a constant k as quantity increases for the above model. Also the demand function Δ will be a straight line for the case of a circle for the above model. Compare Figures 1a) and 1b) with Figure 22.

The process of free entry, profit elimination, and market size alteration should be clearly understood. Consider a group of firms whose circular market areas fill a bounded market area. Fill does not mean interstices will not exist. If any firm is enjoying positive profits, competitors will enter adjacently and sell at a lower price. Eventually we will have a



Equilibrium for a Firm with Hexagonal Market Area $(p,u)_H$ = (2.752, 0.254)

FIGURE 1a)



Equilibrium for a Firm with Circular Market Area $(p,u)_C$ = (2.745, 0.267)

Figure 1b)

group of firms with circular market areas all producing at zero profits. The market demand curve for each firm will be tangent to the average cost curve.[7] Now each firm will find that by circumscribing a hexagon about its circular market area, and expanding output to sell in that hexagon, positive profits can be obtained.[8] Now all firms will change to hexagonal market areas and earn the profits. But entrants will proceed into the market to capture the excess profits and a process of market area packing or compressing will ensue. In the end all firms will be earning zero profits and have hexagonal market areas slightly smaller than those which they previously circumscribed about their circular market areas when zero profits were being earned with those circular market areas.

In the following section it will be shown that an equilibrium with zero profits for a firm with a circular market area always occurs with a radius U_C larger than the parameter U_H satisfying a zero profit equilibrium for a firm with a hexagonal market area. In other words the process of free entry will take place in the manner outlined immediately above. This process is sketched tersely in Lösch.

2. Derivation of Equilibrium Conditions

For an equilibrium for a firm in Lösch, we must have zero profits and average revenue tangent to average cost as in Figure 22. The zero profit condition is obtained by setting Y equal to zero in (3) and substituting for x from equation (1) for a hexagon (s = 6) and from equation (2) for a circle. Integrating, we get

$$Y_H = 12DU^2\left(\frac{a}{2\sqrt{3}} - \frac{bp}{2\sqrt{3}} - 0.2027\ btU\right)(p-k) - A = 0 \qquad (4a)$$

$$Y_C = 2\pi DU^2(\frac{a}{2} - \frac{bp}{2} - \frac{btU}{3})(p-k) - A = 0 \tag{4b}$$

$$p,U > 0 \quad \text{and} \quad \frac{a-bp}{t} > U.$$

The tangency between average costs and average revenue is obtained as follows:

$$\sigma_H^{AR} = \frac{1}{\frac{dx}{dp}} \quad \text{or} \quad R_H = x_H^2 + \frac{dx_H}{dp} A = 0 \tag{5a}$$

where $\sigma_H^{AR} = -\frac{A}{x_H^2}$ and $\frac{dp}{dx_H}$ is the slope of the demand curve.

$$\sigma_C^{AR} = \frac{1}{\frac{dx}{dp}} \quad \text{or} \quad R_C = x_C^2 + \frac{dx_C}{dp} A = 0 \tag{5b}$$

where $\sigma_C^{AR} = \frac{A}{x_C^2}$ and $\frac{dp}{dx_C}$ is the slope of the demand curve.

A Lösch equilibrium for the firm is the (p,U) which simultaneously satisfies 4a) and 5a) for the hexagon case or 4b) and 5b) for the circle case. Equations in 4) are third order or cubics in U and equations in 5) are sixth order in U.

These equations were solved by graphical techniques and an equilibrium for a firm is graphically presented in Figure 1 . For the numerical example in question the equilibrium values for the hexagon are $(p,U)_H = (2.752, 0.254)$ and for the circle are $(p,U)_C = (2.745, 0.267)$.[9] This example indicates that there does in fact exist an equilibrium where $U_H < U_C$. Hence there will be more firms per unit area given hexagonal market areas then with circular ones and in fact given a multi-firm spatial equilibrium in bounded space with circular market areas, free entry will always result in the change to hexagonal market areas provided U_H always is less then U_C in the equilibrium for a firm.

It will now be demonstrated that the U for the hexagon will always be less than the U for the circle and thus that space filling market areas always obtain in Losch's model. The numerical example indicated only that hexagons could obtain rather than circle. The proof will proceed by demonstrating that the graphs of (4) and (5) in (p, U) space are as indicated in Figure 2 below.

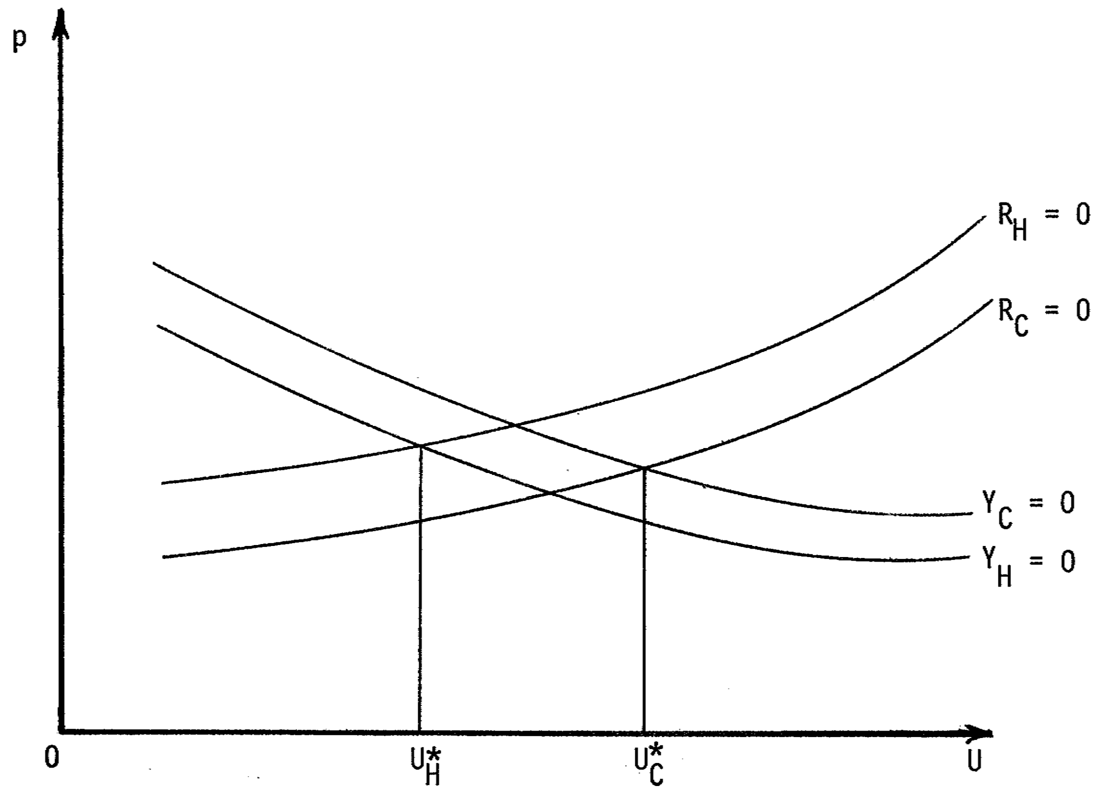


FIGURE 2.

Observe in Figure 2 that as long as R_H lies above R_C and Y_C lies above Y_H and the slopes are as indicated in Figure 2 then if there exists[10] (p,U) satisfying (4) and (5) for the hexagon and circle cases, $U_H^* < U_C^*$. We will now show:-

ia) schedule $Y_C = 0$ lies above $Y_H = 0$ for $p,U > 0$ and $\frac{a-bp}{t} > U$.

$$Y_C = (p_C-k)x_C - A = 0 \tag{6}$$

$$Y_H = (p_H-k)x_H - A = 0 \tag{7}$$

Note first that $\frac{\partial x}{\partial U} > 0$ and $\frac{\partial x}{\partial p} < 0$.

Also note that $x_H > x_C$ for $(p,U)_H = (p,U)_C$.

Thus if $p_C = p_H$, in order for (6) and (7) to be both satisfied $U_H < U_C$ for any $p > 0$ and where $\frac{a-bp}{t} > U$. Thus the $Y_C = 0$ schedule lies above the $Y_H = 0$ schedule for all feasible values of p and U.

ib) schedule $R_H = 0$ lies above schedule $R_c = 0$ for $p,U > 0$ and $\frac{a-bp}{t} > U$.

$$R_C = x_C^2 - \frac{12DU_C^2b}{2\sqrt{3}} A = 0 \tag{8}$$

$$R_H = x_H^2 - \frac{12DU_H^2b}{2\sqrt{3}} A = 0 \tag{9}$$

Recall the comments in ia) concerning x, p, and U. For $U_C = U_H$ in (8) and (9) then $p_H > p_C$ in order for (8) and (9) to be satisfied. Thus the $R_H = 0$ schedule lies above the $R_C = 0$ schedule for all feasible values of p and U.

ii) Y_H and Y_C have negative slopes for $p > 0$ and $U > 0$ subject to $\frac{a-bp}{t} < U$ or $\frac{dp}{dU} > 0$.

We shall demonstrate the result for Y_H. For Y_t the same procedure yields the desired result. We take the total derivative of Y_H with respect to p and U.

$$Y_H = (p-k)x - A = 0$$

$$= 12DU^2(\frac{a}{2\sqrt{3}} - \frac{bp}{2\sqrt{3}} - .2027\ btU)(p-k) - A = 0$$

$$dY_H = \{\frac{2x - 12DU^2(.2027\ btU)}{U}\}dU + \{x - (p-k)\frac{12DU^2b}{2\sqrt{3}}\}dp = 0$$

The coefficients in curly brackets of both dU and dp are positive for all U and p positive. Hence -

$$\frac{dp}{dU} = - \frac{\{\frac{2x - 12DU^2(.2027btU)}{U}\}}{\{x - (p-k)\frac{12DU^2b}{2\sqrt{3}}\}} < 0.$$

iii) R_H and R_C have positive slopes for $p > 0$ and $U > 0$ subject to $\frac{a-bp}{t} > U$ or $\frac{dp}{dU} > 0$

Again we shall demonstrate the result for R_H. For R_C the same procedure yields the desired result.

$$R_H = x^2 - \frac{12DU^2b}{2\sqrt{3}}A = 0$$

We take the total derivative of R_H.

$$dR_H = 2x\{(\frac{2x - .2027btU\ 12DU^2}{U})dU - (\frac{12DU^2b}{2\sqrt{3}})dp - \frac{2}{U}\left(\frac{12DU^2bA}{2\sqrt{3}}\right)dU = 0 \quad (10)$$

In the neighborhood of the zero profit condition

$$(p-k)x = A$$

Substitute this value of A in (10),

$$dR_H = 2x\left\{\left(\frac{2x - .2027btU\ 12DU^2 - \frac{12DU^2b(p-k)}{2\sqrt{3}}}{U}\right)dU + -\left(\frac{12DU^2b}{2\sqrt{3}}\right)dp\right\} = 0$$

The term in the first bracket is positive and in the second is negative. Hence

$$\frac{dp}{dU} = + \frac{\left\{\frac{2x - .2027btU(12DU^2) - \frac{b(p-k)12DU^2}{2\sqrt{3}}}{U}\right\}}{\left\{\frac{12DU^2b}{2\sqrt{3}}\right\}} > 0$$

It is clear that the slope remains positive for all feasible p and $U > 0$ outside the neighborhood $(p-k)x = A$. To verify this consider R_H equal to zero and change p. The change in U required to maintain the value equal to zero indicates the positive slope indicated in Figure 2.

Figure 2 has now been shown to accurately depict a Lösch equilibrium for the firm with either a hexagonal market area or a circular one. What we can now say is that $U_H < U_C$ for a Lösch equilibrium and that free entry will result in hexagonal market areas obtaining for all firms when the total market area for all firms together is bounded.

We noted that for any (p^*, U^*) yielding a Losch equilibrium for the firm the individual demands in the market area must all be non-negative or

$$U^* < \frac{a-bp^*}{t}$$

Now for $U = 0$, the maximum value of p which is consistent with a non-negative individual demand is $\frac{a}{b}$, a positive number. For an equilibrium to exist, average cost is $\frac{A}{x} + k$ and average cost approaches k asymptotically from above as x increases. Thus if $\frac{a}{b} < k$, or marginal costs are prohibitively high, there will never be a Lösch equilibrium, i.e. zero profits for the firm when the average revenue is tangent to the average cost curve.

The parameters of the demand function place clearly defined limits on the values of p and U for an equilibrium. We will never observe one firm expanding its market area indefinitely in space. Since $\frac{\partial AC}{\partial x} = -\frac{A}{x^2} < 0$, the average cost curve will always be convex to the origin in the average cost, total output plane. The demand curve will always be a straight line. Hence a unique tangency point will obtain if an equilibrium exists when the problem is viewed in the plane with dollars per unit output on the vertical axis and total output on the horizontal axis. We will never observe an equilibrium with positive profits or where the demand curve lies above the average cost curve for some range of output since competitors will enter and force the market area frontier, U*, to decrease.[11]

3. Alternative Specifications and Different Equilibria

The analysis of the existence of equilibrium revealed the importance of the magnitudes of certain coefficients of the individual demand function and of the cost function in the solution to the model. The linear demand function is specified by Lösch [3 ; p. 111] and employed by Mills and Lav. On the cost side, however, Lösch had in mind a long run U-shaped average cost curve [3 ; p. 108] while we have worked, following Mills and Lav, with a continually declined average cost curve. The specification of the demand curve is the crucial element in the model.

If we specify a constant elasticity of demand function for each individual in the plane and continue to assume continually declining average cost curve, we will find exactly how dependent the above analysis was on the form of the demand curve. Let the demand function now be:

$$q_F = a(p + tU)^{\lambda} \qquad \lambda < 0 \tag{11}$$

where q_F is quantity demanded per individual or family per unit time, U is distance from the plant or firm, t is transportation cost per unit distance for a unit of the good, p is price at the plant f.o.b., λ is the elasticity of demand and a is a parameter. The demand function in (11) has the property that no matter how far an individual lives from the plant, some positive quantity of the good will be bought. This differs from the linear demand function case where quantity demanded declines to zero (and then negative but we assume zero demanded for values which are mathematically negative) at some distance from the plant.

In the same way in which we determined the market demand for the linear case, i.e. equations (1), (2), (4a), (4b), we can determine the market demand for the constant elasticity case. Numerical investigations reveal that in general the demand curve is tangent to the average cost curve only asymptotically as $x \to 0$ and $p \to \infty$. Elsewhere the demand curve either intersects the average cost curve once as in Figure 3 below, or does not.

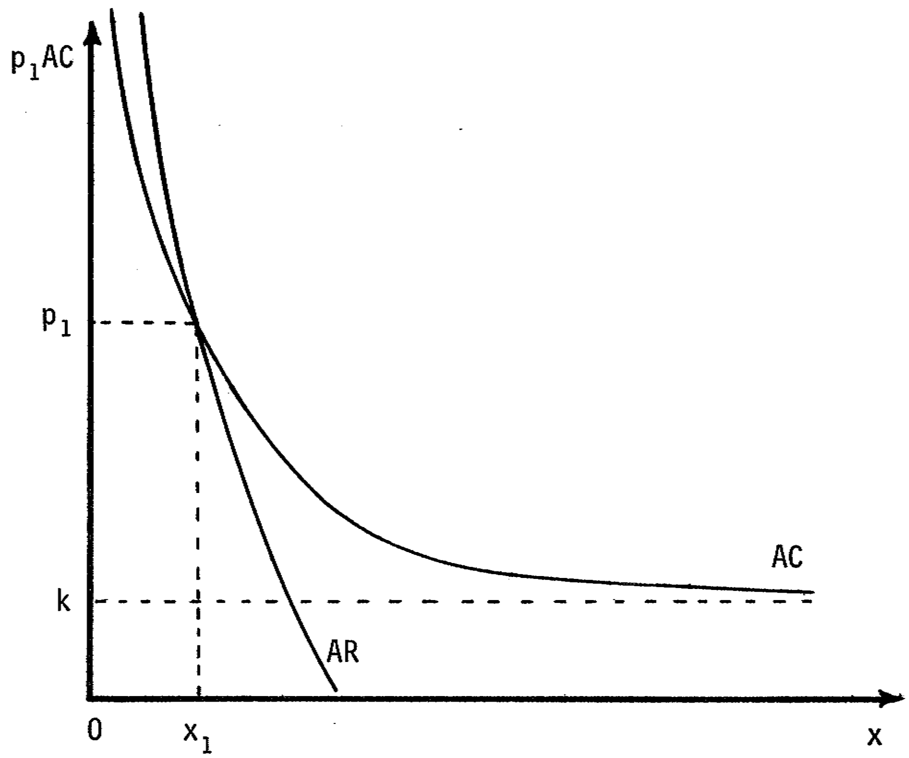


FIGURE 3.

What is implied by an equilibrium like that in Figure 3 is that a firm could set prices "high" and output "low" and make positive profits. In fact profits could be made infinitely high if price was set infinitely high. However a competitor could enter next to the first producer and set a lower price and gain the whole of the first producer's market. Price would at first be driven down to p_1 where zero profits accrue to a producer. However free entry would lead to a situation where a competitor located nearby, selling a larger volume of output in a larger market area at a lower price f.o.b. The process of entry, expansion of output and market area would continue until a single firm was supplying the whole of the bounded market area at a price f.o.b. slightly above marginal cost k. The threat of entry by a competitor would keep profits at zero level or equilibrium would obtain where the average cost curve intersected the demand curve. The question of the form of the market areas becomes inconsequential since a single firm always supplies the whole market.

The one firm equilibrium situation is not novel to economics but is a well-known consequence of continually declining average costs. However it has received little attention in location economics. The equilibrium is not actually a special case of Lösch's analysis since he explicitly limited himself to cases where the average cost curve was U-shaped.

Consider now that we have a U-shaped average cost curve facing each firm. Let the demand function be such that there is no point where the demand curve is tangent to the average cost curve in contrast to the potential Lösch equilibrium. We found that the constant elasticity of demand function had this property with the average cost function $\frac{A}{X} + k$. Now our cost function is assumed to be U-shaped however. With free entry, an equilibrium for a firm will occur when profits are zero and average costs are at a minimum for the firm as in Figure 4.

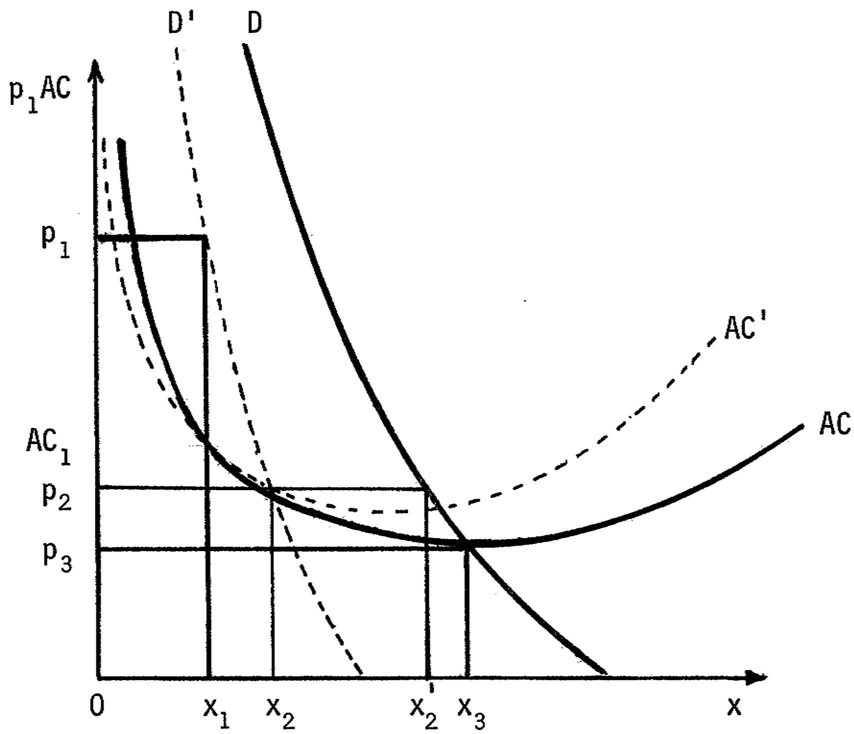


FIGURE 4.

The reason why (p_3, x_3) is the final equilibrium point can be seen in the following way. Let a producer be supplying x_1 at price p_1 and be making a profit of $(AC_1)p_1$ per unit output. With free entry a competitor will enter adjacently and sell at a lower price. Eventually a situation of zero profits will result, i.e. position (p_2, x_2). But competitors can enter and erect larger plants and sell at price p_2 and supply x_2' at a profit. Free entry and competition will ultimately drive the price to p_3 at which point x_3 will be supplied at zero profit.

What about market areas? Let us assume that in the absence of specific competitive forces which constrain the shape of market areas, circular market areas will always be chosen by a firm. Let us also assume that the equilibrium (p_3, x_3) for a firm in competition always occurs with a smaller "radius" U_H for a hexagonal market area than radius U_C for a circular market area when the same parameters in the demand and cost functions are exhibited for the circular and hexagonal cases under consideration.[12] Consider the free entry of many firms to a bounded total market area. Free entry and the forces of competition will drive all firms to a zero profit -- minimum average cost point with circular market areas. The total market will be covered by producers with the exception of the interstices. A single producer could circumscribe a hexagon about his market area and attempt to expand production to meet demand in the hexagon. Losses would result. Alternatively a producer could attempt to reduce the size of his circular market area to gain excess profits but free entry and competition would prevent this occurance. We assume that a producer is not motivated to <u>inscribe</u> a hexagon in his circular market. If he did in fact switch to supplying a small hexagonal market area within his zero profit circular one, he could temporarily gain excess profits. But these would be eliminated by free entry and competition. If this phenomenon of switching to smaller hexagons did occur, a final equilibrium would obtain with all firms producing at the minimum point of their average cost curve and earning zero profits. But we assume this switch not to occur. Consider the reverse adjustment.

If all firms were earning zero profits and producing in hexagonal market areas at the point of minimum average cost and the total market was being supplied, then each firm would observe that by switching to circular market areas, where the circle was contained in the hexagon defining the equilibrium market area, positive profits could be earned. Free entry and competition would force firms to expand their market areas and lower prices until each firm was earning zero profits and producing at minimum average cost with circular market areas. Thus by modifying the functional form of Lösch's demand function[13], circular market areas become the multi-firm equilibrium situation. We have not found it necessary to remove the boundedness assumption to have this situation obtain.

4. Concluding Remarks

We have demonstrated that Lösch's conjecture that free entry of firms to a bounded market area would result in hexagonal market areas for each firm was indeed a true theorem under his specific assumptions. After relaxing the assumption of individual demand functions being linear, the possibility of circular market areas obtaining in equilibrium with free entry of firms to a bounded market area has been analyzed and found plausible. With continuously declining average costs, an equilibrium has been found to obtain by means of numerical situations where one firm supplies the whole bounded market. An effort has been made to define the process of free entry at some length.

Given the multiplicity of possible equilibria outlined above one is hard pressed to make strong statements about the welfare implications of various equilibria. Lösch had some thoughts on these considerations [3; pp. 112-114] and Mills and Lav examined a Lösch equilibrium with free entry and many competitors in order to make some strong statements concerning welfare. These latter authors asked "whether firms of the size and shape that result from profit maximization and free entry use a socially efficient combination of production and transportation costs", [4; p. 286]. This criterion of the welfare quality of spatial equilibrium is similar to the one Hotelling [1] utilized in his celebrated analysis of duopoly and location. The salient issue is: in what sense is the process of free entry and competition socially efficient. A quite different efficiency criterion was utilized by Isard [2] in his analysis of hexagonal market areas. He sought to determine a pattern of market areas which would include all individuals in the plane (i.e. be space-filling) and minimize transportation costs. He concluded: "Since a regular hexagon is more efficient (requires less distance inputs) in serving any area of given size than any other six-sided polygon, the regular hexagonal pattern corresponds to a minimum transport cost, or maximum surplus arrangement". [2; p. 16]. The Isard criterion seems more suitable for a central planning bureau since it has explicitly that element of a utility function which says a priori that all potential customers must be served. Mills and Lav asked at the outset of their analysis: would free entry and competition even result in all customers being served?

There does not appear to be a welfare criterion which permits us to analyze the three types of equilibrium with free entry developed in Sections 2 and 3. Free entry to a bounded market consistently results in equilibria with zero profits. However in some cases potential customers who live in interstices will not be served. In other cases one producer will dominate the whole market area. Mills and Lav's analysis of welfare is relevant to the Lösch equilibrium but fails to deal with problems of unserved potential customers.

FOOTNOTES

1. Isard's general location principle is as follows: "for an optimum space economy which is continuous transportationwise and continuous to some extent with respect to market and supply areas and in which transport rates are proportional to weight and distance, the marginal rate of substitution for any pair of distance inputs must equal the reciprocal of the corresponding transport rates, social surplus (however defined) less transport costs on all other distance inputs held constant". [2 ; p. 11]. "A distance input has been defined as the movement of a unit weight of a particular good over a unit distance". [2; p. 11].

2. They proved the first one in two lines. Isard did not spell this result out. "Irregular shapes (of market areas) are precluded by their inefficient transportation patterns. Any irregular market area can be replaced by a regular one of smaller area but of the same profitability, thus permitting more firms per square mile." [4 ; p. 279].

3. As well as by Isard although Mills and Lav did not remark on his contribution.

4. Lösch assumed average cost curves were U-shaped. [3; p. 108, footnote 4]. However as long as individual demand curves are linear, the fact that the average cost curve has a minimum is not relevant. However in Section 3, where alternative specifications of demand are considered, the U-shape is crucial.

5. Mills and Lav's analysis contains a significant error. They consider the possibility of the equilibrium being such that the circular market area is profitable but the hexagonal market area results in negative profits. "The basic notion in this demonstration is simply that, if the circle is the most profitable market area (for all constellations of demand and cost parameters (my insertion)), then it must be possible for the industry to be sufficiently unprofitable for the most profitable market area to be the only one in which firms can break even (or earn some positive profits? (my insertion))." [4; p. 283]. We will see below that every break even (zero profit) circular market area can be circumscribed by a hexagon with the same U which when supplied by the firm will result in positive profits. Mills and Lav's demonstration should read as follows: Determine a group of coefficients for demand and cost functions such that profit, $\pi_c(p_c, U_c)$ is maximized for a circular market area and $\pi_c = 0$. Then the p_c U_c which maximize profit $\pi_H(p_H, U_H)$ for the hexagon market area are such that π_H is always negative. Lösch's theorem proven in Section 2 is: the U_H satisfying $\pi_H(p, U) = 0$ and the linear demand curve tangent to the average cost curve is always less than the U_C satisfying $\pi_C(p, U) = 0$ and the linear demand curve tangent to the average cost curve for the same groups of coefficients for demand and cost, and free entry to a bounded market results in the zero profit condition and in the display of the above tangency.

   Mills and Lav's demonstration contradicts the Lösch theorem proven below in Section 2. But Mills and Lav's analysis underlying their Figure 1 explicitly contains the condition $a-bk > 0$ which is too weak. The conditions depicted in their Figure 1 are incorrect. We require the condition $a-b(p + tu) \geq 0$ for all $0 < u \leq U^*$, the perimeter of a firm's market area. We know that p(fob) must be greater than or equal to marginal cost k and that $tu > 0$ for $u > 0$. Hence for a market area of non-zero area, i.e. $U^* > 0$, $a \geq b(p + tU^*) > bk$. This condition cannot be built into Mills and Lav's analysis since it depends on the solution U. Hence Mills and Lav's proof is in error. However the proof in Section 2 indicates that their conjecture, quoted immediately above, is in fact incorrect.

6. "We assume that potential customers ("families") are spread with a uniform density of D per square mile over an infinite plain" [4 ; p. 278].

7. We are glossing over problems of having the market areas of all firms not quite fill the complete bounded market area. That is an additional function of a form or market area is required at various stages. Such problems become trivial if the bounded market area supports a large number of firms, eg. $n > 1000$, in the neighborhood of multi-firm equilibrium with free entry as opposed to a small number, eg. $n < 5$.

8. Twelve sided market areas circumscribed about the circle would also permit an expansion of output which would result in positive profits but the potential profits would be largest with hexagons.

9. The coefficients used for this example were

   $a = 25$, $D = 1.0$, $b = 5.0$, $t = 1.0$, $k = 0.6$, $A = 5.0$.

10. The question of non-existence of solutions is analyzed at the end of Section 2.

11. Recall the qualification in footnote 7.

12. For specific choices of cost and demand functions displaying the asymptotic tangency only as in Figure 3, this assumption could be proven or disproven.

13. Lösch invoked the linear form of the demand only to illustrate the mathematical proof of what he considered to be a special case [3; p. 111]. We are not really changing Lösch's assumptions when we utilize a demand and cost function which are never tangent except in the limit. Lösch simply neglected to consider this possibility.

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