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# DEMAND THEORY AND EMPIRICAL DEMAND CURVES

Dan Usher
Queen's University

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

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The theory of demand as received from Marshall, clarified by Hicks and many others, and reproduced in all our textbooks is not well suited as a guide to econometric work, for the theory is set in a context which hardly ever occurs in practice. The demand curve as described in text books of economic theory is a relation between the quantity purchased of a commodity (for instance loaves of bread) and its money price (dollars per loaf) when money income and all other prices are held constant. The empirical demand curve as estimated in econometric work is a relation between a different set of variables; "price" in the empirical demand curve is relative price, the ratio of the percentage change in the bread price to the average percentage change in all other prices since some designated base year; "quantity" in the empirical demand curve is what might be called relative quantity, the ratio of the percentage change in the quantity of bread consumed to the average percentage change in all other quantities since the base year; and "income" when it appears in econometric work is not money income as suggested by the theory of demand but real income. The difference between the textbook demand curve and the empirical demand curve is easily overlooked because the variables in the two curves are given the same names, and because the variables in the empirical demand curves are chosen in accordance with index number conventions which are not normally looked upon as part of the theory of demand. The argument of this paper is that the gulf between the theory of demand as described in textbooks and the procedure followed in estimating demand curves is wider than need be, that a theory of demand can be formulated about the variables used in practice, that the grounds for supposing demand curves are downward sloping are at least as strong in the new theory as in the old one, and that the new theory is more efficiently connected to empirical evidence.

The textbook demand curve is derived by supposing that a consumer chooses a set of quantities $Q_i$ to maximize utility.

$$U = U\ (Q_1,\ Q_2,\ \ldots\ldots,\ Q_n)\ \ldots\ldots\ldots\ldots\ldots\ldots \tag{1}$$

subject to a budget constraint

$$Y = P_1Q_1 + P_2Q_2 + \ldots\ldots + P_nQ_n\ \ldots\ldots\ldots\ldots \tag{2}$$

where money income, Y, and all prices, $P_j$, are given exogenously. From the first order conditions, one can derive demand functions

$$Q_i = f^i\ (Y,\ P_1,\ P_2\ \ldots\ldots\ldots,\ P_n)\ \ldots\ldots\ldots\ldots \tag{3}$$

The demand curve illustrated on the standard demand and supply diagram is the function

$$Q_i = g^i\ (P_i,\ Y)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{4}$$

connecting $Q_i$ and $P_i$, when $P_i$ varies, all $P_j$ $(j \neq i)$ are held constant, and Y is treated as a parameter that shifts the demand curve up or down.

At first sight, equation 4 seems to be exactly what is required for empirical work. Suppose that we are estimating the demand curve for bread, that the income elasticity of demand and the own-price elasticity of demand are assumed constant, and that all lags and cross-elasticities can be ignored. The demand curve for bread becomes

$$Q = D\ P^{\varepsilon}\ Y^{\eta}\qquad \ldots\ldots\ldots\ldots\ldots\ldots \tag{5}$$

where $\varepsilon$ and $\eta$ are elasticities and D is an unknown constant. The empirical demand curve of equation 5 appears to be a special case of equation 4.

The appearance is deceiving because the similarity in form between the theoretical demand curve (4) and the empirical demand curve (5) hides substantial differences in content and interpretation, differences that came to light when one considers the conventions by which the variables P, Q and Y in equation 5 are measured. In equation 4, the variables $P_i$, $Q_i$ and Y are respectively $ per loaf, number of loaves, and money income. The corresponding variables in equation 5 are complex index numbers. Theory and measurement are made to look alike because differences are relegated to rules of measurement which are not counted as part of the theory, and are discussed, if at all, in the econometrics chapter rather than in the theory chapter of the standard text.

Consider how P, Q, and Y would be measured in estimating a demand curve for bread from time series data for the years 1940 to 1970. Price must surely be relative price, for we would not wish to say that a rise in price causes a reduction in quantity consumed, unless the rise in the bread price were greater than the rise in the general price level. Thus P in equation 5 is not the bread price itself but the ratio of the bread price measured as a multiple of its value in the base year to the value of a price index. For convenience, we choose a Laspeyres index. The value of P in equation 5 becomes

$$P \equiv \frac{P_i^B / P_i^A}{\sum_{j=1}^{n} V_j^A \; P_j^B / P_j^A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

where i refers to bread, $P_j^A$ is the price of the j commodity in the base year (1940), $P_j^B$ is the price of the j commodity in the current year and $V_j^A$ is the proportion of income spent on the j commodity in the base year.

Since D is a free parameter in equation 5, the quantity of bread consumed can be expressed as an index number with a value 1 in the base year i.e.

$$Q = Q_i^B / Q_i^A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Finally if we are to avoid money illusion, income must refer to real income, which will be represented by a Laspeyres quantity index.

$$Y \equiv \sum_{j=1}^{n} V_j^A \frac{Q_j^B}{Q_j^A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

When index number conventions are brought into the open by plugging values of P, Q and Y into equation 5, the equation becomes

$$\frac{Q_i^B}{Q_i^A} = D\left(\frac{P_i^B / P_i^A}{\sum_{j=1}^{n} V_j^A P_j^B / P_j^A}\right)^{\varepsilon} \left(\sum_{j=1}^{n} V_j^A Q_j^B / Q_j^A\right)^{\eta} \quad \ldots\ldots \quad (9)$$

or

$$\left(\frac{Q_i^B / Q_i^A}{\sum_{j=1}^{n} V_j^A Q_j^B / Q_j^A}\right) = D\left(\frac{P_i^B / P_i^A}{\sum_{j=1}^{n} V_j P_j^B / P_j^A}\right)^{\varepsilon} \left(\sum_{j=1}^{n} V_j^A Q_j^B / Q_j^A\right)^{\eta - 1} \quad \ldots \quad (9')$$

We shall refer to the term on the left hand side of equation 9' as "relative quantity" - relative in the sense that what counts in this definition is the extent to which quantity has increased since the base year relative to all other commodities. A comparison of equations 4 and 9' brings out the difference between the theoretical demand curve and the empirical demand curve. The theoretical demand curve connects absolute quantity and money price. The empirical demand curve connects relative quantity and relative price. A negative price elasticity in equation 9 means that if the price of bread increases by more than the average of all prices, then the quantity of bread consumed increases by less than the average of all quantities. The quantity of bread consumed may well have increased for the price rise may have taken place over a period in which the purchaser has become better off; a negative price elasticity implies no more than that there would be a substitution away from bread if its relative price had risen. An income elasticity of 1 means that a rise in income has no effect on the relation between relative quantity and relative price.

To assume that the empirical demand curve has the form of equations 5 or 9, is to suppose that the quantity demanded $Q_i^B / Q_i^A$ is the product of three expressions that may be interpreted as substitution effect, income effect and scale effect. The substitution effect, S,

$$S = \left( \frac{P_i^B \;/\; P_i^A}{\sum_{j=1}^{n} V_j^A \;\; P_j^B / P_j^A} \right)^{\varepsilon} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (10)$$

represents the impact on quantity of the change in relative price since the base period. The scale effect, C,

$$C = \sum_{j=1}^{n} V_j^A \;\; Q_j^B \;/\; Q_j^A \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (11)$$

indicates the extent to which the quantity of bread would increase if it increased as much as the average of all other goods since the base period. The income effect, I,

$$I = \left( \sum_{j=1}^{n} V_j^{\ A} \ Q_j^{\ B} / \ Q_j^{\ A} \right)^{\eta-1} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (12)$$

indicates the extent to which the consumer would substitute bread for other goods as income increased and relative prices remained constant.

That these effects are nicely multiplicative is a consequence of the log-linear form of the demand curve of equation 5, but any change in quantity can be resolved, in one way or another into these three effects as long as cross-elasticities of demand are treated as a part of the substitution effect. It is particularly important to distinguish between the income effect (as defined here) and the scale effect, because the economic consequences of these effects are quite different, and because these effects are lumped together under the name of income effect in the ordinary theory of demand.

## The Context of a More Relevant Theory of Demand

The textbook demand curve is a relation between money price and absolute quantity when money income and all other prices are held constant. The empirical demand curve is a relation between relative price and relative quantity when, as is the case with economic time series, money income and all prices may vary substantially from one observation to the next. The contexts of these demand curves are illustrated in a two-commodity case in figures 1 and 2. Both figures are about the demand curve for bread, $Q_1$, measured on the horizontal axis. The numeraire good is cheese, $Q_2$, measured on the vertical axis. The dashed lines are indifference curves, the point A represents consumption in the base period, and the point B represents consumption in the current period.

In the textbook theory of demand, the consumer is given an initial stock, M, of cheese, part of which he consumes and part of which he exchanges for bread at the given market price. Initially when the price of bread is $P_1^A$, his budget constraint is $C(P_1^A)$ and he consumes a bundle of goods indicated as A. When the price falls to $P_1^B$, his budget constraint becomes $C(P_1^B)$, and he consumes a bundle of goods indicated as B.

When a demand curve is estimated from time series data, one can be sure that neither money income nor any price remains the same in the current period as it was in the base period. As illustrated in figure 2, there is a shift of the production possibility curve from $T^A$ to $T^B$. What remains constant, what must remain constant if a demand curve is to be estimated at all, is the structure of taste represented by the field of indifference curves and by the condition that the production possibility curve is tangent to an indifference curve at the chosen bundle of goods. The lines S(A) and S(B) are the common tangents of indifference curves and production possibility curves at A and B, and they appear to people in the situations A and B as budget constraints.

Comparing equilibria A and B in figure 2, it is not surprising to find that more bread is consumed at B despite the fact that bread is more expensive, for real income is greater at B than at A. If the theory of demand is to be applied at all to a comparison of equilibria A and B in figure 2, the theory must be interpreted as pertaining to a relation between relative price and relative quantity as these terms are defined in equation 9. All relative prices and relative quantities are set at unity in the base peridd represented by A. The relative price of bread, $P_{1\,(rel)}^B$, in situation B is

$$P_1^B(\text{rel}) = \frac{P_1^B / P_1^A}{V_1 P_1^B/P_1^A + V_2 P_2^B/P_2^A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where $V_1$ and $V_2$ are proportions of expenditure on bread and cheese in the base period. Similarly,

$$P_2^B(\text{rel}) = \frac{P_2^B / P_2^A}{V_1 P_1^B/P_1^A + V_2 P_2^B/P_2^A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

and the relative quantities of bread and cheese are

$$Q_1^B(\text{rel}) = \frac{Q_1^B / Q_1^A}{V_1 Q_1^B/Q_1^A + V_2 Q_2^B/Q_2^A} \quad \ldots\ldots\ldots \quad (15)$$

and

$$Q_2^B(\text{rel}) = \frac{Q_2^B / Q_2^A}{V_1 Q_1^B/Q_1^A + V_2 Q_2^B/Q_2^A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

It turns out that there is a negative relation between relative price and relative quantity for both goods in figure 2.

Relative quantity is illustrated in figure 3 which is constructed out of figure 2 as follows: Project the line from the origin to A until it meets the budget constraint S(B) at C. The points $X_1$, $Y_1$ and $Z_1$ are the distances of A B and C from the vertical axis, and the points $X_2$, $Y_2$ and $Z_2$ are the distances of A B and C from the horizontal axis. The value of $Q_1^B/Q_1^A$ in equation 15 is $Y_1/X_1$ and it is easily shown that the value of $V_1 Q_1^B/Q_1^A + V_2 Q_2^B/Q_2^A$ is OC/OA or $Z_1/X_1$. Therefore the relative quantity of bread, $Q_1^B$ (rel), is simply $Y_1/Z_1$. Similarily the relative quantity of cheese, $Q_2^B$ (rel), is $Y_2/Z_2$. Of necessity the relative quantity of bread is greater than one whenever the relative quantity of cheese is less than one, and vice versa.

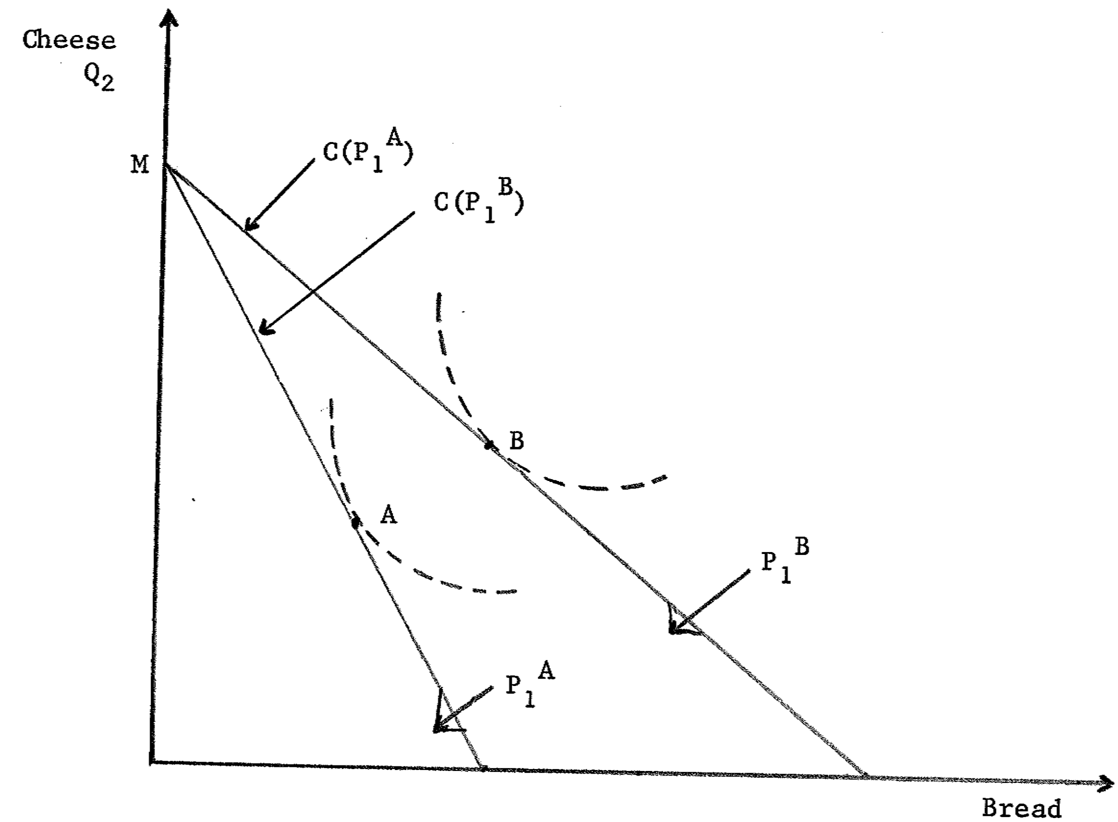


Figure 1.

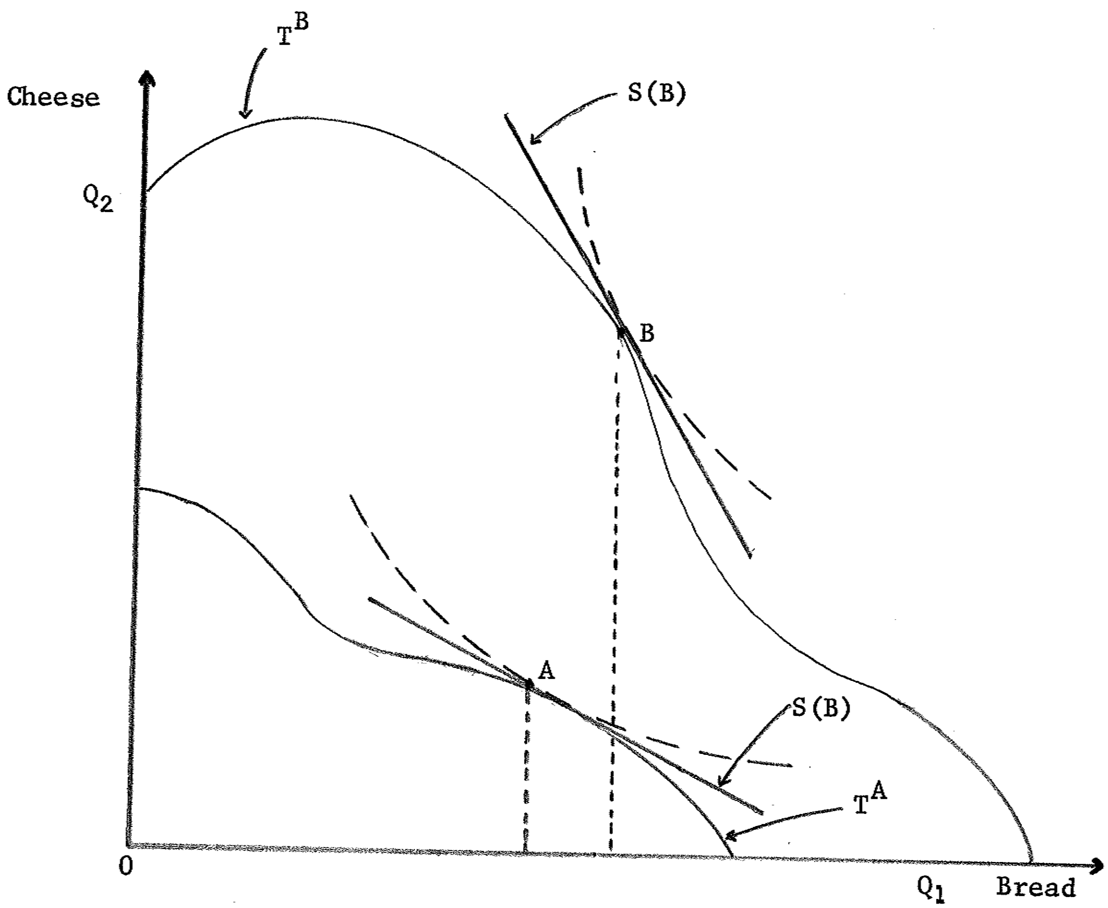


Figure 2.

Relative price is illustrated in figure 4 which is constructed out of figure 2 as follows: Project the budget constraint S(A) to meet the vertical axis at $s_1$ and to meet the horizontal axis at $r_2$. Through A, draw a line parallel to S(B), the budget constraint at B, to meet the vertical axis at $q_1$ and to meet the horizontal axis at $q_2$. Through $r_2$, draw a line parallel to S(B) meeting the vertical axis at $r_1$. The value of $P_1^B/P_1^A$ in equation 13, is the ratio $r_1/s_1$, and the value of the expression $V_1P_1^B/P_1^A + V_2P_2^B/P_2^A$ is $q_1/s_1$ (1), so that the relative price of bread, $P_1^B(\text{rel})$, is $r_1/q_1$. Similarly the relative price of cheese $P_2^B(\text{rel})$ is $q_2/r_2$. Of necessity the relative price of bread is greater than one when the relative price of cheese is less than one, and vice versa.

We say that the demand curve for bread is downward sloping if

$$Q_{1\,(\text{rel})}^{B} \lessgtr 1 \quad \text{whenever} \quad P_{1\,(\text{rel})}^{B} \gtrless 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

In the two-dimensional case, the demand curves for bread and for cheese are either both downward sloping or both upward sloping. These demand curves allow for scale effects, but not for income effects as defined in equation 12.

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(1) Let the price of cheese, $P_2$, be unity in both periods; $P_2^A = P_2^B = 1$. Draw a horizontal line through A meeting the vertical axis at $t_1$. The price of bread, $P_1^A$, at A is $s_1t_1/At_1$; the price of bread, $P_1^B$, at B is $q_1t_1/At_1$; and the value of bread in terms of cheese at A is $s_1t_1$. Therefore $V_1 = s_1t_1/Os_1$, $V_2 = Ot_1/Os_1$ and $V_1P_1^B/P_1^A + V_2P_2^B/P_2^A$ equals

$$\left(\frac{s_1t_1}{Os_1} \times \frac{q_1t_1}{At_1} \div \frac{s_1t_1}{At_1}\right) + \left(\frac{Ot_1}{OS_1} \times \frac{1}{1}\right) = \frac{q_1}{S_1}$$

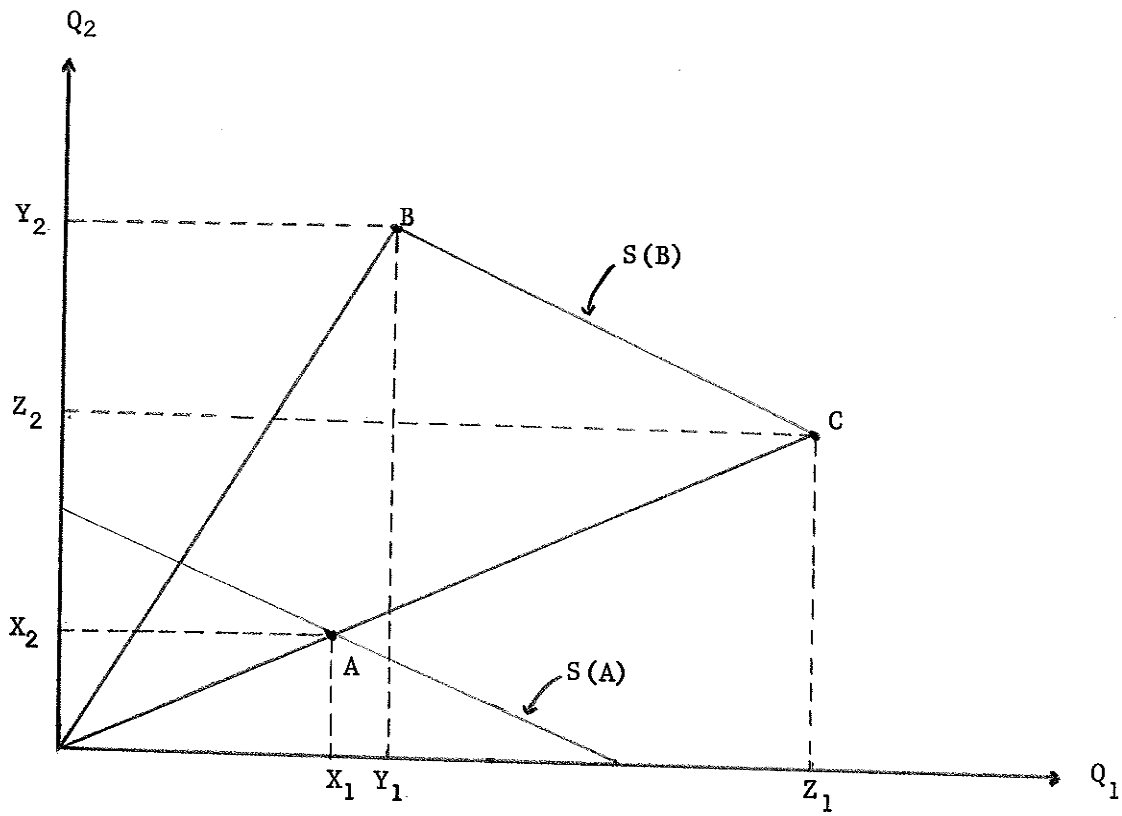


Figure 3.

$Q_1^B(\text{rel}) = \frac{Y_1}{Z_1}$ $Q_2^B(\text{rel}) = \frac{Y_2}{Z_2}$

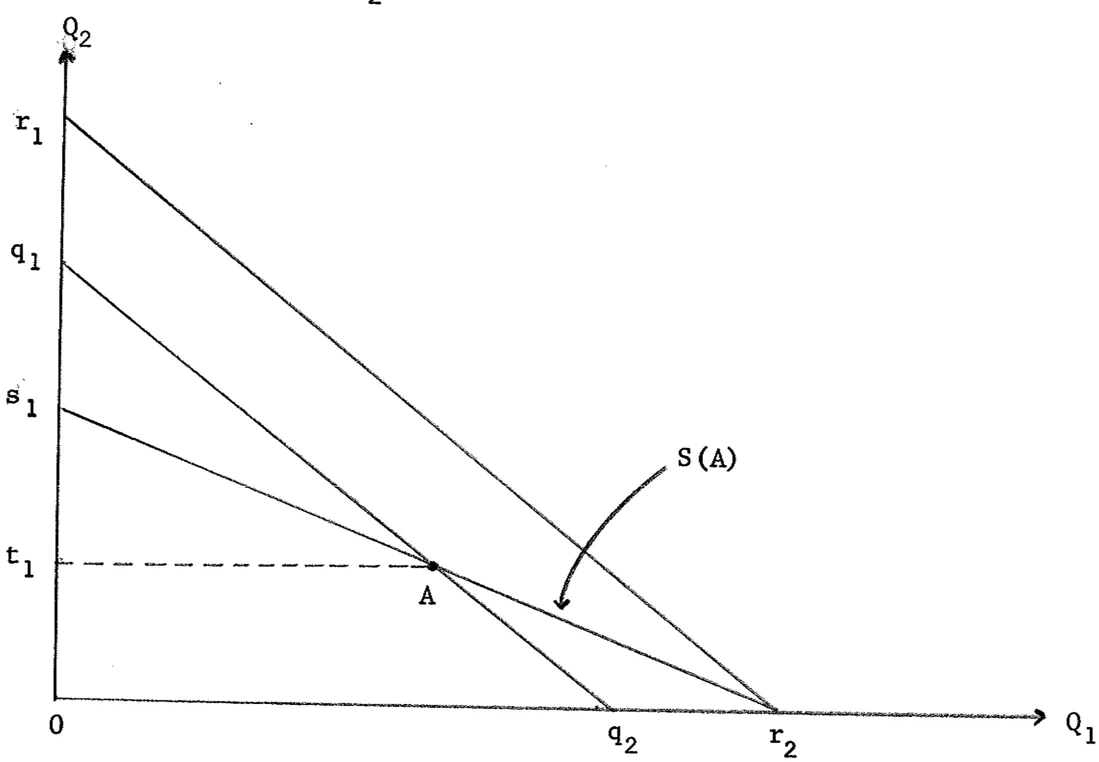


Figure 4.

$P_1^B(\text{rel}) = \frac{r_1}{q_1}$ $P_2^B(\text{rel}) = \frac{q_2}{r_2}$

Demand curves are downward sloping if $Q_1^B(\text{rel}) \lessgtr 1$ whenever $P_1^B \gtrless 1$

Convexity of Indifference Curves and the Slope of the Demand Curve

We have argued that, a) the circumstances in which demand curves and measured are substantially different from those postulated in the theory of demand as set out in textbooks, and that b) the definitions of price and quantity required for estimating a demand curve are not those of the ordinary theory of demand. Figure 2 illustrates how a negative relation might be shown to exist between relative price and relative quantity in a comparison of equilibria that cannot be said to lie on a demand curve of the ordinary kind.

But, as yet, we have not formulated a theory of demand that might be applied to our new variables. By a theory of demand, I mean a demonstration that characteristics of taste as reflected in shapes of indifference curves either guarantee that demand curves must be downward sloping under certain conditions or create a presumption that most demand curves should be downward sloping most of the time. What is required is a generalization of Slutsky's equation or of the inequalities of the theory of revealed preference.

The law of demand that we shall derive may be thought as a generalization of what Hicks calls the first substitution theorem

$$\sum_{i=1}^{n} (P_i - P_i')(Q_i - Q_i') < 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (18)$$

where $P_i$ and $Q_i$ are ordinary prices and quantity of the i commodity, where the primed and unprimed variables refer to two positions on the same indifference curve, and where prices always reflect rates of substitution in use between commodities. The theorem does not state that there is a negative relation between price changes and quantity changes for every and the Paasche index is

$$P = \frac{\sum_{i=1}^{n} P_i^B Q_i^B}{\sum_{i=1}^{n} P_i^B Q_i^B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Equation 19 equivalent to the statement that $L > P$.[1] The proof of the generalized law of demand is based on a familiar procedure in the theory of index numbers.

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[1] Equation 19 states that

$$1 \geqslant \sum_{i=1}^{n} V_i \quad P_i^B(\text{rel}) \; Q_i^B(\text{rel}) =$$

$$= \sum_{i=1}^{n} \left( \frac{P_i^A \; Q_i^A}{\sum_{j=1}^{n} P_j^A \; Q_j^A} \cdot \frac{P_i^B \,/\, P_i^A}{\sum_{j=1}^{n} V_j \; P_j^B/P_j^A} \cdot \frac{Q_i^B \,/\, Q_i^A}{\sum_{j=1}^{n} V_j \; Q_j^B/Q_j^A} \right)$$

$$= \frac{\sum_{i=1}^{n} P_i^B \; Q_i^B}{\sum_{j=1}^{n} P_j^A \; Q_j^A} \cdot \frac{\sum_{k=1}^{n} P_k^A \; Q_k^A}{\sum_{j=1}^{n} Q_j^A \; P_j^B} \cdot \frac{\sum_{k=1}^{n} P_k^A \; Q_k^A}{\sum_{j=1}^{n} P_j^A \; Q_j^B}$$

$$= \frac{\sum_{i=1}^{n} P_i^B \; Q_i^B}{\sum_{j=1}^{n} P_j^B \; Q_j^A} \cdot \frac{\sum_{k=1}^{n} P_k^A \; Q_k^A}{\sum_{j=1}^{n} P_j^A \; Q_j^B} = P \,/\, L$$

commodity and for every pair of equilibria. It states that price and quantity changes tend to be correlated and that a compensated fall in the price of just one commodity must bring about an increase in the demand for that commodity.

The generalization of this law applicable to the demand curve connecting relative quantity and relative price is

$$\sum_{i=1}^{n} V_i^A \; P_i^B(\text{rel}) \; Q_i^B(\text{rel}) \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (19)$$

where $V_i^A$ is the share of total expenditure on the good i in the base period represented in figure 2 by the point A. Recall that all relative quantities and relative prices are normalized to equal 1 in the base period and that the average value of $P_i^B(\text{rel})$ or $Q_i^B(\text{rel})$ over all commodities is always equal to unity. Consequently if the average of the products of $P_i^B(\text{rel})$ and $Q_i^B(\text{rel})$ is less than unity, the highs of $P_i^B(\text{rel})$ must correspond to the laws of $Q_i^B(\text{rel})$ and vice versa, which is to say that demand curves tend to be downward sloping - exactly the content of Hicks' first substitution theorem.

To show how and under what circumstances this law may be derived from assumptions about the convexity of indifference curves, and to bring to hear a considerable mass of empirical evidence which is not normally thought of as pertaining to slopes of demand curves, we first observe that our generalized law of demand is equivalent to the statement that the Laspeyres quantity index exceeds the Paasche quantity index. The Laspeyres quantity index is

$$L = \frac{\sum_{i=1}^{n} P_i^A \; Q_i^B}{\sum_{i=1}^{n} P_i^A \; Q_i^A} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (20)$$

Assume that indifference durves are convex and that relative prices reflect rates of substitution in use between commodities. Formally, convexity is the property that

$$U(\lambda\ \hat{Q}^{A} + (1-\lambda)\ \hat{Q}^{B}) \geqslant \lambda\ U(\hat{Q}^{A}) + (1-\lambda)\ U(\hat{Q}^{B}) \quad \ldots\ldots\ldots\ldots \quad (22)$$

where $\hat{Q}^{A}$ and $\hat{Q}^{B}$ are any vectors of quantities consumed, $U$ is the utility function, and $0\leqslant\lambda\leqslant1$. The property of convex functions that we shall use in our proofs is (1)

$$\sum_{i=1}^{n} U_i^{A} . (Q_i^{B} - Q_i^{A}) \geqslant U(\hat{Q}^{B}) - U(\hat{Q}^{A}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

where $Q_i^B$ is the quantity of the i commodity in the vector $\hat{Q}^{B}$ and $U_i^A$ is the marginal utility of the i commodity when consumption of all commodities is represented by the vector $\hat{Q}^{A}$. At any point in the field of indifference curves, prices are proportional to marginal utilities

$$P_i^{B} = N(B)\ \frac{\partial\ U(\hat{Q}^{B})}{\partial\ Q_i^{B}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

where N(B) is an arbitrary constant which may vary from time to time or from one equilibrium to another, but which is the same for all prices at a given equilibrium and at a given time.

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(1) K. Lancaster Mathematical Economics Macmillan, 1968, pp. 331-333. Note that in conformity with practice in the writing about the theory of demand, we have used the term convex to describe what Lancaster calls concave.

In a comparison of real income in situations such as those represented by A and B in figure 2, there are two types of "true" quantity indices.(1) One type, which may be called "the true index of real income at A prices", answers the question. "What multiple of the income at A would a man require to be as well off as he would at B." This index, designated as $T_A$, is given by the formula

$$T_A = \frac{\sum_{j=1}^{n} U_j^{A} \; Q_j^{C}}{\sum_{j=1}^{n} U_j^{A} \; Q_j^{A}} \qquad \text{.......................... (25)}$$

where $Q_j^C$ is the amount of the good j in a bundle of goods $\hat{Q}^C$ chosen such that the utility of $\hat{Q}^C$ is the same as the utility of $\hat{Q}^B$

$$U(\hat{Q}^C) = U(\hat{Q}^B) \qquad \text{.............................. (26)}$$

and relative prices at C are the same as relative prices at A

$$\frac{U_i(\hat{Q}^C)}{U_j(\hat{Q}^C)} = \frac{U_i(\hat{Q}^A)}{U_j(\hat{Q}^A)} \qquad \text{............................. (27)}$$

for all i and j. At the point C, the indifference curve through B is tangent to a budget constraint parallel to that at A. The point C represents the cheapest of all bundles of goods yielding as much satisfaction as the bundle represented by the point B, when the valuation is at A prices. The

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(1) R. Frisch "Annual Survey of Economic Theory: The Problem of Index Numbers "*Econometrica* 1935, pp. 1-38.

"true index of real income at B prices" is

$$T_B = \frac{\sum_{j=1}^{n} U_j^B \quad Q_j^B}{\sum_{j=1}^{n} U_j^B \quad Q_j^D} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

where the point D is chosen so that

$$U(\hat{Q}_D) = U(\hat{Q}_A) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

and

$$\frac{U_i(\hat{Q}_D)}{U_j(\hat{Q}_D)} = \frac{U_i(\hat{Q}_B)}{U_j(\hat{Q}_B)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

the relation between A,B,C and D in a two commodity world is illustrated in figure 5.

From the property of convexity described in equation 23, it follows that

$$L \geqslant T_A \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

and that

$$T_B \geqslant P \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

so that our generalized law of demand is true[1] whenever

$$(T_B - T_A) < \min\ (L - T_A,\ T_B - P) \quad \ldots\ldots\ldots\ldots \quad (33)$$

This can come about in several ways:

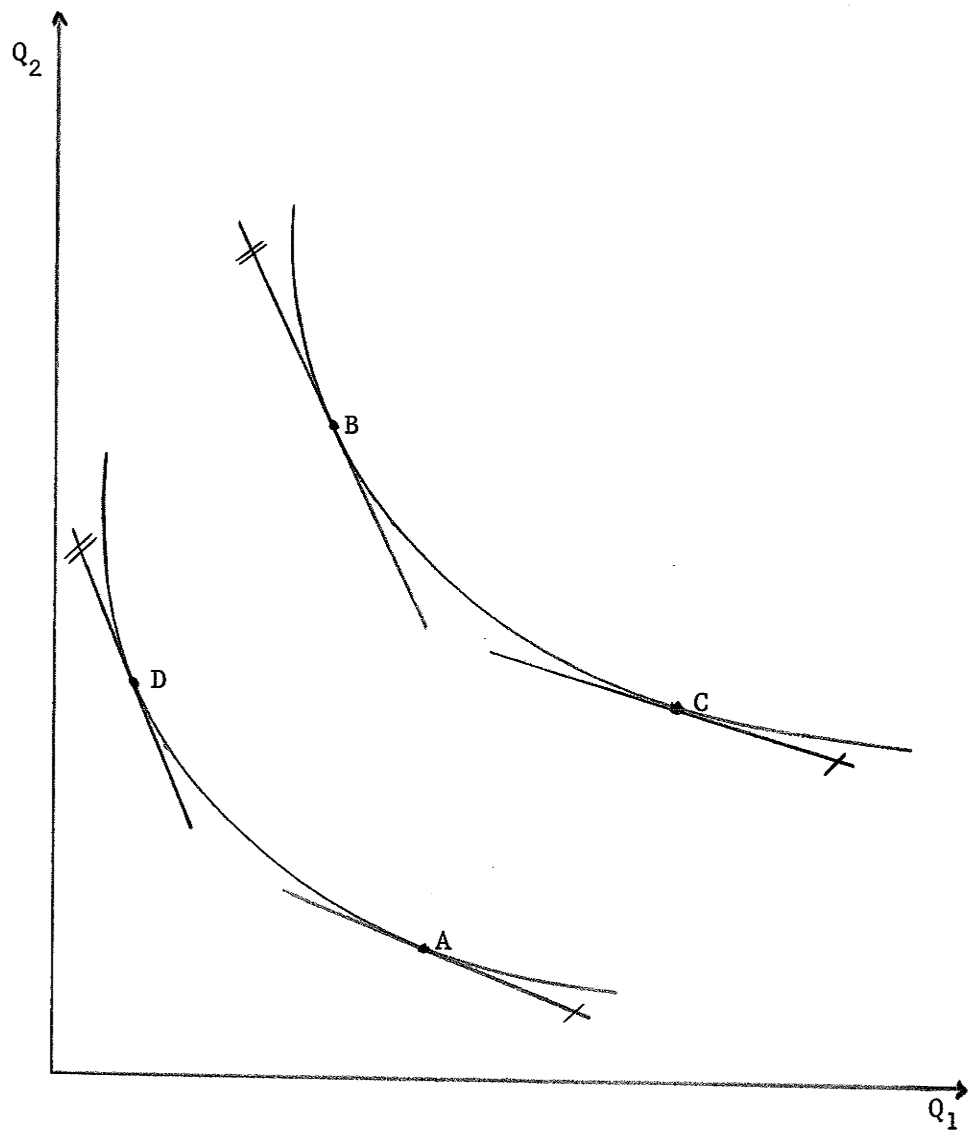


Figure 5.

First, the generalized law of demand is true whenever the points compared lie on the same indifference curve. In this case the points C and B coincide, the points A and D coincide and $T_A = T_B$, which assures that $L > P$.

Second, the generalized law of demand is true if the utility function is homogeneous to any degree. In this case as well,(2) $T_B = T_A$ so that $L > P$.

Third, the generalized law of demand holds whenever price changes from one situation to the next are positively correlated with income elasticities so that income effects and substitution effects work in the same direction.(3)

Finally, there is a substantial body of evidence on international and intertemporal comparison of income which reveals that the Laspeyres index is almost always greater than the Paasche index, so that our generalized law of demand has been demonstrated to hold in fact and not merely in principle subject to qualifications about the shape of U or about correlations between price changes and income elasticities.(4)

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(1) Equation 23 implies that

$$\sum_{j=1}^{n} U_j^C (Q_j^B - Q_j^C) \geqq U(Q^B) - U(Q^C) = 0$$

From equation 27, it follows that

$$0 \leqq \sum_{j=1}^{n} U_j^C (Q_j^B - Q_j^C) = \left[ \sum_{j=1}^{n} U_j^A (Q_j^B - Q_j^C) \right] \frac{U_i^C}{U_i^A}$$

Therefore

$$\frac{\sum_{j=1}^{n} U_j^A Q_j^B}{\sum_{j=1}^{n} U_j^A Q_j^A} \quad \frac{\sum_{j=1}^{n} U_j^A Q_j^C}{\sum_{j=1}^{n} U_j^A Q_j^A}$$

(2) Homogeneity of degree p implies that for any $\lambda$, $U_i^{\,B} = \lambda^{\sigma-1}\, U_i^{\,D}$ and $U_i^{\,C} = \lambda^{\sigma-1}\, U_i^{\,A}$ when $\hat{Q}^B = \lambda\hat{Q}^D$ and $\hat{Q}^C = \lambda\hat{Q}^A$.

Therefore by Euler's theorem

$$T_A = \frac{\sum_{j=1}^{n} U_j^{\,A} Q_j^{\,C}}{\sum_{j=1}^{n} U_j^{\,A} Q_j^{\,A}} = \frac{1}{\lambda^{\sigma-1}} \frac{U(\hat{Q}^C)}{U(\hat{Q}^A)} =$$

$$= \frac{1}{\lambda^{\sigma-1}} \frac{U(\hat{Q}^B)}{U(\hat{Q}^D)} = \frac{1}{\lambda^{\sigma-1}} \frac{\sum_{j=1}^{n} U_j^{\,B} Q_j^{\,B}}{\sum_{j=1}^{n} U_j^{\,D} Q_j^{\,D}} = T_B$$

(3) This fact is established by decomposing the difference L - P into income effects and substitution effects. For convenience, subscripts are omitted.

$$L - P = \frac{\Sigma P^A\, Q^B}{\Sigma P^A\, Q^A} - \frac{\Sigma P^B\, Q^B}{\Sigma P^B\, Q^A}$$

$$= \frac{\Sigma P^A (Q^D - Q^A)}{\Sigma P^A\, Q^A} - \frac{\Sigma P^B (Q^D - Q^A)}{\Sigma P^B\, Q^A}$$

$$+ \frac{\Sigma P^A (Q^B - Q^D)}{\Sigma P^A\, Q^A} - \frac{\Sigma P^B (Q^B - Q^D)}{\Sigma P^B\, Q^A}$$

Of these four expressions, the first two are always positive and may be thought of as a substitution effect. The second pair of terms may be positive or negative and are called income effect I.

$$I = \Sigma \left[ \left( \frac{P^A\, Q^A}{\Sigma P^A\, Q^A} \right) - \left( \frac{P^B\, Q^A}{\Sigma P^B\, Q^A} \right) \right] \left[ \frac{Q^B - Q^D}{Q^A} \right]$$

$$= \Sigma\, V^A \left( 1 - \frac{P^B}{P^A} \left( \frac{\Sigma P^A\, Q^A}{\Sigma P^B\, Q^A} \right) \right) \left( \frac{Q^B - Q^D}{Q^A} \right)$$

Since all relative prices are the same at B and at D, the term $(Q^B - Q^D)/Q^A$ may be thought of as an approximation to an income elasticity, and the term

$$\frac{P^B}{P^A}\left(\frac{\Sigma P^A Q^A}{\Sigma P^B Q^A}\right)$$ is a measure

of change in relative price between A and B.

This demonstration is taken from Yasushi Toda <u>On International Comparison of Consumption: Studies in Index Number Theory and Measurement</u> Ph.D., Harvard, 1969.

(4) This evidence is reviewed by Yasushi Toda's thesis.

Types of Demand Curves

We began this paper by pointing out that the variables entering into empirical demand curves are not those discussed in the theory of demand. Then it was shown that the convexity of indifference curves makes it likely that there will be a negative correlation between relative price and relative quantity observed at any two points in the field of indifference curves. What remains to be done is to define a theoretical demand curve in our new variables.

There are really two issues here. The first is that our new variables are defined with reference to a base period, the period for which shares of expenditure $V_1^A$, $V_2^A$, ........ , $V_n^A$ are defined. Any demand curve in our new variables has to be with reference to value shares, and each point in the field of indifference curve gives rise to its own set of value shares and its own demand curve.

Second, any demand curve - whether in our new variables, relative price and relative quantity, or in the original variables, money price and absolute quantity - need be defined with reference to a price-consumption curve, as illustrated in figure 6. The issue is that to any quantity, $\bar{Q}_1$, there corresponds a multiplicity of prices, represented by slopes of all indifference curves crossing a vertical line, L, at or above $\bar{Q}_1$. To define a demand curve, one requires a price-consumption curve that picks out the price appropriate to each quantity. If the price-consumption curve is constructed by fixing income in units of the good $Q_2$ and allowing $P_1$ to vary, one arrived at Hicks' constant money income demand curve. Friedman's

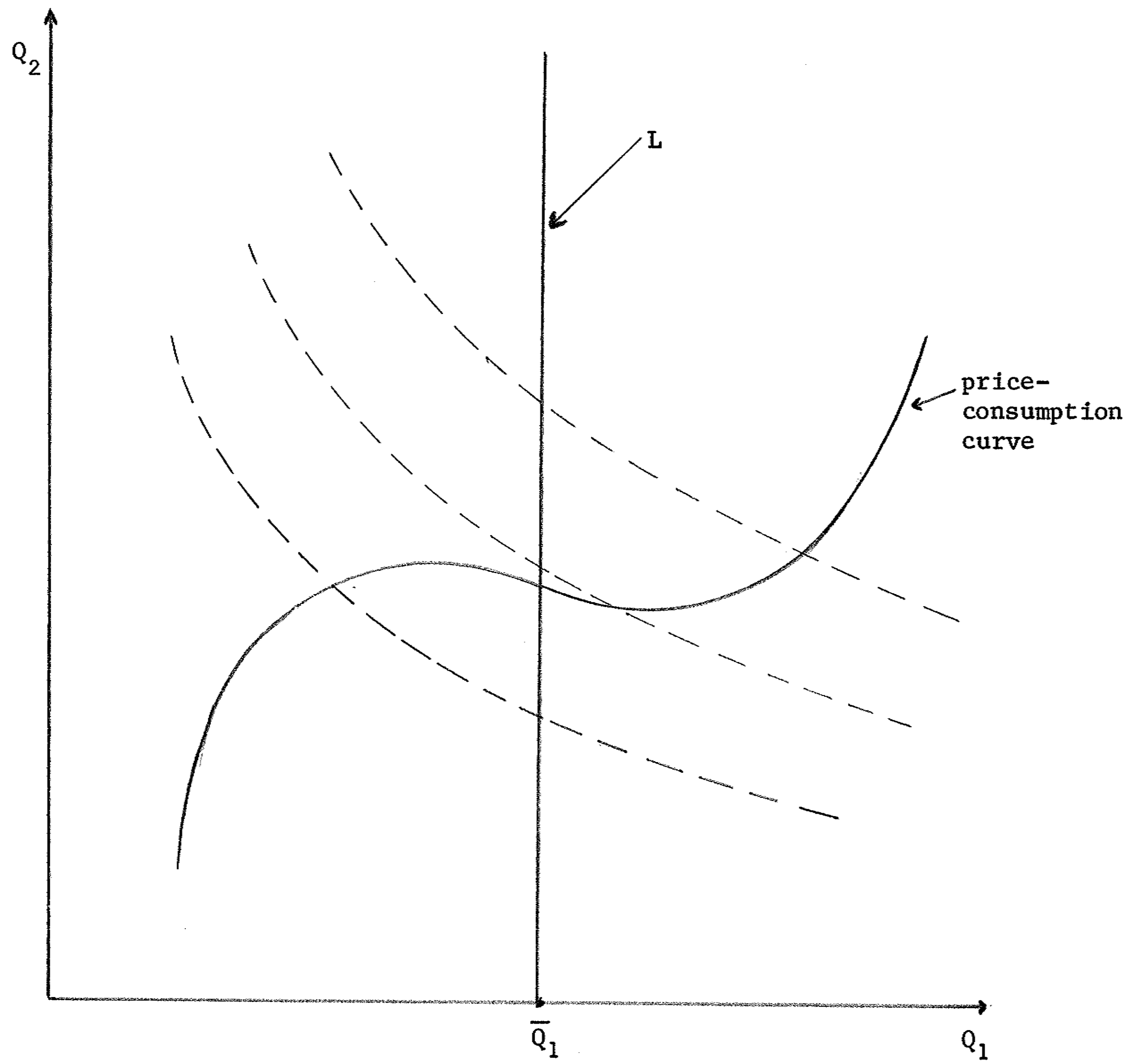


Figure 6.

Marshallian demand curve is constructed by letting one of the indifference curves do double-duty as the price-consumption curve. Another useful demand curve is constructed by letting a production possibility curve serve as a price-consumption curve, and, though some constructions are more useful than others, there is no end to the possibilities.(1)

The choice of a price-consumption curve is of the utmost importance in the definition of demand curves correcting money price and absolute quantity, for only within a limited range of price-consumption curves can the demand curve be expected to have a negative slope. In a two commodity economy, the Friedman demand curve for which an indifference curve serves as a price consumption curve must have a negative slope and other demand curves are progressively less likely to have negative slopes the more their price-consumption curves diverge from an indifference curve. The Giffen case in the constant money income demand curve has its counterpart in all demand curves except those with price-consumption curves that trace out indifference curves. A positively sloped price-consumption curve such as that drawn in figure 6 is almost certain to generate a positively sloped demand curve even if indifference curves are homogeneous.

The dependence of a demand curve on a price-consumption cannot be eliminated by changing the variables from money price and absolute quantity to relative price and relative quantity, but the issue becomes much less important in the new framework, because all price-consumption-curves yield one and the same demand curve in the new variables, whenever the indifference

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(1) The classification of ordinary demand curves is discussed in my paper "The Derivation of Demand Curves from Indifference Curves", Oxford Economic Papers, Nov. 1965.

curves are homogeneous. Even when indifference curves are not homogeneous, shapes of demand curves connecting relative price and relative quantity need not be affected greatly by the choice of the price-consumption curve, and a wide range of price-consumption curves is consistent with a downward sloping demand curve. With non-homogenity of indifference curves, one could devise a price-consumption curve to generate a positively-sloped demand curve, but there is evidence that this possibility is almost never realized in practice. Price-consumption curves are artificial and arbitrary in the sense that time-series or cross-section data cannot be expected to conform to a price-consumption curve chosen in advance. Thus the relative independence of our new demand curves of the choice of price-consumption curve is a valuable property for a theory of demand designed to explain relations among empirical observations.

The main advantages of the demand curve connecting relative price and relative quantity are these: Its variables are appropriate for empirical studies of demand. It fits the circumstances of empirical work in that it pertains to any sequence of observations of prices and quantities in the field of indifference curves. It is as well-grounded theoretically as the textbook demand curve in the sense that convexity of indifference curves guarantees a negative slope to both curves under similar conditions. Its law of demand is more general than that of the textbook demand curve became the law is strictly true along an indifference curve or whenever indifferences curves are homogeneous to any degree. Its law of demand is directly verified in international or intertemporal comparisons of income.