Queen's Economics Department Working Paper No. 16

NOTES ON THE ISARD AND CHENERY-MOSES INTERREGIONAL INPUT-OUTPUT MODELS

John M. Hartwick
Queen's University

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

1-1970

NOTES ON THE ISARD AND CHENERY-MOSES INTERREGIONAL INPUT-OUTPUT MODELS

John M. Hartwick

Queen's University

Discussion Paper # 16

January, 1970
(Revised, July, 1970)

NOTES ON THE ISARD AND CHENERY-MOSES INTERREGIONAL INPUT-OUTPUT MODELS

John M. Hartwick*

Queen's University

Introduction

Interregional input-output models have seldom been empirically articulated owing no doubt to the inordinate demands made on research teams to collect data on economic flows and to prepare the coefficients required in the models. The first interregional input-output model, published in 1951 by Isard [8] has been empirically articulated only once and even then on a very small scale [6]. Chenery [4] and Moses [15] developed a model, independently of one another, which is similar to the Isard model but requires considerably less data in order to be statistically implemented. Their model has in fact been statistically implemented for systems of regions in, for example, Italy[1], the United States of America[2], and Canada [7]. Efforts have been directed to constructing interregional input-output models which require a minimum of data on interregional commodity flows. Leontief [9] developed one model. A more recent model is that of Leontief and Strout [11] which makes use of the gravity theory for estimating interregional flows. Miller [13], [14] has performed some interesting empirical tests which suggest that the increase in accuracy from the use of interregional as opposed to a

*This paper is part of my Ph.D. dissertation [7], Johns Hopkins, 1969. I am indebted to my supervisors, Edwin S. Mills and Peter Newman, for patient guidance. R.E. Miller made helpful suggestions on an earlier draft of this paper.

and the detailed comparison of the Isard and Chenery-Moses models.

1. The Isard Model and a Reformulation.

Table 1 contains the basic data on economic flows required to define the Isard model. There are m regions and n sectors and nm x nm intersectoral and interregional trade flows.

Table 1
Regional Accounts for Isard Interregional
Input-Output Model

	Region 1		Region ℓ		Region m	Final Demand	Total
Region 1	X^{11}	...	$X^{1\ell}$	...	X^{1m}	y^1	x^1
	.		.		.	.	.
	.		.		.	.	.
	.		.		.	.	.
Region k	X^{k1}	...	$X^{k\ell}$	...	X^{km}	y^k	x^k
	.		.		.	.	.
	.		.		.	.	.
	.		.		.	.	.
Region m	X^{m1}	...	$X^{m\ell}$	...	X^{mm}	y^m	x^m
Primary Inputs	$g^{1'}$	...	$g^{\ell'}$	...	$g^{m'}$		
Total	$x^{1'}$	...	$x^{\ell'}$	...	$x^{m'}$		

series of separate regional input-output models is not significant compared to the increase in costs required for the more detailed models. Interregional models capture feedback effects of increased activity in region A on region B and B's induced increase in A and so on. Regional input-output models cannot capture these feedback effects. In this paper we are concerned only with two interregional input-output models, the Isard and the Chenery-Moses.

The Isard model captures observable interregional intersectoral trade relationships in a more precise way than does the Chenery-Moses model. It is the purpose of this paper to reformulate the Isard model in a manner which permits us to make a precise comparison of the two models and to explicitly define the averaging implicit in the Chenery-Moses model vis-a-vis the Isard Model. Specifically the representative coefficient in the original Isard model $a_{ij}^{k\ell}$ expressing the flow of commodity i from region k to region ℓ required in order to produce a unit of j in region ℓ will be separated into two coefficients, a supply coefficient and a technical coefficient analogous to the Chenery-Moses supply and technical coefficients. The reformulation of the Isard model will also permit one to introduce supply capacities in specific sectors in specific regions into the model in a way which Chenery illustrated to the Chenery-Moses model. This property will be described in Section 3.

Section 1 contains a presentation of the Isard model and the reformulation of the Isard model which we shall call the revised Isard model. Section 2 contains a presentation of the Chenery-Moses model

Where

X^{11} is an nxn matrix in which each component is a flow from one of n sectors to another of n sectors in region .

$X^{\ell m}$ is an nxn matrix containing analogous intersectoral flows but from region ℓ to region m.

$X^{k\ell}$ is an nxn matrix of intersectoral flows moving from region k to region ℓ.

y^k is a column vector of n components defining final demands by sector in region k. It excludes consumption since we consider this activity endogenous and hence one of our n activities by region.

x^k is a column vector of n components defining the sum by sector of intermediate outputs originating in region k plus final demands in region k. That is, all of region k's exports to its related regions are included in x^k.

$g^{\ell'}$ is a row vector of n components defining the primary inputs by sector flowing into region ℓ. These primary inputs comprise non-competitive imports, and indirect taxes. Wages and salaries are endogenous and are hence included as one of the n activities by region corresponding to consumption above in x^k.

$x^{\ell'}$ is a row vector of n components defining total input by sector for region ℓ. $x^{\ell'}$ or total inputs is defined to equal x^{ℓ} or total output above.

Each of the matrices along the diagonal defines flows originating and terminating in the same respective regions. The off-diagonal matrices define exports and imports. Imports or supplies are defined in the column arrays of matrices and exports are defined in the row arrays associated with each separate region.

Given the known flows in Table 1, we can proceed to define an operator matrix analogous to Leontief's matrix A (not I-A) of technical coefficients.

Define,

$$a_{ij}^{k\ell} = \frac{x_{ij}^{k\ell}}{\sum_k \sum_i x_{ij}^{k\ell} + g_j^{\ell}} \tag{1}$$

where $x_{ij}^{k\ell}$ is the i,j^{th} component of $X^{k\ell}$ defining the flow from sector i to sector j and from region k to region ℓ; g_j^{ℓ} is the j^{th} component of vector $g^{\ell'}$. In defining this coefficient (1) we have directly followed Isard.[3] The procedure for defining coefficients resembles that developed by Leontief [10] for simple national models. We divide each element in a column in Table 1 by the sum of all the elements in the column. Matrix $A^{k\ell}$ will be composed nm x nm elements $a_{ij}^{k\ell}$ as in (1) above $i,j = 1, \ldots, n$; $k,\ell = 1, \ldots, m$. <u>Note that throughout the paper k and ℓ sum from 1 to m, that is over all regions, and i and j sum from 1 to n, that is over all sectors.</u> One exception will be noted.

One of the drawbacks of the above procedure for defining the coefficients of $A^{k\ell}$ is that the coefficients contain elements of both interregional trade characteristics and technical production characteristics. The coefficients $a_{ij}^{k\ell}$ are imprecisely related to the technical coefficients in a simple Leontief model. Thus if the coefficients change over time we cannot say whether there has been a simple rearrangement of trading patterns, a change in technology or both.

With two regions and two industries the Isard matrix $A^{k\ell}$ (not I-A or $(I-A)^{-1}$) will be as follows:

$$(a_{ij}^{k\ell}) = \begin{bmatrix} a_{11}^{11}, & a_{12}^{11}, & a_{11}^{12}, & a_{12}^{12} \\ a_{21}^{11}, & a_{22}^{11}, & a_{21}^{12}, & a_{22}^{12} \\ a_{11}^{21}, & a_{12}^{21}, & a_{11}^{22}, & a_{12}^{22} \\ a_{21}^{21}, & a_{22}^{21}, & a_{21}^{22}, & a_{21}^{22} \end{bmatrix} \qquad \begin{pmatrix} i,j = 1,2 \\ k,\ell = 1,2 \end{pmatrix} \qquad (1A)$$

Note that no coefficients are identical in this formulation.

The inadequacies of a single coefficient can be to a large extent overcome by defining two coefficients for each observed cell from the flows in Table 1.

Define:

$$(s_{ij}^{k\ell})(a_{ij}^{\ell}) = \left[\frac{x_{ij}^{k\ell}}{\sum\limits_{k} x_{ij}^{k\ell}}\right] \left[\frac{\sum\limits_{k} x_{ij}^{k\ell}}{\sum\limits_{k}\sum\limits_{i} x_{ij}^{k\ell} + g_j^{\ell}}\right] \qquad (2)$$

where $x_{ij}^{k\ell}$ and g_j^{ℓ} are the same as in (1). The elements in the two sets of brackets on the left correspond with the elements in the two sets of brackets on the right.

We might remark that the a_{ij}^{ℓ} is a technical coefficient defined in a manner identical to those in simple Leontief system. Competitive imports have been incorporated in the technical coefficients in a common way.[4] Now we have a square nxn technical coefficient matrix for each of

They suggested consigning interregional flows to sectors through the use of the known intersectoral flows as weighting coefficients; that is, partitioning the aggregated interregional flows into intersectoral interregional flows through the use of the known intersectoral flows.

Chenery and Moses have expressed a different interpretation of the procedure in constructing coefficients. Their interpretation is that all producers in each region consider the imports from a specific region as homogenous and thus all producers import from a specific region in proportion to their total use patterns rather than importing in different proportion from different regions. For example, if ten million dollars' worth of bricks are imported to region eight from region ten and eleven respectively, then all sectors in region eight will import bricks from regions ten and eleven in the ratio one to one. This was not the case in section 1, although this above assumption can be considered a special case of the approach in section 1. In section 1, region eight could hypothetically import ten million dollars' worth of bricks from regions ten and eleven respectively, but we would not expect all sectors to import from these two regions with flows in the ratio one to one.

The coefficients of the Chenery-Moses matrix operator are defined as follows:

$$(s_i^{k\ell})(a_{ij}^{\ell}) = \left[\frac{x_i^{k\ell}}{\sum\limits_k x_i^{k\ell}}\right] \cdot \left[\frac{\hat{x}_{ij}^{\ell\ell}}{\sum\limits_i \hat{x}_{ij}^{\ell\ell} + g_j^{\ell}}\right] \qquad (3)$$

where $\hat{x}_{ij}^{\ell\ell}$ is the i,j^{th} component of $X^{\ell\ell}$ and g_i^{ℓ} is the j^{th} component

2. The Chenery-Moses Model and Its Relationship to the Isard Model.

The accounting system for the Chenery-Moses model differs from that for the Isard model in the respect that for inter regional flows in the Chenery-Moses model, the sector of destination of the flows is unknown. That is there is a system of vectors of inter regional flows in the Chenery-Moses model rather than of matrices as in the Isard accounting framework presented in Table 1.

In Table 2, we have the interregional accounting framework for the Chenery-Moses model.

Table 2

Regional Accounts for Chenery-Moses
Interregional Input-Output Model

	Region 1		Region ℓ		Region m	Final Demand	Total
Region 1	x^{11}	...	$x^{1\ell}$	...	x^{1m}	y^1	x^1
	.		.		.	.	.
Region k	x^{k1}	...	$x^{k\ell}$	...	x^{km}	y^k	x^k
	.		.		.	.	.
Region m	x^{m1}	...	$x^{m\ell}$	...	x^{mm}	y^m	x^m
Primary Inputs	$g^{1'}$	...	$g^{\ell'}$	...	$g^{m'}$		

where

$x^{k\ell}$ is a column vector with n components, $x_i^{k\ell}$, $i = 1, \ldots , n$.

$x_i^{k\ell}$ is the flow of commodity i from region k to region ℓ.

y^k is a column vector with n components, y_i^k, $i = 1, \ldots , n$.

y_i^k is the final demand in region k for commodity i. It includes exports to regions not included in the set of m regions, but excludes consumption which is assumed to be one of the n activities within the system.

x^k is a column vector of n components, x_i^k, $i = 1, \ldots , n$.

x_i^k is the total output of i from region k including exports to other regions plus final demand in region k.

$g^{\ell\prime}$ is a row vector of n components, g_j^ℓ, $j = 1, \ldots , n$. g_j^ℓ is the primary input including non-competitive imports and indirect taxes. Households are assumed endogenous so that wages and salaries are considered as one of the n activities within the vectors in the upper left hand part.

In addition to the flows in Table 2, it is assumed that we know a matrix $\hat{X}^{kk}$ for each region $k = 1, \ldots , m$ which contains intersectoral flows from each of n sectors to each of n sectors aggregated to include all competitive imports. Thus $\hat{x}_{ij}^{kk}$, the i,j^{th} component of $\hat{X}^{kk}$ describes the total amount of i <u>including imports</u> required to produce an observed flow of j. $\hat{X}^{kk}$ differs from X^{kk} in Table 1 in that competitive imports have been excluded from the flows in X^{kk}.

$\hat{X}^{kk}$ forms the accounting flows of the interindustrial portion of a simple Leontief system. In other words we could form a technical production coefficient for an isolated economy by dividing such element of $\hat{X}^{kk}$ by the column sum. That is for region ℓ rather than k.

$$\hat{a}^{\ell}_{ij} = \frac{\hat{x}^{\ell\ell}_{ij}}{\sum_{i} \hat{x}^{\ell\ell}_{ij}}$$

where $\hat{a}^{\ell}_{ij}$ is the amount of input i required to produce a unit of output j in region or economy ℓ. The $\hat{a}^{\ell}_{ij}$ is the familiar Leontief technical coefficient. In order to define this identical coefficient from an Isard accounting framework, we must sum over industrial sectors and regions, the latter being already disaggregated in the accounting scheme. Thus from the Isard accounting framework, the technical coefficient is defined as follows:

$$a^{\ell}_{ij} = \frac{\sum_{k} x^{k\ell}_{ij}}{\sum_{k} \sum_{i} x^{k\ell}_{ij}}$$

Now $\hat{a}^{\ell}_{ij}$ is by definition equal to a^{ℓ}_{ij} for all i and j.

Chenery and Moses independently arrived at the same procedure for of g^{ℓ}. There are nm x nm pairs of coefficients as in (3) forming an **operator** matrix ${}^{s}\hat{A}^{k\ell}$ analogous to matrices $A^{k\ell}$ and ${}^{s}A^{k\ell}$ in section 1.

Consider the second coefficient first. a_{ij}^{ℓ} is identical to the a_{ij}^{ℓ} in (2). We examined this relationship in detail below Table 2. a_{ij}^{ℓ} is a technical coefficient in the sense that it defines the total amount of flow i required per unit output of j in region k.

The coefficient $s_i^{k\ell}$ indicates the fraction of commodity i supplied by region k to region ℓ. Obviously summing over regions $\sum_k s^{k\ell} = 1$ and if region k imports no i, $s_i^{kk} = 1$ with all other supply coefficients of commodity i related to region k equal to zero.

The Chenery-Moses matrix analogue to (1A) and (2A) for the two region, two sector case is as follows:

$$(s_i^{k\ell})(a_{ij}^{\ell}) = \begin{bmatrix} s_1^{11}\ a_{11}^{1}, & s_1^{11}\ a_{12}^{1}, & s_1^{12}\ a_{11}^{2}, & s_1^{12}\ a_{12}^{2} \\ s_2^{11}\ a_{21}^{1}, & s_2^{11}\ a_{22}^{1}, & s_2^{12}\ a_{21}^{2}, & s_2^{12}\ a_{22}^{2} \\ s_1^{21}\ a_{11}^{1}, & s_1^{21}\ a_{12}^{1}, & s_1^{22}\ a_{11}^{2}, & s_1^{22}\ a_{12}^{2} \\ s_2^{21}\ a_{21}^{1}, & s_2^{21}\ a_{22}^{1}, & s_2^{22}\ a_{21}^{2}, & s_2^{22}\ a_{22}^{2} \end{bmatrix} \quad \begin{pmatrix} i,\ j = 1,\ 2 \\ k,\ \ell = 1,\ 2 \end{pmatrix} \qquad (3A)$$

It was a matrix like (3A) which Chenery and Moses developed for regional analysis. Observe that we have two sets of technical coefficients of production of the traditional Leontief type defined - one for each region. These technical coefficients are weighted by supply or trade coefficients.

REFERENCES

[1] Beckenbach, Edwin F., and Richard Bellman, Inequalities, Second Revised Printing, New York: Springer-Verlag, 1965.

[2] Chenery, H.B., "Interregional and International Input-Output Analysis", in Tibor Barna, ed., The Structural Interdependence of the Economy, New York: Wiley, 1956, 341-356.

[3] Chenery, Hollis B. and Paul G. Clark, Interindustry Economics, New York: Wiley, 1959.

[4] Chenery, H.B., P. G. Clark, and Vera Cao Pinna, The Structure and Growth of the Italian Economy, United States Mutual Security Agency, Rome, 1953, Chapter 5 written by Chenery.

[5] Dorfman, R., P.A. Samuelson, R.M. Solow, Linear Programming and Economic Analysis, New York: McGraw-Hill, 1958.

[6] Heady, Earl O. and H.O. Carter, "Input-Output Models as Techniques in the Analysis of Interregional Competition", Journal of Farm Economics, Vol. 41, 1960, 978-991.

[7] Hartwick, John M., Regional Analysis by Means of Interregional Input-Output Models with Applications to Eastern Canada, Unpublished Ph.D. Dissertation, Johns Hopkins University, 1969.

[8] Isard, W., "Interregional and Regional Input-Output Analysis: A Model of a Space Economy", Review of Economics and Statistics, Vol. XXXIII, No. 4, 1951, 318-28.

[9] Leontief, Wassily, "Interregional Theory", in Leontief et. al, Studies in the Structure of the American Economy, New York: Oxford University Press, 1953, 93-115.

[10] Leontief, W., "Quantitative Input and Output Relations in the Economic System of the United States", Review of Economic Statistics, 18, 1936, 105-125.

[11] Leontief, Wassily, in collaboration with Alan Strout, "Multiregional Input-Output Analysis", in Tibor Barna, ed., Structural Interdependence and Economic Development, New York: St. Martin's Press, 1963, 119-50 reprinted in Wassily Leontief, Input-Output Economics, New York: Oxford University Press, 1966.

In the exposition of the Isard procedure using both supply and technical coefficients (revised Isard) we weighted each technical coefficient separately with a set of supply coefficients which summed to unity. In this Chenery-Moses procedure we weight each technical coefficient in the same row of a region's technical coefficient matrix by the same supply coefficient. Thus, there is an implicit averaging involved in the Chenery-Moses procedure which is not found in the more precise Isard procedure. We can make this explicit by relating the supply coefficients of the two procedures.

In section 1 we showed how we could represent our known flows in terms of coefficients and total outputs. What can we say about the flows we derive from multiplying total outputs by our operating coefficients? They clearly do not restate known flows since we have no knowledge of the actual interregional intersectoral flows when we construct our coefficients. We can, however, compare the approximation in Chenery-Moses formulation to the known flows expressed in terms of the revised Isard formulation with supply and technical coefficients.

Consider the following identity:

$$\frac{x_{i1}^{kk}}{\sum_{\ell} x_{i1}^{\ell k}} \cdot \frac{\sum_{\ell} x_{i1}^{\ell k}}{\sum_{j} \sum_{\ell} x_{ij}^{\ell k}} + \frac{x_{i2}^{kk}}{\sum_{\ell} x_{i2}^{\ell k}} \cdot \frac{\sum_{\ell} x_{i2}^{\ell k}}{\sum_{j} \sum_{\ell} x_{ij}^{\ell k}} + \dots +$$

$$\frac{x_{in}^{kk}}{\sum_{\ell} x_{in}^{\ell k}} \cdot \frac{\sum_{\ell} x_{in}^{\ell k}}{\sum_{j} \sum_{\ell} x_{ij}^{\ell k}} \equiv \frac{\sum_{j}^{n} x_{ij}^{kk}}{\sum_{j} \sum_{\ell} x_{ij}^{\ell k}} \qquad (4)$$

$\tilde{x}^{kk}_{ij}$ =

$$[s^{kk}_{i} \cdot a^{k}_{ij} \cdot x^{k}_{j}] = (x^{kk}_{ij}) \cdot r^{k}_{ij} + s^{kk}_{i1} \cdot r^{k}_{i1} \cdot a^{k}_{ij} \cdot x^{k}_{j} + \ldots + s^{kk}_{i,j-1} \cdot r^{k}_{i,j-1} \cdot a^{k}_{i,j} \cdot x^{k}_{j} + s^{kk}_{i,j+1} \cdot r^{k}_{i,j+1} \cdot a^{k}_{ij} \cdot x^{k}_{j} + \ldots + s^{kk}_{in} \cdot r^{k}_{in} \cdot a^{k}_{ij} \cdot x^{k}_{j} .$$

Observe in the above equation that the term in square brackets is flow from sector i to sector j in region k as expressed in terms of coefficients in the Chenery-Moses model. That flow does not equal the Isard flow x^{kk}_{ij} alone but equals the Isard flow (in round brackets) weighted by r^{k}_{ij} plus n-2 terms.

What we can do now is to reconstruct the actual flows which the Chenery-Moses procedure estimates as proxies for the precise flows in the Isard accounting framework. The i, j flow in region k is represented by the term in square brackets in the above equation. Let us call it $\tilde{x}^{kk}_{ij}$, the counterpart of x^{kk}_{ij} in the Isard accounting framework. For the two region, two sector case the estimated flows will be

$$\begin{bmatrix} \tilde{x}^{11}_{11}, & \tilde{x}^{11}_{12}, & \tilde{x}^{12}_{11}, & \tilde{x}^{12}_{12} \\ \tilde{x}^{11}_{21}, & \tilde{x}^{11}_{22}, & \tilde{x}^{12}_{21}, & \tilde{x}^{12}_{22} \\ \tilde{x}^{21}_{11}, & \tilde{x}^{21}_{12}, & \tilde{x}^{22}_{11}, & \tilde{x}^{22}_{12} \\ \tilde{x}^{21}_{21}, & \tilde{x}^{21}_{22}, & \tilde{x}^{22}_{21}, & \tilde{x}^{22}_{22} \end{bmatrix} \qquad \begin{bmatrix} x^{11}_{11}, & x^{11}_{12}, & x^{12}_{11}, & x^{12}_{12} \\ x^{11}_{21}, & x^{11}_{22}, & x^{12}_{21}, & x^{12}_{22} \\ x^{21}_{11}, & x^{21}_{12}, & x^{22}_{11}, & x^{22}_{12} \\ x^{21}_{21}, & x^{21}_{22}, & x^{22}_{21}, & x^{22}_{22} \end{bmatrix}$$

Chenery-Moses Proxy Flows   Isard Flows (from Table 1)

The sum of elements in any row of the left hand matrix will equal the sum of elements in the corresponding row of the right hand matrix. A review of the accounts in Tables 1 and 2 and of the definitions of coefficients will show this to be so.

F is an information theoretic value measuring how much cell entries deviate from values predictable from column and row entries.[5]

The proof of the result is a straightforward application of a theorem of Beckenbach and Bellman [1 ; p. 17].[6] We first develop the expressions for the information content of the Isard and Chenery-Moses models separately, then transform these expressions, and compare their relative magnitude with the aid of the Beckenbach and Bellman theorem. The manipulation is much less cluttered if regional superscrips are omitted and so we shall omit them, and let the subscrips range over sectors and regions. For the Isard model, the information content F_I is

$$F_I = \sum_i \sum_j \frac{x_{ij}}{\sum\sum_{ij} x_{ij}} \log \left\{ \frac{\left(\frac{x_{ij}}{\sum\sum_{ij} x_{ij}} \right)}{\left(\frac{\sum_j x_{ij} \sum_i x_{ij}}{(\sum\sum_{ij} x_{ij})^2} \right)} \right\}$$

$$= \sum\sum_{ij} \frac{x_j}{K} s_{ij} a_{ij} \log \frac{K}{x_{i.}} s_{ij} a_{ij}$$

where $K = \sum\sum_{ij} x_{ij}$, x_j is the total output including final demand of commodity j, and $x_{i.}$ is the row sum of the i^{th} row of intermediate goods flows.

For the Chenery-Moses model, we get F_{CM} by a similar transformation.

$$F_{CM} = \sum_i \sum_j \frac{x_j}{K} (s_i a_{ij}) \log \frac{K}{x_{i.}} (s_i a_{ij})$$

where the terms have the same definitions as for the ones in F_I. The crucial fact required for the application of the Beckenbach-Bellman theorem is that s_i is the arithmetic mean of the s_{ij} for each row i. That is, $s_i = \left(\frac{1}{n}\right)\sum_{j=1}^{n} s_{ij}$; this by construction. Hence by the Beckenbach-Bellman theorem,[7]

$$F_I \geqq F_{CM}.$$

The Isard model captures these special contractural relationships and incorporates them into gross outputs generated by different final demands. In other words, Chenery and Moses felt that the averaging involved in their procedure had the advantage of not incorporating "noise" or very particular economic relationships which would not persist when the economic system was generating different gross outputs.

Their contention is reasonably persuasive, although it is not unlikely that the particular elements captured by the Isard procedure would persist under different final demands. Such elements as special market relationships between firms aggregated in these models to industries, the particular spatial distribution of firms, and the product heterogeneity involved both in statistically implementing interregional input-output models and in the outputs of actual firms can be adduced to cast doubt on the contention of Chenery and Moses concerning the superiority of their procedure.

3. Procedures for Introducing Variable Supply Coefficients in Interregional Input-Output Models

Chenery [2] points out how one could introduce responses to regional capacity constraints by varying supply coefficients in preassigned magnitudes. Similar approaches were developed in order to introduce increasing returns to scale in Leontief models. In this case, technical coefficients were varied [12]. In this section, we will review procedures for solving for activity levels with given final demands and show how the Chenery coefficient adjustment procedure can be applied to the revised Isard model.

We are familiar with the fact that with well-behaved, non-negative coefficient matrices, say A, and non-negative final demand sectors y, we can solve for non-negative activity levels, a vector x.[8] That is:

$$x = (I-A)^{-1} y$$

Moreover, if A has all characteristic roots with absolute values less than unity, then:

$$(I-A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots$$

Hence,

$$x = (I-A)^{-1} y = [I + A + A^2 + \ldots + A^n + \ldots] y$$

or

$$x = (I-A)^{-1} y = x_0 + x_1 + x_2 + \ldots + x_n + \ldots \qquad (5)$$

where $x_0 = Iy$, $x_1 = Ay$, $x_2 = A^2y$ etc. Equation (5) can be written:

$$x = (I-A)^{-1} y = x_0 + Ax_0 + Ax_1 + Ax_2 + \ldots + Ax_{n-1} + \ldots \qquad (6)$$

given the definitions for x_0, x_1, x_2 etc.

Now, equation (6) has some desirable qualities when considered as a computational scheme for solving for activity levels x, given final demands y. First, it requires no matrix inversion. Secondly, one can solve in steps, that is, determine x_0, then x_1, etc. One can test the intermediate sums, say $x_0 + x_1 + x_2$ in order to see if they exceed pre-assigned capacity constraints on any of the activity levels. If they have, then coefficients in matrix A can be changed to accommodate the new real world situation. If coefficients were adjusted at each step, then (6) would become:

$$x = x_0 + A_1 x_0 + A_2 x_1 + A_3 x_2 + \ldots + A_n x_{n-1} + \ldots$$

It is obvious that by abandoning the postulate of proportionality, we preclude the existence of a general solution where the vector of activity levels, x, are a linear transformation of the final demand vector y, the matrix of transformation being $(I-A)^{-1}$. However, we are able to build into our applied work the quality that our results will more closely approximate observable phenomena. This is at the sacrifice of much of the elegance of the basic Leontief model.

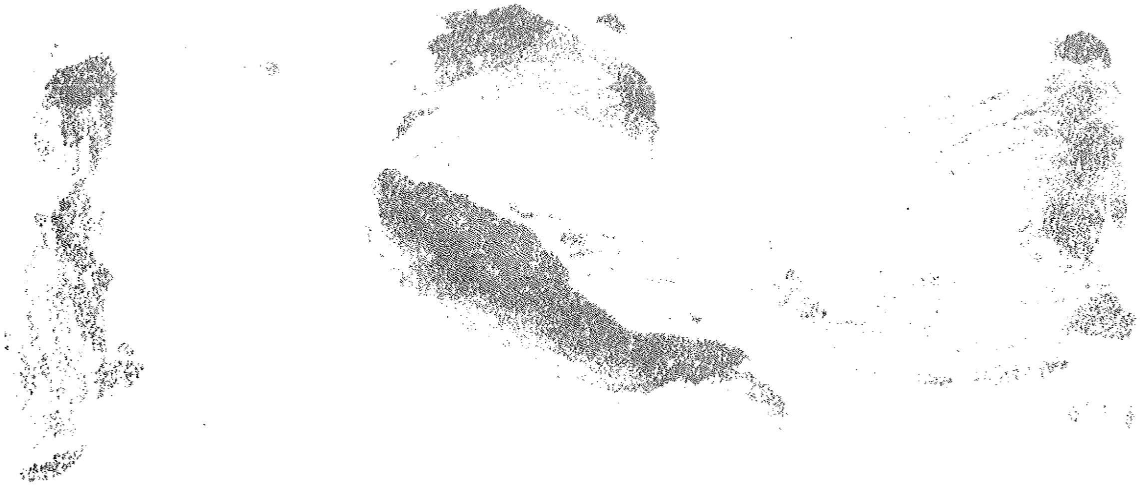
If we consider matrix A to be the Chenery-Moses coefficient matrix described above as ${}^{s}A^{k\ell}$ then we can easily introduce Chenery's adjustable coefficient procedure. Recall that the Chenery-Moses coefficient matrix was essentially a diagonal array of regional Leontief technical coefficient matrices with the rows of each matrix weighted by supply coefficients which summed to unity over all regions in the model. The matrix was nm by nm where n was the number of commodities in each region and m was the number of regions.

Chenery[9] suggested introducing pre-assigned capacity constraints into the program of solving for activity levels. If, in solving for x in steps as in (6), a capacity constraint was met, he suggested that the supply coefficient of the sending region for the industry at capacity be set at zero and the coefficients of the other supplying regions be adjusted in order to reflect the fact that they now were supplying the share which the "incapacitated" region was supplying before. The original Chenery-Moses coefficient matrix, say A_1, would have a different set of supply coefficients introduced for the relevant sector for the relevant receiving region and become then A_2, and the solution would be continued with matrix A_2 until another capacity constraint was met. The supply coefficients would be then altered in some predetermined way and A_3 would emerge.

Provided that the new matrices formed at each alteration have the same properties as the original coefficient matrix, this procedure will converge and give meaningful results. This is assured in the inter-regional input-output system since the column sums of our coefficient matrices are unchanged at each coefficient adjustment. Since ranges on the magnitude of the eigen values can be determined from the column sums of a matrix, we are thus assured of the convergence of our solution procedure with intermediate adjustments in supply coefficients.

The revised Isard model introduced in section 2 has a coefficient matrix with supply coefficients also. Thus adjustments in response to capacity constraints can be built into this model's solution if the series expansion outlined above is followed. The introduction of more general types of non-proportionalities than a once over capacity constraint adjustment is possible. The non-linear nature of the real world can be more closely approximated with the step-functions implicit in these adjustment procedures than with the linear functions inherent in the fixed coefficient models outlined above. It might be remarked that the original Isard model lacked a system of coefficients that was easily amenable to adjustment along the lines described above. Since there was only one coefficient for each intersectoral interregional flow relating both trade and technical phenomena, a rerouting of trade flows would involve an adjustment in the single coefficient in such a way as to leave unknown the part of the adjustment which was in trade relations and the part that was in technical relations. The introduction of flexibility into the model would be more difficult than with the revised Isard or Chenery-Moses models.

Not any vector of final demands elicits a meaningful vector of gross output in these solution sequences involving variable coefficients and capacity constraints. The capacity constraints associated with various coefficients impose upper bounds on the size of the gross outputs which an economy can generate. This is of course how actual economies are and it is the behavior of these actual economies which we want to simulate. That is, when a producer expands production and requires additional inputs which his supplier is temporarily lacking, he simply orders from another supplier. In flow terms, the supply coefficient of the first firm (or sector in a region) becomes zero and the new one becomes positive. The restriction on the final demands compatible with the structure is, then, not a fault of the procedures outlined above.



The case of theorem illustrated says that the areas

$$ox_2cy_2 + ox_1ay_1 \geqq 2\left(0\ \left(\frac{x_1 + x_2}{2}\right)\ b\ y_{\overline{x}}\right)$$

a result by no means intuitively obvious.

7. In our application of the theorem

$$f(s) = us \log ws \qquad u, w > 0$$

$$f''(s) = \frac{u}{s} > 0.$$ Thus the theorem applies since f(s) is convex.

8. See for example Dorfman, Samuelson, and Solow [5], pp. 254-257.

9. See Chenery [2].

[12] Matuszewski, T.I., "Un Systeme Rectangulaire D'Exchange Inter-Industries à Rendements Non-proportionnels", unpublished paper read to the Econometric Society, Rome, 1965.

[13] Miller, Ronald E., "Interregional Feedbacks in Input-Output Models: Some Experimental Results", Western Economic Journal, Vol. VII, No. 1, March 1969, 41-50.

[14] Miller, Ronald E., "Interregional Feedbacks in Input-Output Models: Some Experimental Results", Papers, Regional Science Association, 17, 1966, 105-25.

[15] Moses, Leon N., "The Stability of Interregional Trading Patterns and Input-Output Analysis", American Economic Review, Vol. XLV, No. 5, December, 1955, 803-832.

[16] Theil, Henri, Economics and Information Theory, Chicago: Rand-McNally, 1967.