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Abstract

We examine forecasting performance of the recent fractionally cointegrated vector autoregressive (FCVAR) model. We use daily polling data of political support in the United Kingdom for 2010-2015 and compare with popular competing models at several forecast horizons. Our findings show that the four variants of the FCVAR model considered are generally ranked as the top four models in terms of forecast accuracy, and the FCVAR model significantly outperforms both univariate fractional models and the standard cointegrated VAR (CVAR) model at all forecast horizons. The relative forecast improvement is higher at longer forecast horizons, where the root mean squared forecast error of the FCVAR model is up to 15% lower than that of the univariate fractional models and up to 20% lower than that of the CVAR model. In an empirical application to the 2015 UK general election, the estimated common stochastic trend from the model follows the vote share of the UKIP very closely, and we thus interpret it as a measure of Euro-skepticism in public opinion rather than an indicator of the more traditional left-right political spectrum. In terms of prediction of vote shares in the election, forecasts generated by the FCVAR model leading into the election appear to provide a more informative assessment of the current state of public opinion on electoral support than the hung government prediction of the opinion poll.

Keywords: forecasting, fractional cointegration, opinion poll data, vector autoregressive model.

1 Introduction

In this paper we investigate the forecasting performance of the recently developed fractionally cointegrated vector autoregressive (FCVAR) model of Johansen (2008) and Johansen

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and Nielsen (2012) relative to a portfolio of competing models at various forecast horizons. The FCVAR model generalizes the concept of cointegration, and in particular generalizes Johansen's (1995) cointegrated VAR (CVAR) model to fractionally integrated time series, and hence allows estimating long-run equilibrium relationships between fractional time series.

The FCVAR model is a very recently developed statistical model, and it is therefore of particular interest to examine the gains this model can deliver for the purposes of forecasting. The choice of data set for applying the model should reflect a current and relevant issue for forecasting. A prominent example is the desire to predict political support and election vote share outcomes. This paper addresses this task by applying the FCVAR model to a novel data set which is comprised of polling results of political support in the United Kingdom for the period 2010–2015 at the business-daily observation frequency. The fractional integration behavior of political opinion polling data has been well established in the literature, albeit for time series at lower frequencies (monthly, quarterly), e.g. Box-Steffensmeier and Smith (1996), Byers et al. (1997, 2000, 2002), Dolado et al. (2002, 2003), and Jones et al. (2014). For a more general reference on fractional integration methods in political time series data, see Box-Steffensmeier and Tomlinson (2000) and Lebo et al. (2000); both in the special issue of *Electoral Studies* edited by Lebo and Clarke (2000). It therefore appears natural to apply a fractional time series model such as the FCVAR to model and forecast political opinion polls.

The industry standard for measuring the current state of political support is through opinion polling. The demand for polling and survey methodology is largely driven by the clients desire to form an accurate understanding of the current state of opinion on a particular question. The poll evidence then serves as an input into the decision making process. When polls are conducted at regular intervals, it seems natural to use a statistical model to extract the full potential of the information contained in these time series of poll results by using them to forecast public opinion beyond the most recent poll date. However, long time series of poll data are scarce, and, to the best of our knowledge, all previous studies that have analyzed time series of political opinion polls have used data observed at the monthly frequency or lower, see, e.g., above references. Authors of these studies have noted that an ideal data set would have all observations contained within a single government regime spanning only one political cycle, while providing a large enough sample to conduct meaningful statistical analysis. The data set used in this paper fully satisfies both desired properties: it spans the entire UK political cycle following the 2010 UK general election, it is conducted at a high observation frequency (business-daily), and it is very recent and on-going, and hence very relevant also for forecasting poll standings which can be viewed as the predicted vote shares for each political party in an election.

The long time series provided in our data set facilitates forecast accuracy evaluation using several forecast evaluation procedures. In particular, it allows the formation of a large number of training sets from which each statistical model can produce forecasts. We apply two standard procedures to assess forecast accuracy: the rolling window and recursive forecasting schemes. The main distinction between the two schemes is how they select the training sets used for estimating the models. The rolling window scheme uses a fixed training set length (commonly referred to as a window) that moves across the data set, and the recursive scheme uses an expanding training set length with a fixed start date. The portfolio of models we consider consists of eight statistical models, four of which are variants of the FCVAR model. These are then evaluated on their forecasting ability relative to a group of four popular competing models. Among the latter, the CVAR model serves as the main multivariate benchmark model. The simple ARFIMA$(0, d, 0)$ and the more general ARFIMA(p, d, q) models serve as the fractional univariate benchmarks, where the former was found by, e.g., Byers et al. (1997) and Dolado et al. (2002), to fit (monthly) UK polling data well. Finally, we include the ARMA(p, q) model as the classical univariate benchmark. Forecast accuracy is assessed at seven out-of-sample forecast horizons: 1, 5, 10, 15, 20, 25, and 50 steps ahead.

The forecasting analysis in this paper shows that the FCVAR model delivers valuable gains in predicting political support. Both forecasting schemes agree on this finding. The accuracy of forecasts generated by the FCVAR model is better than all multivariate and univariate models in the portfolio, and overall the four variants of the FCVAR model are ranked as the four top performing models. Not only do they perform better relative to the other models, but the forecasting performance of all FCVAR variants is within very close range of each other. When compared to the multivariate benchmark model, the FCVAR model substantially outperforms the CVAR model in 56 of 56 cases, and the relative forecast improvement is highest at the 15–50 steps ahead forecast horizons, where the root mean squared forecast error (RMSFE) of the FCVAR model is up to 20% lower than that of the CVAR benchmark model. Previous literature, as cited above, has documented the superiority of fractional (ARFIMA) models for forecasting polling data. Compared to this more difficult benchmark, the RMSFE of the FCVAR model is as much as 15% lower, and the advantage of the FCVAR model again appears to be increasing with the forecast horizon.

As an empirical application, we apply the FCVAR model to the full data set, comprising observations until the day before the 2015 UK general election. We first consider estimation of the model, with interpretations of both the estimated cointegrating relations and estimated common stochastic trend. It appears that the latter can be interpreted a measure of Euro-skepticism, rather than an indicator of the more traditional left-right political spectrum, reflecting public opinion and debate in the sampling period which was to a great extent focused on the European Union question. Specifically, the estimatoed common stochastic trend from the model follows the vote share of the UKIP (as measured by the polls) very closely throughout the sampling period. Finally, we also consider prediction of vote shares for the election from a range of forecast horizons. In this application, we find that the FCVAR model has advantages for predicting vote shares and complements the industry standard of basing predictions on the latest opinion poll standings.

The remainder of the paper is structured as follows. Section 2 introduces the concept of fractional integration, the classic arguments for fractional integration in polling data, and describes our data set. In Section 3 we describe the FCVAR methodology and Section 4 presents the main forecasting analysis. Section 5 presents the empirical application to the 2015 UK general election, and finally Section 6 provides some concluding remarks.

2 Fractional integration, polling data, and summary statistics

In important early contributions, Box-Steffensmeier and Smith (1996) and Byers et al. (1997, 2002) show that political popularity, as measured by public opinion polls, can be modeled as fractional time series processes. The fractional (or fractionally integrated or just integrated) time series models are based on the fractional difference operator,

$$\Delta^d X_t = \sum_{n=0}^{\infty} \pi_n(-d) X_{t-n}, \tag{1}$$

where the fractional coefficients $\pi_n(u)$ are defined in terms of the binomial expansion $(1-z)^{-u} = \sum_{n=0}^{\infty} \pi_n(u) z^n$, i.e.,

$$\pi_0(u) = 1 \text{ and } \pi_n(u) = \frac{u(u+1)\cdots(u+n-1)}{n!}. \tag{2}$$

For details and for many intermediate results regarding this expansion and the fractional coefficients, see, e.g., Johansen and Nielsen (2015, Appendix A). Efficient calculation of fractional differences, which we apply in our analysis, is discussed in Jensen and Nielsen (2014).

With the definition of the fractional difference operator in (1), a time series X_t is said to be fractional of order d, denoted $X_t \in I(d)$, if $\Delta^d X_t$ is fractional of order zero, i.e. if $\Delta^d X_t \in I(0)$. The latter property can be defined in the frequency domain as having spectral density that is finite and non-zero near the origin or in terms of the linear representation coefficients if the sum of these is non-zero and finite, see, e.g., (Johansen and Nielsen, 2012, p. 2672). An example of an $I(0)$ process is the stationary and invertible ARMA model.

The standard reasoning for political opinion poll series being fractional relies on Robinson's (1978) and Granger's (1980) aggregation argument, and can briefly be described as follows. Suppose individual level voting or polling behavior is governed by the (possibly binary) autoregressive process

$$x_{i,t} = \delta_{i,1} + \delta_{i,2} x_{i,t-1} + u_{i,t}, \tag{3}$$

where $i = 1, \ldots, N$ denotes individuals and $t = 1, 2, \ldots$ as usual denotes time. The important point here is that the autoregressive coefficients $\delta_{i,2}$ differ across individuals. Some individuals have coefficients $\delta_{i,2} \approx 0$ and are referred to as "floating" voters, whereas others have coefficients $\delta_{i,2} \approx 1$ and are referred to as "committed" voters.[1] If it is assumed that the distribution of $\delta_{i,2}$ across individuals in the population follows a $\text{Beta}(u, v)$ distribution, then the aggregate vote share or polling share $X_t = N^{-1} \sum_{i=1}^{N} x_{i,t}$ is fractionally integrated of order $d = 1-v$ when N is large, i.e., $X_t \in I(1-v)$. For more details, see Box-Steffensmeier and Smith (1996) or Byers et al. (1997, 2002).

The above theoretical argument in favor of modeling opinion poll data as fractional time series has been supported in empirical work by a large number of authors. For example, Box-Steffensmeier and Smith (1996) estimate fractional models for US data, Byers et al. (1997) and Dolado et al. (2002) analyze UK data, Dolado et al. (2003) analyze Spanish data, Byers et al. (2000) analyze data for eight countries, and Jones et al. (2014) analyze Canadian data. All find strong evidence in support of fractional integration with estimates of d around $0.6 - 0.8$. In addition, Byers et al. (2007) analyze an updated version of the sample in Byers

[1] "Floating" voters are defined as those who do not have a strong alliance to one party and may be more easily swayed by current events, media, etc., and "committed" voters, on the other hand, are those who consistently vote for a particular party and are less inclined to change their voting preference.

Table 1: Summary statistics (data in percentage)

Series	Mean	SD	Min	Max	Skew	Kurt	Start date	End date	Obs
Conservative (CP)	34.67	3.29	27	44	0.76	2.89	2010/05/14	2015/05/06	1227
Labour (LP)	39.39	3.32	30	45	−0.44	2.33	2010/05/14	2015/05/06	1227
Lib. Dem. (LD)	9.41	1.87	5	21	1.42	7.75	2010/05/14	2015/05/06	1227
UKIP (IP)	11.65	2.82	5	19	−0.18	2.30	2012/04/16	2015/05/06	771
Green (GP)	3.40	1.75	1	10	0.91	2.85	2012/06/18	2015/05/06	729

Notes: The table presents summary statistics for the polling data (expressed in percentages). The statistics presented are the sample mean, standard deviation, minimum, maximum, skewness, kurtosis, start and end dates, and number of observations.

et al. (1997) and show that the change to phone interviews had no effect on estimates of d and did not appear to constitute a structural break.

The aggregate polling data set we analyze is from the on-going YouGov daily poll of voting intention for political parties in the United Kingdom. Each business day survey participants are asked the question:

> "If there were a general election tomorrow, which party would you vote for? Conservative, Labour, Liberal Democrat, Scottish Nationalist/Plaid Cymru, some other party, would not vote, don't know?"

If the respondent replied "some other party", he/she would then be presented with a list of prompted alternatives, at which point they would be able to select United Kingdom Independence Party, Green Party, and so on.

This poll, and hence the data series, is business-daily and began on May 14$^{\text{th}}$, 2010 (so shortly after the 55th UK general election of 2010 held on May 6$^{\text{th}}$). With the next general election held on May 7$^{\text{th}}$, 2015, this on-going survey provides a long series of polling data all within the tenure of a single government regime. The results presented in this paper use May 6$^{\text{th}}$, 2015, as the end date, which was the last day the poll was conducted before the election, for a total of 1227 business-daily observations.[2] Previous empirical studies of political support have analyzed monthly and quarterly data spanning several decades and election cycles. Thus, this daily frequency data set is particularly attractive for estimating models within a single election cycle.

Our analysis focuses on the three major political parties in the United Kingdom: the Conservative Party (CP) and the Liberal Democrats (LD), which together constitute the British government over the sample period (coalition formed on May 12$^{\text{th}}$, 2010), and the Labour Party (LP)—the official opposition. Until April 15$^{\text{th}}$, 2012, YouGov reported all outcomes from "some other party" in the residual time series, so that no distinction was made between, e.g., United Kingdom Independence Party (UKIP or just IP) and the Green Party (GP).

On April 16$^{\text{th}}$, 2012, YouGov changed the way they reported the outcomes of their polls in their UK Polling Report, and started reporting the United Kingdom Independence Party

[2] Starting April 7$^{\text{th}}$, 2015, i.e. for the last month before the 2015 election, YouGov changed their polling frequency to daily, including non-business days. We ignore this minor change, as well as the break in polling over the Christmas holiday, and in our analysis we treat all observations as equi-distant as is standard.

Table 2: Summary statistics (logit transformed data)

							ELW(m)		
Series	Mean	SD	Min	Max	Skew	Kurt	$m = T^{0.6}$	$m = T^{0.7}$	$m = T^{0.8}$
Conservative (CP)	−0.63	0.14	−0.99	−0.24	0.67	2.81	0.79	0.70	0.65
Labour (LP)	−0.43	0.14	−0.84	−0.20	−0.50	2.41	0.88	0.71	0.64
Lib. Dem. (LD)	−2.28	0.20	−2.94	−1.32	0.50	4.59	0.85	0.66	0.64
UKIP (IP)	−2.05	0.29	−2.94	−1.45	−0.59	2.54	0.75	0.69	0.62
Green (GP)	−3.47	0.51	−4.59	−2.19	0.21	2.24	0.73	0.63	0.48

Notes: The table presents summary statistics for the logit transform of the polling data. The start and end dates are the same as in Table 1, as are the statistics presented, with the addition of ELW(m), which denotes the exact local Whittle estimator of Shimotsu and Phillips (2005) with bandwidth parameter m, whose asymptotic standard error is $(4m)^{-1/2}$.

as a separate time series (rather than it being included in the residual category). On June 18$^{\text{th}}$, 2012, they also started reporting the Green Party as a separate series rather than as part of the residual category. This facilitates extending the analysis to four or even five political parties, albeit for a substantially shorter data set that spans only the second half of the 2010 to 2015 political cycle in the UK. However, unlike the three major political parties, neither the UKIP nor the Green Party are stated explicitly as choices in the survey question posed to the poll participants as quoted above, and for a survey respondent to indicate that they wish to vote for either of these parties, they must first choose "some other party" after which they are presented with a list of prompted alternatives, at which point they are able to select UKIP, Green Party, and so on. Although we include the UKIP and the Green Party in our empirical application to the 2015 UK general election, this characteristic of the survey, together with the substantially shorter sample size, leads us to exclude these two parties from our main forecasting analysis. In Table 1 we present some summary statistics for the polling data, where these are given in percentage vote shares.

The analysis proceeds after converting the polling data to log-odds. This is done to map variables on the unit interval into the real line, in order to use error terms with unbounded support in our models (for more details and background, see e.g. Byers et al. (1997)). The log-odds or logit transformation for a variable $Y_t \in (0, 1)$ is

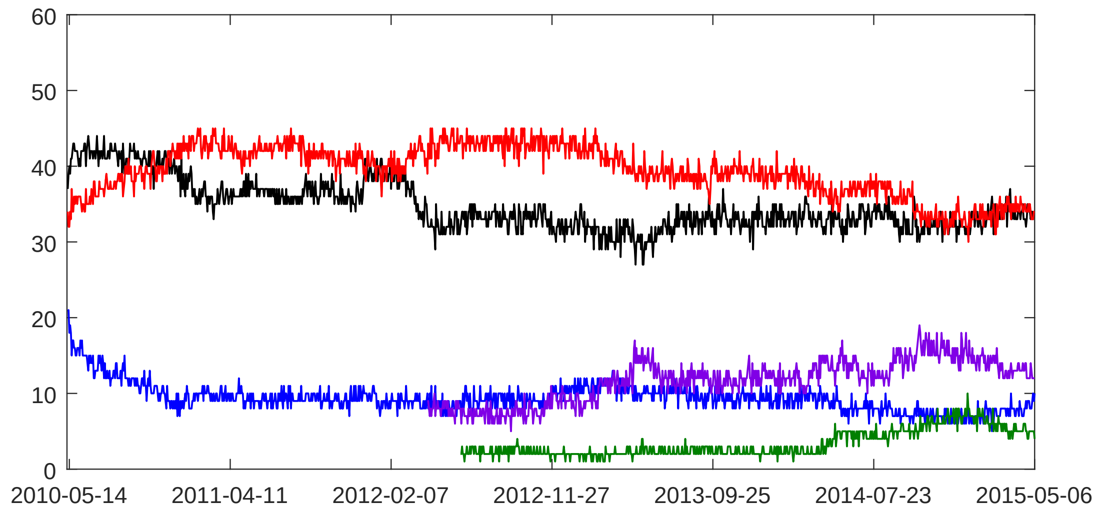
$$y_t = \log\left(\frac{Y_t}{1 - Y_t}\right),$$

where Y_t is the original series and y_t is the logit transformed series with support $(-\infty, \infty)$. Table 2 presents summary statistics for the logit transformed data. The original data and the logit transform of the data are shown in Figures 1(a) and 1(b), respectively.

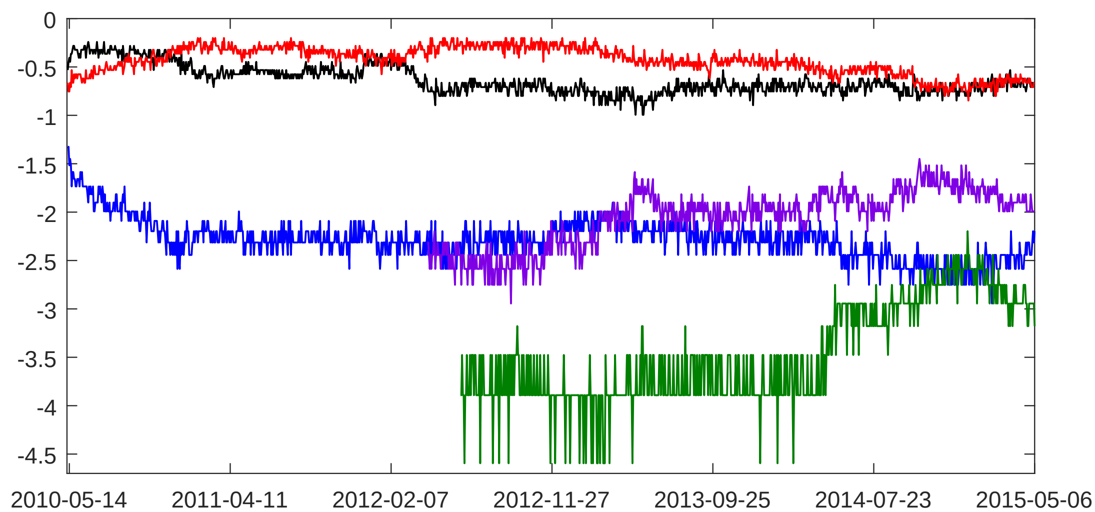
As mentioned earlier, the fractional integration characteristic of political opinion polling data has been well established in the literature for monthly and quarterly data. To add to this body of literature, we computed the sample autocorrelation functions and estimated spectral density functions of each of the three series, and we display these in Figures 2(a) and 2(b), respectively. For a fractionally integrated time series, we would expect that the sample autocorrelation functions decay very slowly (hyperbolically, as opposed to exponentially)

Figure 1: Time series plots of data 2010-05-14 – 2015-05-06

(a) Percentage



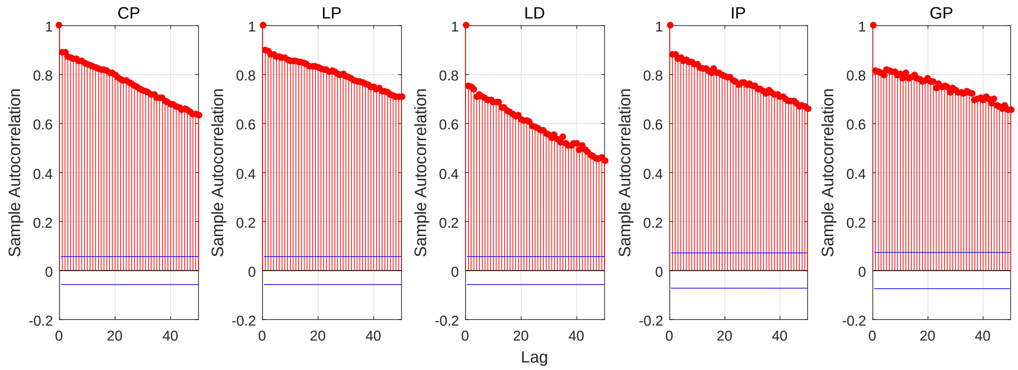
(b) Logit transform



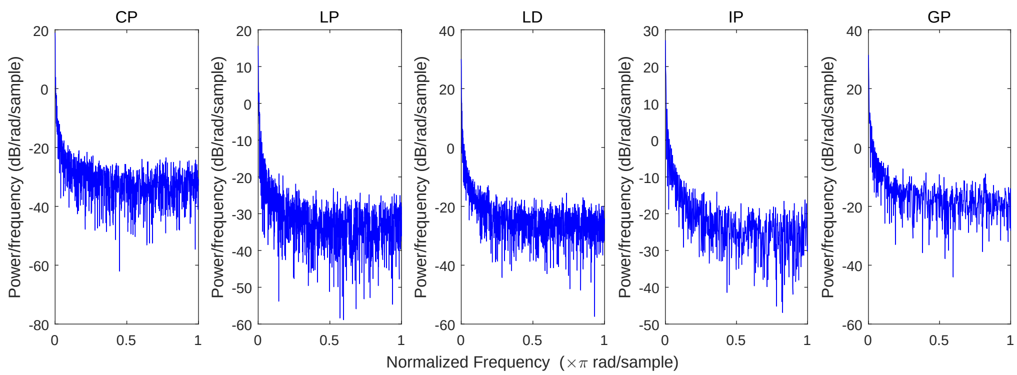
Note: Black line is Conservative (CP), red line is Labour (LP), blue line is Liberal Democrats (LD), purple line is UKIP (IP), and green line is the Green Party (GP).

Figure 2: Serial dependence structure

(a) Sample autocorrelation functions



(b) Estimated spectral density functions



and that the estimated spectral density functions have mass concentrated near the origin. See, e.g., Baillie (1996) for a review covering these properties. Indeed, both these features appear clearly in Figure 2.

Finally, for each univariate series we have computed semiparametric estimates of the fractional integration parameter, d, that do not rely on the specification of a parametric model or lag structure. Specifically, we computed the exact local Whittle (ELW) estimates of Shimotsu and Phillips (2005), which are displayed in the last three columns in Table 2 for three different choices of bandwidth parameter, $m = T^{0.6}, m = T^{0.7}$, and $m = T^{0.8}$ (when this is not an integer, the integer part of the result is used). In each case, the asymptotic standard error of the estimate is $(4m)^{-1/2}$, so, for example, when $m = T^{0.6}$ the asymptotic standard error of the estimate is 0.059 for the first three series where $T = 1227$. The results from the ELW estimates suggest that each series is fractionally integrated with a fractional integration parameter that is statistically significantly less than one.

More generally, the ELW estimates in Table 2 are in line with estimates from the literature, e.g., Box-Steffensmeier and Smith (1996), Byers et al. (1997, 2000), Dolado et al. (2002, 2003), and Jones et al. (2014), where estimates around $0.6 - 0.8$ are commonly found for polling data at the monthly and quarterly frequencies. An important property of fractional processes is self-similarity, in other words that the autocorrelation structure is independent of sampling frequency (unlike exponential decay models such as those of the ARMA type). The fact that the fractional parameters estimated here using daily data are generally very close to estimates obtained using monthly and quarterly data is thus another important reason for favouring the fractional approach. The evidence presented here clearly shows fractional integration characteristics for political opinion polls at the daily frequency, as suggested by both the self-similarity property and the theoretical (aggregation-based) arguments discussed earlier.

3 Statistical methodology: FCVAR model

Our analysis applies the FCVAR model of Johansen (2008) and Johansen and Nielsen (2012). This model is a generalization of Johansen's (1995) CVAR model to allow for fractionally integrated and fractionally cointegrated time series.

3.1 Variants of the FCVAR model and interpretations

For a time series Y_t of dimension p, the well-known CVAR model with a so-called "restricted constant" term is given in error correction form as

$$\Delta Y_t = \alpha(\beta' Y_{t-1} + \rho') + \sum_{i=1}^{k} \Gamma_i \Delta Y_{t-i} + \varepsilon_t = \alpha L(\beta' Y_t + \rho') + \sum_{i=1}^{k} \Gamma_i L^i \Delta Y_t + \varepsilon_t, \tag{4}$$

where, as usual, ε_t is a p-dimensional independent and identically distributed error term with mean zero and covariance matrix Ω. The simplest way to derive the FCVAR model from the CVAR model is to replace the difference and lag operators, Δ and $L = 1 - \Delta$, in (4) by their fractional counterparts, Δ^b and $L_b = 1 - \Delta^b$, respectively, and apply the resulting model to $Y_t = \Delta^{d-b} X_t$. We then obtain[3]

$$\Delta^d X_t = \alpha \Delta^{d-b} L_b(\beta' X_t + \rho') + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t, \tag{5}$$

where Δ^d is the fractional difference operator, and $L_b = 1 - \Delta^b$ is the fractional lag operator.[4]

Model (5) nests Johansen's (1995) CVAR model in (4) as the special case $d = b = 1$. Some of the parameters are well-known from the CVAR model and these have the usual interpretations also in the FCVAR model. The most important of these are the long-run parameters α and β, which are $p \times r$ matrices with $0 \le r \le p$, and ρ, which is an r-vector. The rank r is termed the cointegration, or sometimes cofractional, rank. The columns of β constitute the r cointegration (cofractional) vectors, such that $\beta' X_t$ are the cointegrating combinations of the variables in the system, i.e. the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables. The restricted constant term ρ is interpreted as the mean level of the long-run equilibria $\beta' X_t$ when these are stationary. The short-run dynamics of the variables are governed by the parameters $(\Gamma_1, \ldots, \Gamma_k)$ in the autoregressive augmentation.

[3] In principle, the restricted constant term should be included as $\rho' \pi_t(1)$, where $\pi_t(1)$ denotes the coefficient in (1). This is mathematically convenient, but makes no difference in terms of the practical implementation because the infinite summation in (1) needs to be truncated in practice.

[4] Both the fractional difference and fractional lag operators are defined in terms of their binomial expansion in the lag operator, L, as in (1). Note that the expansion of L_b has no term in L^0 and thus only lagged disequilibrium errors appear in (5).

The FCVAR model has two additional parameters compared with the CVAR model, namely the fractional parameters d and b. Here, d denotes the fractional integration order of the observable time series, while the parameter b determines the degree of fractional cointegration, i.e. the reduction in fractional integration order of $\beta' X_t$ compared to X_t itself. Both fractional parameters are estimated jointly with the other parameters, see Section 3.2. The FCVAR model (5) thus has the same main structure as the standard CVAR model (4), in that it allows for modeling of both cointegration and adjustment towards equilibrium, but is more general since it accommodates fractional integration and cointegration.

We note that the fractional difference as defined in (1) is an infinite series, but any observed sample will include only a finite number of observations. This makes calculation of the fractional differences as defined in (1) impossible. In practice, therefore, the summation in (1) would need to be truncated at $n = t-1$, and the bias introduced by application of such a truncation is analyzed by Johansen and Nielsen (2015) using higher-order expansions in a simpler model. They show, albeit in a simpler model, that this bias can be avoided by including a level parameter, μ, that shifts each of the series by a constant. We follow this suggestion and also consider the unobserved components formulation

$$X_t = \mu + X_t^0, \quad \Delta^d X_t^0 = \alpha \Delta^{d-b} L_b \beta' X_t^0 + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b^i X_t^0 + \varepsilon_t, \tag{6}$$

from which we easily derive the model

$$\Delta^d (X_t - \mu) = \alpha \beta' \Delta^{d-b} L_b (X_t - \mu) + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b^i (X_t - \mu) + \varepsilon_t. \tag{7}$$

The formulation (7) includes the restricted constant, which may be obtained as $\rho' = -\beta' \mu$. More generally, the level parameter μ in (7) is meant to accommodate a non-zero starting point for the first observation on the process, i.e., for X_1; see Johansen and Nielsen (2015).

Our forecasting analysis applies the versions of the FCVAR model given in (5) and (7) and we provide comparisons with the CVAR model in (4) as our multivariate benchmark model. Following the work of Jones et al. (2014) we also consider the sub-models obtained by setting $d = b$ in (5) and (7), which results in disequilibrium errors that are $I(0)$. Thus, the four variants of the FCVAR model that we consider are

1. $\text{FCVAR}_{d,b,\rho}$: model (5) with restricted constant ρ and fractional parameters d and b,
2. $\text{FCVAR}_{d,b,\mu}$: model (7) with level parameter μ and fractional parameters d and b,
3. $\text{FCVAR}_{d=b,\rho}$: model (5) with restricted constant ρ and fractional parameter $d = b$,
4. $\text{FCVAR}_{d=b,\mu}$: model (7) with level parameter μ and fractional parameter $d = b$.

In each model the fractional parameters are estimated as described in the next subsection, possibly with the restriction $d = b$ imposed as appropriate.

3.2 Maximum likelihood estimation

The models (5) and (7) are estimated by conditional maximum likelihood. It is assumed that a sample of length $T+N$ is available on X_t, where N denotes the number of observations used for conditioning, for details see Johansen and Nielsen (2012, 2015). For the standard CVAR model, the arguments are well-known and conditioning on the first $N \geq k+1$ observations leads to reduced rank regression estimation. For the FCVAR model, we proceed similarly by maximizing the conditional log-likelihood function

$$\log L_T(\lambda) = -\frac{Tp}{2}(\log(2\pi)+1) - \frac{T}{2}\log\det\left\{T^{-1}\sum_{t=N+1}^{T+N}\varepsilon_t(\lambda)\varepsilon_t(\lambda)'\right\}, \tag{8}$$

where the residuals are defined as

$$\varepsilon_t(\lambda) = \Delta^d X_t - \alpha\Delta^{d-b}L_b(\beta' X_t + \rho') - \sum_{i=1}^{k}\Gamma_i\Delta^d\ L_b^i X_t, \quad \lambda = (d, b, \alpha, \beta, \Gamma_i, \rho), \tag{9}$$

for model (5), and hence also for sub-models of model (5), such as that obtained under the restriction $d = b$. For model (7) the residuals are

$$\varepsilon_t(\lambda) = \Delta^d(X_t-\mu) - \alpha\beta'\Delta^{d-b}L_b(X_t-\mu) - \sum_{i=1}^{k}\Gamma_i\Delta^d\ L_b^i(X_t-\mu), \quad \lambda = (d, b, \alpha, \beta, \Gamma_i, \mu), \tag{10}$$

and similarly for sub-models of model (7).

It is shown in Johansen and Nielsen (2012) how, for fixed (d, b), the estimation of model (5) reduces to regression and reduced rank regression as in Johansen (1995). In this way the parameters $(\alpha, \beta, \Gamma_i, \rho)$ can be concentrated out of the likelihood function, and numerical optimization is only needed to optimize the profile likelihood function over the two fractional parameters, d and b. In model (7) we can similarly concentrate the parameters $(\alpha, \beta, \Gamma_i)$ out of the likelihood function resulting in numerical optimization over (d, b, μ), thus making the estimation of model (7) somewhat more involved numerically than that of model (5).

The asymptotic analysis of the FCVAR model is provided in Johansen and Nielsen (2012). For model (5), Johansen and Nielsen (2012) show that asymptotic theory is standard when $b < 0.5$, and for the case $b > 0.5$ asymptotic theory is non-standard and involves fractional Brownian motion of type II. Specifically, when $b > 0.5$, Johansen and Nielsen (2012) show that under i.i.d. errors with suitable moment conditions, the conditional maximum likelihood parameter estimates $(\hat{d}, \hat{b}, \hat{\alpha}, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_k)$ are asymptotically Gaussian, while $(\hat{\beta}, \hat{\rho})$ are locally asymptotically mixed normal. These results allow asymptotically standard (chi-squared) inference on all parameters of the model, including the cointegrating relations and orders of fractionality, using quasi-likelihood ratio tests. As in the CVAR model, see Johansen (1995), the same results hold for the same parameters in the model (7), whereas the asymptotic distribution theory for the remaining parameter, μ, is currently unknown.

Likelihood ratio (trace-type) tests for cointegration rank can be calculated as well, and hypotheses on the cointegration rank can be tested in the same way as in the CVAR model. In the FCVAR model, the asymptotic distribution of the tests for cointegration rank depends on the unknown (true value of the) scalar parameter b, which complicates empirical analysis compared to the CVAR model. However, the distribution can be simulated on a case-by-case basis, or the computer programs by MacKinnon and Nielsen (2014) can be used to obtain either critical values or P values for the rank test. The calculation of maximum likelihood estimators and test statistics is discussed in detail in Johansen and Nielsen (2012) and Nielsen and Popiel (2016), with the latter providing Matlab computer programs that we apply in our empirical analysis.

3.3 Forecasting from the FCVAR model

We now discuss how to forecast the (logit transformed) polling data, that is X_t, from the FCVAR model (since the CVAR model is a special case obtained as $d = b = 1$, forecasts from that model are derived in the same way). Because the model is autoregressive, the best (minimum mean squared error) linear predictor takes a simple form and is relatively straightforward to calculate. We first note that

$$\Delta^d(X_{t+1} - \mu) = X_{t+1} - \mu - (X_{t+1} - \mu) + \Delta^d(X_{t+1} - \mu) = X_{t+1} - \mu - L_d(X_{t+1} - \mu)$$

and then rearrange (7) as

$$X_{t+1} = \mu + L_d(X_{t+1} - \mu) + \alpha\beta'\Delta^{d-b}L_b(X_{t+1} - \mu) + \sum_{i=1}^{k}\Gamma_i\Delta^d L_b^i(X_{t+1} - \mu) + \varepsilon_{t+1}. \quad (11)$$

Since $L_b = 1-\Delta^b$ is a lag operator, so that $L_b^i X_{t+1}$ is known at time t for $i \geq 1$, this equation can be used as the basis to calculate forecasts from the model.

We let conditional expectation given the information set at time t be denoted $E_t(\cdot)$, and the best (minimum mean squared error) linear predictor forecast of any variable Z_{t+1} given information available at time t be denoted $\hat{Z}_{t+1|t} = E_t(Z_{t+1})$. Clearly, we then have that the forecast of the innovation for period $t+1$ at time t is $\hat{\varepsilon}_{t+1|t} = E_t(\varepsilon_{t+1}) = 0$, and $\hat{X}_{t+1|t}$ is then easily found from (11). Inserting coefficient estimates based on data available up to time t, denoted[5] $(\hat{d}, \hat{b}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_k)$, we have that

$$\hat{X}_{t+1|t} = \hat{\mu} + L_{\hat{d}}(X_{t+1} - \hat{\mu}) + \hat{\alpha}\hat{\beta}'\Delta^{\hat{d}-\hat{b}}L_{\hat{b}}(X_{t+1} - \hat{\mu}) + \sum_{i=1}^{k}\hat{\Gamma}_i\Delta^{\hat{d}} L_{\hat{b}}^i(X_{t+1} - \hat{\mu}). \quad (12)$$

This defines the one-step ahead forecast of X_{t+1} given information at time t.

Multi-period ahead forecasts can be generated recursively. That is, to calculate the h-step ahead forecast, we first generalize (12) as

$$\hat{X}_{t+j|t} = \hat{\mu} + L_{\hat{d}}(\hat{X}_{t+j|t} - \hat{\mu}) + \hat{\alpha}\hat{\beta}'\Delta^{\hat{d}-\hat{b}}L_{\hat{b}}(\hat{X}_{t+j|t} - \hat{\mu}) + \sum_{i=1}^{k}\hat{\Gamma}_i\Delta^{\hat{d}} L_{\hat{b}}^i(\hat{X}_{t+j|t} - \hat{\mu}), \quad (13)$$

where $\hat{X}_{s|t} = X_s$ for $s \leq t$. Then forecasts are calculated recursively from (13) for $j = 1, 2, \ldots, h$ to generate h-step ahead forecasts, $\hat{X}_{t+h|t}$.

Clearly, one-step ahead and h-step ahead forecasts for the model (5) with a restricted constant term instead of the level parameter can be calculated entirely analogously. We will apply the forecasts (12) and (13) for both models (5) and (7) in our analysis below for several forecast horizons, h.

4 Forecasting analysis

In this section we present and discuss our main forecasting analysis. In the first subsection we briefly discuss some preliminary estimation results to introduce and compare the different variants of the FCVAR model, and the next two subsections then present the forecasting procedure and the corresponding results.

[5] To emphasize that these estimates are based on data available at time t, they could be denoted by a subscript t. However, to avoid cluttering the notation we omit this subscript and let it be understood in the sequel.

4.1 Preliminary estimation results

Before we move on to the forecasting analysis, we briefly discuss some preliminary estimation results which are presented in Table 3 for the three-party data set that includes CP, LP, and LD. In Section 5.1 we consider a more detailed analysis of estimation results from the FCVAR model for a five-party data set. For now, the results in Table 3 will serve only as illustrations of the FCVAR model variants.

Each panel of Table 3 shows FCVAR estimation results for one of the four variants considered. Specifically, Panel A shows results for model (5) with two fractional parameters, (d, b), and a restricted constant term, ρ, Panel B shows results for model (7) with two fractional parameters, (d, b), and the level parameter, μ, while Panels C and D show the corresponding results for the models with only one fractional parameter, $d = b$. In addition, the multivariate Portmanteau Q-test for serial correlation up to order six in the residuals is reported as $Q_{\hat{\varepsilon}}$ and the maximized log-likelihood value is reported as $\log(\mathscr{L})$. Standard errors are in parentheses below $\hat{d}$ and $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}}$.

Since it is infeasible to present results for all the many models and training sets on which our forecasting analysis is based, the results in Table 3 are based simply on the full sample of size $T + N = 1227$. In the models with a restricted constant term in Panels A and C, the first $N = 20$ observations are used as initial values in the estimation, while in the models with a level parameter in Panels B and D, we follow Johansen and Nielsen (2015) and set $N = 0$.

For each model we initially chose the lag order by the Bayesian information criterion, and then conducted cointegration rank tests. The model was then estimated and the residuals tested for white noise (multivariate Portmanteau Q-test reported), which was not rejected for three of the four models with a lag order of $k = 1$ and cointegration rank of $r = 2$. For the $\text{FCVAR}_{d=b,\rho}$ model, the Q-test P value is 0.023, but after adding an additional lag the model estimation failed to converge to sensible parameter values, so we chose to maintain the $k = 1$ structure as chosen by the Bayesian information criterion, and we conclude that the model appears to be correctly specified.

We note that the FCVAR models impose the same value of the fractional integration parameter, d, for each time series in the system. This restriction can be tested in a local Whittle or exact local Whittle framework, see Robinson and Yajima (2002) and Nielsen and Shimotsu (2007), respectively. Both require the selection of several bandwidth parameters and can be quite sensitive to these. We applied the latter methodology because it allows for nonstationary values of d, and we were not able to reject the hypothesis of equality of the d parameters for the three series using a range of bandwidth parameters (unreported T_0 statistics are all less than 0.84 in the notation of Nielsen and Shimotsu (2007)). Furthermore, we note that if the d parameters for each of the univariate time series were in fact different, imposing the same d in the FCVAR model would be an important source of mis-specification, which would lead to neglected serial correlation (of the fractional integration type) in the residuals. Since the residuals in our estimated models do not display such strong signs of serial correlation, this does not appear to be an issue.

We will discuss and interpret estimated parameters in detail for a larger model in Section 5.1 below, in particular $\hat{\alpha}$ and $\hat{\beta}$. For now, we note from Table 3 that $\hat{d}$ is very strongly significantly different from one and that $\hat{b}$ is strongly significantly different from both zero

Table 3: Preliminary estimation results: three parties

Panel A: $\text{FCVAR}_{d,b,\rho}$ model, $k = 1$

$$\hat{d} = \underset{(0.023)}{0.774}, \quad \hat{b} = \underset{(0.026)}{0.094}, \quad \hat{\alpha} = \begin{bmatrix} 0.532 & -0.810 \\ 0.405 & 0.034 \\ 5.517 & 10.748 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \\ 2.247 & -1.962 \end{bmatrix},$$

$$\hat{\rho} = \begin{bmatrix} 0.019 & -1.536 \end{bmatrix}, \quad \widehat{\Gamma} = \begin{bmatrix} -6.011 & 0.970 & -2.485 \\ -0.250 & -5.686 & -0.684 \\ -5.875 & -10.224 & 2.080 \end{bmatrix},$$

$$Q_{\hat{\varepsilon}} = \underset{(0.625)}{50.111}, \quad \log(\mathscr{L}) = 4898.335$$

Panel B: $\text{FCVAR}_{d,b,\mu}$ model, $k = 1$

$$\hat{d} = \underset{(0.026)}{0.624}, \quad \hat{b} = \underset{(0.084)}{0.273}, \quad \hat{\alpha} = \begin{bmatrix} 0.009 & -0.156 \\ 0.448 & 0.647 \\ 0.064 & -0.354 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \\ 0.947 & -0.748 \end{bmatrix},$$

$$\hat{\mu} = \begin{bmatrix} -0.396 \\ -0.660 \\ -1.597 \end{bmatrix}, \quad \widehat{\Gamma} = \begin{bmatrix} -1.278 & 0.283 & -0.033 \\ -0.367 & -2.105 & 0.074 \\ -0.308 & 0.478 & -1.924 \end{bmatrix},$$

$$Q_{\hat{\varepsilon}} = \underset{(0.359)}{57.148}, \quad \log(\mathscr{L}) = 4965.089$$

Panel C: $\text{FCVAR}_{d=b,\rho}$ model, $k = 1$

$$\hat{d} = \underset{(0.013)}{0.627}, \quad \hat{\alpha} = \begin{bmatrix} -0.024 & -0.024 \\ -0.013 & 0.005 \\ -0.100 & 0.132 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \\ 0.370 & -0.922 \end{bmatrix},$$

$$\hat{\rho} = \begin{bmatrix} 1.955 & -1.652 \end{bmatrix}, \quad \widehat{\Gamma} = \begin{bmatrix} -0.552 & 0.054 & 0.029 \\ 0.049 & -0.587 & 0.031 \\ 0.049 & -0.031 & -0.554 \end{bmatrix},$$

$$Q_{\hat{\varepsilon}} = \underset{(0.023)}{76.692}, \quad \log(\mathscr{L}) = 4875.513$$

Panel D: $\text{FCVAR}_{d=b,\mu}$ model, $k = 1$

$$\hat{d} = \underset{(0.015)}{0.572}, \quad \hat{\alpha} = \begin{bmatrix} -0.022 & -0.018 \\ 0.079 & 0.055 \\ -0.058 & -0.048 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \\ 2.403 & -3.679 \end{bmatrix},$$

$$\hat{\mu} = \begin{bmatrix} -0.411 \\ -0.653 \\ -1.574 \end{bmatrix}, \quad \widehat{\Gamma} = \begin{bmatrix} -0.497 & 0.075 & 0.027 \\ -0.032 & -0.652 & 0.022 \\ -0.061 & 0.119 & -0.694 \end{bmatrix},$$

$$Q_{\hat{\varepsilon}} = \underset{(0.263)}{60.163}, \quad \log(\mathscr{L}) = 4962.473$$

Notes: The table shows estimation results for the four variations of the FCVAR model. The multivariate Portmanteau Q-test for serial correlation up to order six in the residuals is reported as $Q_{\hat{\varepsilon}}$ and the maximized log-likelihood value is reported as $\log(\mathscr{L})$. Standard errors are in parentheses below $\hat{d}$ and $\hat{b}$ and the P value is in parenthesis below $Q_{\hat{\varepsilon}}$. The sample size is $T + N = 1227$, and the first $N = 20$ ($N = 0$) observations are used as initial values in the models with a restricted constant term (level parameter).

and one in all models. This suggests very clearly that the FCVAR model is more appropriate for this data than the non-fractional CVAR model since the latter has $d = b = 1$ imposed. Comparing across the models, it appears that the estimates of d are fairly close, ranging from 0.57 to 0.77, whereas the estimates of b are quite different in the models with $d \neq b$ in Panels A and B of Table 3.

Further comparison across models leads to consideration of the likelihood ratio test statistic for the null hypothesis that $d = b$. That is, for the models with a restricted constant term (and $N = 20$), we can test the null of the model in Panel C against the alternative of the model in Panel A, and for the models with a level parameter (and $N = 0$), we can test the null of the model in Panel D against the alternative of the model in Panel B. In the first case, the likelihood ratio test statistic is 45.644, and in the second case, the likelihood ratio test statistic is 5.232. In both cases, this is asymptotically chi-squared distributed with one degree of freedom, so the conclusions of these tests differ somewhat and consequently we proceed with the consideration of all four models.

4.2 Forecasting methodology

The four variants of the FCVAR model presented above are evaluated on their forecasting ability relative to a group of popular competing models. The CVAR model (4) serves as the multivariate benchmark model. The main univariate benchmark is the ARFIMA(p, d, q) model,

$$A(L)\Delta^d(X_t - \mu) = B(L)\varepsilon_t, \tag{14}$$

where $A(L)$ and $B(L)$ are the autoregressive and moving average polynomials, satisfying standard regularity conditions. A special case of (14) is the simple ARFIMA$(0, d, 0)$ model, which is also included because it was found by, e.g., Byers et al. (1997) and Dolado et al. (2002), to fit (monthly) UK polling data well. Finally, another special case of (14) is the standard ARMA(p, q) model, which we include as the classical univariate benchmark. Estimation of the univariate models is by minimization of the conditional sum-of-squares, see Box and Jenkins (1970) for ARMA models and Hualde and Robinson (2011) and Nielsen (2015) for ARFIMA models, while forecasting is done using the best (minimum mean squared error) linear predictor which appears standard for these models.

The forecasting procedure applies the standard rolling window and recursive forecasting schemes to examine forecasting accuracy. The main distinction between the two schemes is how they select the training sets used for estimating each model to produce forecasts. The rolling window scheme uses a fixed training set length (usually referred to as a window) that moves across the data set. The recursive scheme uses an expanding training set length with a fixed start date at the beginning of the data set. In order to assess the forecasting capability of each model, it is necessary to generate predictions from a sufficiently large number of training sets used to estimate each model, and it is preferable that each training set is long enough for reliable estimation and forecasting.

For the rolling window scheme, each statistical model uses training sets with a fixed length of $T + N = 600$ observations, approximately half the length of the data set. For the recursive scheme, the first training set includes $T + N = 600$ observations and each subsequent training set includes one more observation, until the last training set which has $T + N = 1227 - h$ observations, where h is the forecast horizon. This implies that for both forecasting schemes, the total number of training sets is $1227 - 600 - h + 1 = 628 - h$, and the first training set is the same for both procedures. For the FCVAR models with a restricted constant term we use the first $N = 20$ observations of each training set as initial values, and for the FCVAR models with the level parameter we follow Johansen and Nielsen (2015) and use $N = 0$ initial values. The CVAR model applies estimation conditional on $N = k + 1$ initial values, such that maximum likelihood estimation reduces to reduced rank regression.

The forecasting programs use all applicable model specification criteria and tests consistently across both the multivariate and univariate model types, and all rejection rules for statistical hypothesis testing are conducted at the five percent level of significance. For all models, the model specification is based on the very first training set, and the same specification is then applied to all training sets. That is, we maintain the same lag orders and cointegration ranks for all training sets, but all the parameters of the models are re-estimated for each training set before forecasts are calculated.

The multivariate specifications involve first selecting the lag order, k. Lag order selection is initially based on the Bayesian information criterion (BIC). Given the lag order, cointegration rank tests are performed, which determine the number of cointegrating relations, r, for each model. The multivariate model is then estimated using these values of k and r. In the next step, the program performs a multivariate Portmanteau Q-test for white noise up to order six on the residuals. If the white noise test rejects, then an additional lag is added and the rank test, estimation, and white noise test are repeated in sequence until the program fails to reject white noise for the residuals. The univariate specification differs from the multivariate only in that two lag orders, p and q, need to be selected conditional on the univariate white noise test failing to reject.

Following the completion of the specification algorithm, the forecasting program estimates the model for all training sets and uses the estimated model parameters to generate h-step ahead forecasts for each time series. All multivariate and univariate models considered generate forecasts recursively, see Section 3.3. The ARFIMA$(0, d, 0)$ model is included in the portfolio due to its popularity for political opinion poll data. Its fixed lag orders make it the only model in the portfolio with lag orders not determined by a decision rule in the forecasting program.

Forecasts are generated for seven out-of-sample horizons, h: 1, 5, 10, 15, 20, 25, and 50 steps ahead. As mentioned above, the number of training sets is different for each forecast horizon, and we denote this number M_h. The models are ranked based on the multivariate (system) root mean squared forecast error,

$$\text{RMSFE}_{\text{sys}} = \sqrt{\frac{1}{pM_h} \sum_{i=1}^{p} \sum_{j=1}^{M_h} \left(\hat{X}_{i,T_j+h|T_j} - X_{i,T_j+h} \right)^2}, \tag{15}$$

as well as the univariate root mean squared forecast errors for each series,

$$\text{RMSFE}_i = \sqrt{\frac{1}{M_h} \sum_{j=1}^{M_h} \left(\hat{X}_{i,T_j+h|T_j} - X_{i,T_j+h} \right)^2}, \tag{16}$$

where, in both cases, h denotes the forecast horizon, $p = 3$ is the number of series, i.e., the dimension of the multivariate system, $M_h = 628 - h$ is the number of training sets, and T_j is the terminal date of training set j. The individual RMSFE$_i$ ($i =$ CP, LP, LD) measures the typical magnitude of forecast errors for each individual time series, while the $\mathrm{RMSFE}_{\mathrm{sys}}$ measures the typical magnitude of all forecast errors produced by each model.

4.3 Forecasting results

This section discusses the forecast performance results and concludes with several figures of forecasts generated by all models in the portfolio.

Tables 4 and 5 report the RMSFE_i ($i = \mathrm{CP}, \mathrm{LP}, \mathrm{LD}$) and $\mathrm{RMSFE}_{\mathrm{sys}}$ values for the rolling and recursive schemes, respectively. Numbers in parentheses are the corresponding rankings at each individual forecast horizon. The models are ranked based on the $\mathrm{RMSFE}_{\mathrm{sys}}$ because for each model it provides a single measurement of forecast accuracy for all three time series. We note that the ARFIMA(p, d, q) model specifies both lag orders to be zero, i.e. $p = q = 0$, for all three series, so that the results for the ARFIMA(p, d, q) and ARFIMA$(0, d, 0)$ models are identical, and therefore we do not report the ARFIMA$(0, d, 0)$ results. Given the results from the literature cited in Section 2 above, this univariate model specification is not surprising. We also note that the ARMA(p, q) model specifies $(p, q) = (0, 1)$ for all three series by a very slim margin over $(p, q) = (1, 0)$ in terms of the BIC; the forecast performance (unreported) with $(p, q) = (1, 0)$ is qualitatively very similar to that reported with $(p, q) = (0, 1)$.

We begin the assessment of the forecast accuracy with a discussion of the one-step ahead forecasts. This seems natural prior to examining performance at other subjectively selected horizons that may be of interest in any given application. In the context of political opinion polls, one can easily imagine the relevance of forecasting poll standings or vote shares at many different horizons.

According to the recursive scheme, all four variants of the FCVAR model outperform all competing models at the one-step ahead forecasting horizon. According to the rolling window scheme, three of the four FCVAR variants outperform all competing models, and the $\mathrm{FCVAR}_{d=b,\rho}$ model is tied with the ARFIMA(p, d, q) model. Both forecasting schemes rank a variant of the FCVAR model with two fractional parameters as the top performing model. The rolling window scheme ranks the $\mathrm{FCVAR}_{d,b,\mu}$ and $\mathrm{FCVAR}_{d,b,\rho}$ models first (tied), while the recursive scheme ranks the $\mathrm{FCVAR}_{d,b,\rho}$ model first and the $\mathrm{FCVAR}_{d=b,\rho}$ model second. An important observation is that the third and fourth ranked FCVAR specifications are very close in performance to the top performing variant, showing that the reliability of one-step ahead forecasts generated by the FCVAR model is robust to the number of fractional parameters and type of deterministic terms used in the specification, at least for this data set.

The longer forecast horizons considered, 5, 10, 15, 20, 25 and 50 steps ahead, deliver results that are in agreement with the findings for the one-step ahead horizon. The model rankings across all forecast horizons determine that the accuracy of both short, medium and long term forecasts generated by the FCVAR model is better than the other models in the portfolio. This can be seen from the fact that for 14 of 14 cases (1 to 50 steps ahead in both the rolling and the recursive schemes), the top two performing models are always variants of the FCVAR model and three of the top four models are always variants of the FCVAR model. Furthermore, with the exception of one forecast horizon (50 steps ahead) in both schemes, all four variants of the FCVAR model are ranked as the top four models. The exception occurs when only one variant of the FCVAR model underperforms by a small margin relative

Table 4: Root mean squared forecast errors – rolling window forecast scheme

Model	Series	1 step	5 step	10 step	15 step	20 step	25 step	50 step
$\text{FCVAR}_{d,b,\rho}$	CP	0.0562	0.0613	0.0637	0.0656	0.0678	0.0704	0.0781
	LP	0.0499	0.0524	0.0569	0.0609	0.0647	0.0690	0.0902
	LD	0.1134	0.1161	0.1233	0.1298	0.1358	0.1415	0.1650
	System	0.0785 (1)	0.0816 (2)	0.0866 (2)	0.0910 (2)	0.0953 (2)	0.0996 (2)	0.1176 (2)
$\text{FCVAR}_{d,b,\mu}$	CP	0.0561	0.0608	0.0632	0.0656	0.0677	0.0702	0.0781
	LP	0.0505	0.0549	0.0607	0.0669	0.0723	0.0781	0.1030
	LD	0.1130	0.1145	0.1201	0.1249	0.1295	0.1350	0.1543
	System	0.0785 (1)	0.0813 (1)	0.0858 (1)	0.0901 (1)	0.0941 (1)	0.0987 (1)	0.1162 (1)
$\text{FCVAR}_{d=b,\rho}$	CP	0.0569	0.0629	0.0664	0.0684	0.0708	0.0729	0.0782
	LP	0.0506	0.0525	0.0569	0.0604	0.0645	0.0687	0.0908
	LD	0.1174	0.1241	0.1329	0.1400	0.1445	0.1499	0.1663
	System	0.0808 (4)	0.0859 (4)	0.0918 (4)	0.0965 (4)	0.1001 (3)	0.1041 (3)	0.1183 (3)
$\text{FCVAR}_{d=b,\mu}$	CP	0.0566	0.0634	0.0682	0.0731	0.0783	0.0828	0.1039
	LP	0.0502	0.0540	0.0597	0.0653	0.0705	0.0828	0.1056
	LD	0.1131	0.1169	0.1242	0.1326	0.1379	0.1450	0.1799
	System	0.0786 (3)	0.0829 (3)	0.0888 (3)	0.0952 (3)	0.1002 (4)	0.1076 (4)	0.1345 (5)
CVAR_{ρ}	CP	0.0605	0.0656	0.0712	0.0748	0.0800	0.0835	0.0953
	LP	0.0547	0.0577	0.0667	0.0744	0.0805	0.0865	0.1114
	LD	0.1200	0.1260	0.1386	0.1489	0.1559	0.1619	0.1828
	System	0.0838 (6)	0.0885 (6)	0.0979 (6)	0.1054 (6)	0.1113 (6)	0.1164 (6)	0.1353 (6)
$\text{ARFIMA}(p,d,q)$	CP	0.0578	0.0623	0.0664	0.0700	0.0730	0.0758	0.0860
	LP	0.0531	0.0638	0.0746	0.0834	0.0909	0.0977	0.1252
	LD	0.1158	0.1244	0.1335	0.1410	0.1466	0.1521	0.1707
	System	0.0808 (4)	0.0884 (5)	0.0963 (5)	0.1029 (5)	0.1081 (5)	0.1132 (5)	0.1319 (4)
$\text{ARMA}(p,q)$	CP	0.0899	0.1167	0.1178	0.1187	0.1187	0.1190	0.1207
	LP	0.1109	0.1582	0.1605	0.1628	0.1650	0.1673	0.1781
	LD	0.1627	0.1947	0.1957	0.1971	0.1983	0.1997	0.2049
	System	0.1250 (7)	0.1597 (7)	0.1612 (7)	0.1627 (7)	0.1640 (7)	0.1653 (7)	0.1715 (7)

Notes: The overall performance of each model is measured by the root mean square forecast error of the entire multivariate system. The $\text{ARFIMA}(0,d,0)$ model is not included because the $\text{ARFIMA}(p,d,q)$ model specifies both lag orders to zero for all three series. The $\text{ARMA}(p,q)$ model specifies $(p,q) = (0,1)$ for all three series. Numbers in parentheses are the corresponding rankings at each individual forecast horizon. The number 1 rank is assigned to the best performing model and the number 7 rank is assigned to the worst performing model. Results are based on h-step ahead forecasts produced using 628-h training sets of length 600.

to the ARFIMA model. Overall, this evidence provides strong support for the application of the FCVAR model for forecasting next day (one-step), next week (5-steps), and all the way up to ten weeks ahead (50-steps) poll standings.

The results for both forecasting schemes suggest that the FCVAR model with two fractional parameters produces the smallest average forecast errors. A variant of the FCVAR model with two fractional parameters always outperforms both sub-models, the $\text{FCVAR}_{d=b,\rho}$ and $\text{FCVAR}_{d=b,\mu}$, although only by very small margins. The recursive scheme determines the $\text{FCVAR}_{d,b,\rho}$ model as the absolute favorite at all forecast horizons, while the rolling window scheme ranks the $\text{FCVAR}_{d,b,\mu}$ model as the favorite at all forecast horizons greater than one-

Table 5: Root mean squared forecast errors – recursive window forecast scheme

Model	Series	1 step	5 step	10 step	15 step	20 step	25 step	50 step
$\text{FCVAR}_{d,b,\rho}$	CP	0.0561	0.0616	0.0642	0.0667	0.0697	0.0727	0.0839
	LP	0.0505	0.0548	0.0611	0.0671	0.0726	0.0784	0.1062
	LD	0.1123	0.1144	0.1197	0.1251	0.1292	0.1342	0.1520
	System	0.0781 (1)	0.0814 (1)	0.0860 (1)	0.0906 (1)	0.0946 (1)	0.0991 (1)	0.1175 (1)
$\text{FCVAR}_{d,b,\mu}$	CP	0.0563	0.0619	0.0653	0.0688	0.0728	0.0765	0.0905
	LP	0.0499	0.0524	0.0564	0.0602	0.0640	0.0686	0.0939
	LD	0.1150	0.1198	0.1288	0.1381	0.1466	0.1567	0.2024
	System	0.0793 (3)	0.0835 (3)	0.0895 (4)	0.0956 (4)	0.1015 (4)	0.1082 (4)	0.1390 (5)
$\text{FCVAR}_{d=b,\rho}$	CP	0.0563	0.0617	0.0646	0.0672	0.0701	0.0732	0.0846
	LP	0.0507	0.0550	0.0612	0.0673	0.0729	0.0786	0.1058
	LD	0.1144	0.1193	0.1267	0.1338	0.1387	0.1439	0.1614
	System	0.0792 (2)	0.0838 (4)	0.0894 (3)	0.0948 (3)	0.0991 (2)	0.1037 (2)	0.1217 (2)
$\text{FCVAR}_{d=b,\mu}$	CP	0.0563	0.0624	0.0662	0.0697	0.0731	0.0766	0.0887
	LP	0.0507	0.0555	0.0622	0.0685	0.0747	0.0808	0.1113
	LD	0.1147	0.1171	0.1236	0.1309	0.1367	0.1441	0.1741
	System	0.0794 (4)	0.0830 (2)	0.0886 (2)	0.0943 (2)	0.0993 (3)	0.1051 (3)	0.1298 (3)
CVAR_{ρ}	CP	0.0615	0.0684	0.0712	0.0830	0.0909	0.0972	0.1160
	LP	0.0551	0.0595	0.0667	0.0828	0.0926	0.1017	0.1388
	LD	0.1208	0.1222	0.1386	0.1427	0.1500	0.1572	0.1789
	System	0.0845 (6)	0.0878 (6)	0.0979 (6)	0.1066 (6)	0.1145 (6)	0.1218 (6)	0.1469 (6)
$\text{ARFIMA}(p,d,q)$	CP	0.0585	0.0627	0.0668	0.0706	0.0739	0.0770	0.0881
	LP	0.0517	0.0566	0.0634	0.0692	0.0744	0.0792	0.0991
	LD	0.1182	0.1241	0.1359	0.1468	0.1560	0.1651	0.2007
	System	0.0818 (5)	0.0867 (5)	0.0948 (5)	0.1022 (5)	0.1085 (5)	0.1147 (5)	0.1389 (4)
$\text{ARMA}(p,q)$	CP	0.1105	0.1503	0.1511	0.1516	0.1515	0.1517	0.1523
	LP	0.1157	0.1693	0.1706	0.1719	0.1731	0.1744	0.1808
	LD	0.1714	0.2124	0.2135	0.2150	0.2165	0.2180	0.2249
	System	0.1354 (7)	0.1792 (7)	0.1803 (7)	0.1815 (7)	0.1824 (7)	0.1834 (7)	0.1884 (7)

Notes: The overall performance of each model is measured by the root mean square forecast error of the entire multivariate system. The $\text{ARFIMA}(0,d,0)$ model is not included because the $\text{ARFIMA}(p,d,q)$ model specifies both lag orders to zero for all three series. The $\text{ARMA}(p,q)$ model specifies $(p,q) = (0,1)$ for all three series. Numbers in parentheses are the corresponding rankings at each individual forecast horizon. The number 1 rank is assigned to the best performing model and the number 7 rank is assigned to the worst performing model. Results are based on h-step ahead forecasts produced using 628-h training sets of length$= 600, 601, \dots, 1227-h$.

step, and ranks the $\text{FCVAR}_{d,b,\mu}$ and $\text{FCVAR}_{d,b,\rho}$ models as the top two performing models for next day forecasting.

Thus, the model rankings strongly suggest that the forecasting accuracy of the FCVAR is better than both the fractional benchmark model (ARFIMA) and the multivariate benchmark model (CVAR). To assess the degree of relative performance, Table 6 reports the RMSFE percentage change,

$$100\left(\frac{\text{RMSFE}_{\text{sys}}(\text{FCVAR})}{\text{RMSFE}_{\text{sys}}(\text{ARFIMA})} - 1\right), \tag{17}$$

Table 6: Percentage change in $\text{RMSFE}_{\text{sys}}$: FCVAR vs. ARFIMA$(p, d, q)$

Model	1 step	5 step	10 step	15 step	20 step	25 step	50 step
Panel A: rolling scheme							
$\text{FCVAR}_{d,b,\rho}$	-2.79	-7.68	-10.07	-11.53	-11.88	-12.05	-10.87
$\text{FCVAR}_{d,b,\mu}$	-2.90	-8.05	-10.87	-12.39	-12.92	-12.77	-11.89
$\text{FCVAR}_{d=b,\rho}$	-0.01	-2.88	-4.62	-6.24	-7.41	-8.05	-10.28
$\text{FCVAR}_{d=b,\mu}$	-2.77	-6.26	-7.82	-7.48	-7.31	-4.94	2.01
Panel B: recursive scheme							
$\text{FCVAR}_{d,b,\rho}$	-4.50	-6.10	-9.29	-11.39	-12.85	-13.63	-15.40
$\text{FCVAR}_{d,b,\mu}$	-3.01	-3.66	-5.58	-6.44	-6.48	-5.68	0.08
$\text{FCVAR}_{d=b,\rho}$	-3.15	-3.35	-5.71	-7.26	-8.66	-9.61	-12.41
$\text{FCVAR}_{d=b,\mu}$	-2.98	-4.22	-6.58	-7.72	-8.43	-8.34	-6.53

Notes: Negative values favor the FCVAR model. Results are based on h-step ahead forecasts produced using 628-h training sets of length 600 (rolling scheme) or length$= 600, 601, \ldots, 1227 - h$ (recursive scheme).

of the FCVAR model relative to the ARFIMA model for the rolling and recursive schemes in Panels A and B, respectively. Negative values favor the FCVAR model and positive values favor the ARFIMA model.

Previous literature, as cited earlier, has extensively documented the superiority of fractional (ARFIMA) models for modeling and forecasting polling data. Compared to this important benchmark, Table 6 shows that the RMSFE of the FCVAR model is as much as 13% lower for the rolling scheme and 15% lower for the recursive scheme. In 54 of 56 cases in Table 6, the multivariate fractional model outperforms the univariate fractional model and in 16 of 56 cases, the FCVAR model delivers more than a 10% reduction in the $\text{RMSFE}_{\text{sys}}$ relative to the ARFIMA model. The gains at the longer horizons are more pronounced, and the gains appear to be larger for the FCVAR models with two fractional parameters.

Similarly, Table 7 reports the RMSFE percentage change of the FCVAR model relative to the CVAR model. Again, this table shows improved forecast accuracy for all variants of the FCVAR model relative the to the CVAR model. In 56 of 56 cases in Table 7, all four variants of the FCVAR model perform better than the CVAR model. For the FCVAR models with a restricted constant, the improvements in performance increase substantially as the forecasting horizon increases. The recursive scheme shows 15%, 17%, 19% and 20% improvement for the 15, 20, 25 and 50 step ahead horizons, attained by the FCVAR model with two fractional parameters and a restricted constant. The rolling scheme shows up to 14% improvement for the FCVAR models with a restricted constant, and up to 15% improvement for the FCVAR models with a level parameter. In 30 of 56 cases, the FCVAR model delivers more than a 10% reduction in the $\text{RMSFE}_{\text{sys}}$ relative to the CVAR model. Overall the fractional models clearly outperform the non-fractional models, with the gains becoming more pronounced at longer forecast horizons.

To conclude the forecast comparison, examples of forecasts generated by all models in the portfolio are presented in Figure 3. The presented forecasts are generated using the first training set, which is common to both the rolling and recursive forecasting schemes.

Table 7: Percentage change in $\text{RMSFE}_{\text{sys}}$: FCVAR vs. CVAR

Model	1 step	5 step	10 step	15 step	20 step	25 step	50 step
Panel A: rolling scheme							
$\text{FCVAR}_{d,b,\rho}$	-6.24	-7.80	-11.50	-13.60	-14.43	-14.48	-13.10
$\text{FCVAR}_{d,b,\mu}$	-6.34	-8.18	-12.29	-14.44	-15.45	-15.19	-14.10
$\text{FCVAR}_{d=b,\rho}$	-3.56	-3.01	-6.14	-8.42	-10.10	-10.60	-12.52
$\text{FCVAR}_{d=b,\mu}$	-6.22	-6.39	-9.29	-9.64	-10.00	-7.58	-0.55
Panel B: recursive scheme							
$\text{FCVAR}_{d,b,\rho}$	-7.53	-7.32	-12.13	-15.07	-17.43	-18.66	-20.00
$\text{FCVAR}_{d,b,\mu}$	-6.08	-4.92	-8.53	-10.32	-11.39	-11.17	-5.36
$\text{FCVAR}_{d=b,\rho}$	-6.23	-4.61	-8.65	-11.11	-13.45	-14.88	-17.18
$\text{FCVAR}_{d=b,\mu}$	-6.06	-5.47	-9.50	-11.55	-13.24	-13.68	-11.61

Notes: Negative values favor the FCVAR model. Results are based on h-step ahead forecasts produced using 628-h training sets of length 600 (rolling scheme) or length$=600, 601, \ldots, 1227-h$ (recursive scheme).

The figure shows the last 9 observations in the training set, followed by the out-of-sample observations and forecasts beginning at the 10$^{\text{th}}$ observation and continuing up to the longest horizon considered, $h = 50$, for the CP, LP and LD series in separate graphs. Panel (a) shows forecasts of the logit transformed series, while in Panel (b) all series (and forecasts) are transformed back to percentage vote shares to make interpretations easier. The figure thus provides an illustration of how all models forecast political support as measured by daily opinion polls. In Panel (a) we also include 90% confidence bands shown using slightly thinner lines.[6]

The forecasts for this particular training set are in agreement with the evidence presented in the model ranking exercise. An interesting observation, which is present in results from other training sets as well, is that even though the CVAR forecasts exhibit more dynamics in the short run (which is a result of a higher lag order selected compared to the FCVAR), this does not translate into more accurate predictions for short horizons. In particular, as the short run dynamics die out for all multivariate models, it is evident that fractional cointegration generates point forecasts that are closer to the subsequently realized observations. These conclusions are strongly supported by the reported confidence bands in Figure 3(a), which show that forecasts based on the FCVAR models are much more accurate than those based on the non-fractional CVAR model. This is especially true at the longer horizons, as one might have expected. Finally, we note that all variants of the FCVAR produce similar predictions, and these are reasonably close to the realized data series in all three panels, while the other models in the portfolio only perform well in some cases.

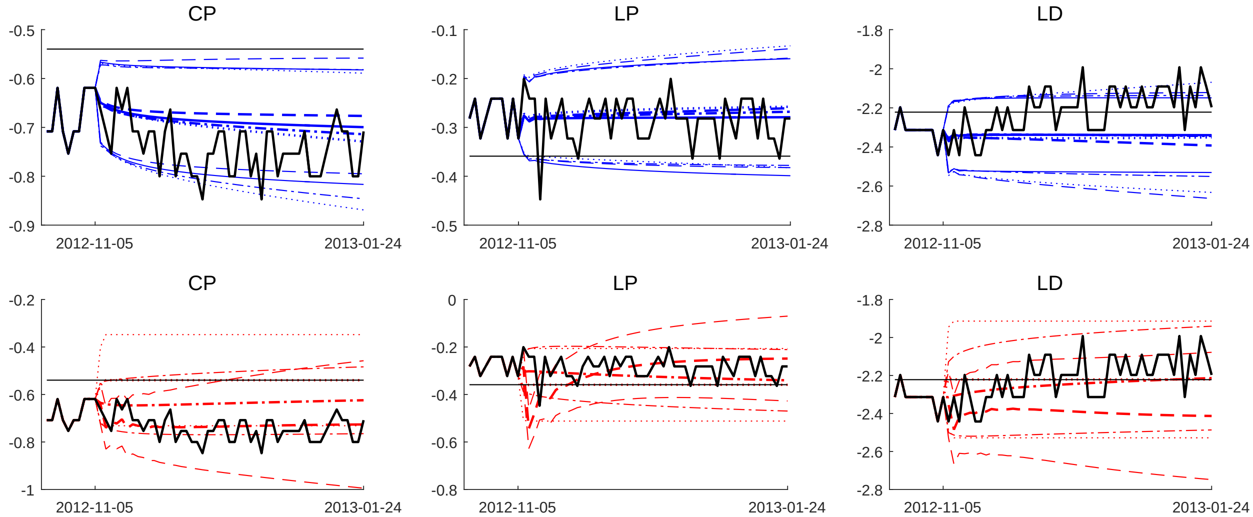
5 Empirical application to the 2015 UK general election

In this section we present an application to the 2015 UK general election, which was deemed the most unpredictable election in decades in the media. Opinion poll agencies predicted a

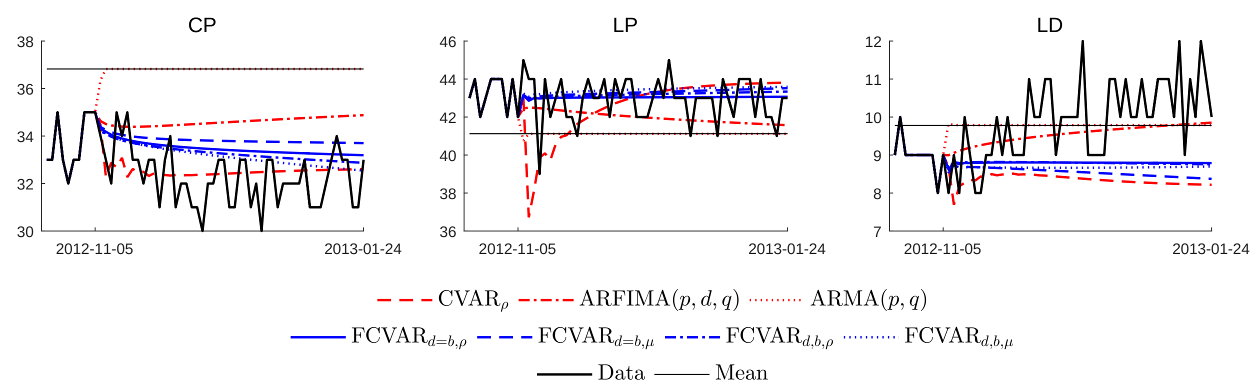
[6]Following the advice of a referee, these were simply calculated from the moving-average representation of the models ignoring estimation uncertainty.

Figure 3: Forecasts for the first training set

(a) Forecasts of logit transformed series



(b) Forecasts of vote shares in percentage



Notes: The training set is the first window, which is shared by the rolling window and recursive window forecasting schemes. Panel (a) shows forecasts of logit transformed series, where 90% confidence bands are shown using with slightly thinner lines, and Panel (b) shows forecasts of vote shares in percentage. The ARFIMA$(0, d, 0)$ model is not included because the ARFIMA(p, d, q) model specifies both lag orders to zero for all three series. The ARMA(p, q) model specifies $(p, q) = (0, 1)$ for all three series.

hung government, but ended up significantly underestimating the Conservative Party vote share; the party that won the election with a majority representation in Parliament. Note, however, that vote shares cannot be mapped into election outcomes in the context of the UK election process. This can be seen from the fact that the realized vote share for the Conservative Party was 36.8%, but the party won 330 out of 650 constituencies in the country; an outcome that the predicted vote share of the previous-day YouGov opinion poll, 34%, does not exclude.

The three political parties represented in the full data set, spanning the entire duration of the survey, are the three major political parties in the UK that have historically had the most representation in government by a strong margin over other parties running in the election. However, as the 2015 general election has shown, major losses incurred by the three major parties can be due to constituencies lost to other political parties not in the top three (as measured by representation in parliament). In this election, the UKIP and the Green Party received 12.7% and 3.8% vote shares, in particular. As discussed earlier, on April 16$^{\text{th}}$, 2012, and June 18$^{\text{th}}$, 2012, respectively, YouGov changed the way they reported the outcomes of their polls and started reporting the UKIP and the Green Party as separate time series (rather than being included in the residual category). Therefore, in this empirical application to the 2015 UK general election, we apply the multivariate and univariate models to both the case of three political parties (based on the full sample spanning the entire 2010 to 2015 political cycle) and to the case of either four or five political parties (based on shorter data sets spanning only the second half of the 2010 to 2015 political cycle).[7]

A key strength of our data set for the purpose of statistical modeling and forecasting, and hence for vote share prediction, is that the observed time series are contained within one political cycle. This should allow application of a relatively simple statistical model, and the previous analysis has shown strong support for the FCVAR model for this task.

5.1 Estimation results prior to the election

In the first part of this empirical application, we analyze and interpret the estimated model coefficients more carefully. To this end, we consider the full five-party data set consisting of CP_t, LP_t, LD_t, IP_t, and GP_t, but for a reduced sample size covering June 18$^{\text{th}}$, 2012, to May 6$^{\text{th}}$, 2015, for a total of $T+N = 729$ observations, which is the period where observations are available for all five parties. The increased dimension of the model tends to cause problems in the multi-dimensional numerical optimization required for the FCVAR models with either two fractional parameters or with the level parameter, and for this reason we consider only the $\text{FCVAR}_{d=b,\rho}$ model for the full data set, since this model requires only one-dimensional numerical optimization and thus remains feasible.

As usual, the estimation begins with lag length selection. The BIC first suggests $k = 1$, but for this choice the Portmanteau Q-test rejects the null of no serial correlation in the residuals with a P value of 0.000. Consequently, we increase the lag length to $k = 2$ for which the Q-test P value is 0.17. Next, the LR test for cointegration rank (the trace test) produces P values of 0.051 and 0.977 for $r = 3$ and $r = 4$, respectively, and we proceed with the specification $r = 4$. As before, all results are conditional upon $N = 20$ initial values.

The estimation results for the $\text{FCVAR}_{d=b,\rho}$ model for the five-party data set are presented in Table 8. We focus our interpretations on the long-run cointegration parameters, α and β.

To interpret the estimated cointegrating relations in $\hat{\beta}$, we find it convenient to re-normalize them on the coefficient for the Conservatives. That is, we re-normalize β such that

$$\beta = \begin{bmatrix} -1 & -1 & -1 & -1 \\ \beta_{\text{LP}} & 0 & 0 & 0 \\ 0 & \beta_{\text{LD}} & 0 & 0 \\ 0 & 0 & \beta_{\text{IP}} & 0 \\ 0 & 0 & 0 & \beta_{\text{GP}} \end{bmatrix}. \tag{18}$$

[7]There were also some regional parties with non-negligible vote shares, but we do not include these in our analysis because they seem to compete on a different basis and with a somewhat different agenda.

Table 8: $\text{FCVAR}_{d=b,\rho}$ estimation results: five parties

Model:

$$\Delta^{\hat{d}} \begin{bmatrix} \text{CP}_t \\ \text{LP}_t \\ \text{LD}_t \\ \text{IP}_t \\ \text{GP}_t \end{bmatrix} = \hat{\alpha}\left(\hat{\beta}' \Delta^{\hat{d}-\hat{b}} L_{\hat{b}} \begin{bmatrix} \text{CP}_t \\ \text{LP}_t \\ \text{LD}_t \\ \text{IP}_t \\ \text{GP}_t \end{bmatrix} + \hat{\rho}'\right) + \widehat{\Gamma}_1 \Delta^{\hat{d}} L_{\hat{b}} \begin{bmatrix} \text{CP}_t \\ \text{LP}_t \\ \text{LD}_t \\ \text{IP}_t \\ \text{GP}_t \end{bmatrix} + \widehat{\Gamma}_2 \Delta^{\hat{d}} L_{\hat{b}}^2 \begin{bmatrix} \text{CP}_t \\ \text{LP}_t \\ \text{LD}_t \\ \text{IP}_t \\ \text{GP}_t \end{bmatrix} + \hat{\varepsilon}_t$$

Parameters:

$$\hat{\alpha} = \begin{bmatrix} -0.147 & 0.140 & 0.063 & 0.046 \\ 0.019 & -0.232 & 0.070 & -0.069 \\ -0.038 & 0.145 & -0.425 & 0.071 \\ -0.012 & -0.300 & 0.018 & -0.083 \\ -0.872 & -1.549 & -0.752 & -0.335 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \\ 0.170 & -0.604 & 0.510 & 2.760 \end{bmatrix}$$

$$\hat{d} = \underset{(0.016)}{0.813}, \quad \hat{\rho} = \begin{bmatrix} 1.068 & -0.555 & 3.872 & 7.821 \end{bmatrix}$$

$$Q_{\hat{\varepsilon}} = \underset{(0.175)}{166.098}, \quad \log(\mathscr{L}) = 3632.605$$

Notes: The table shows FCVAR estimation results for model (5) with one fractional parameter, $d = b$, and a restricted constant term, ρ. The multivariate Portmanteau Q-test for serial correlation up to order six in the residuals is reported as $Q_{\hat{\varepsilon}}$ and the maximized log-likelihood value is reported as $\log(\mathscr{L})$. The standard error is in parenthesis below $\hat{d}$ and the P value is in parenthesis below $Q_{\hat{\varepsilon}}$. The sample size is $T + N = 729$ and the first $N = 20$ observations are used as initial values.

We note that this is simply a re-normalization because a similar rotation of the α matrix implies that the product $\alpha\beta'$, and hence the likelihood, is unchanged. The normalization in (18) implies that each cointegrating relation takes the form

$$\text{CP}_t = \beta_S S_t \quad \text{for } S \in \{\text{LP, LD, IP, GP}\}.$$

Applying the normalization in (18) to $\hat{\beta}$ in Table 8, we find

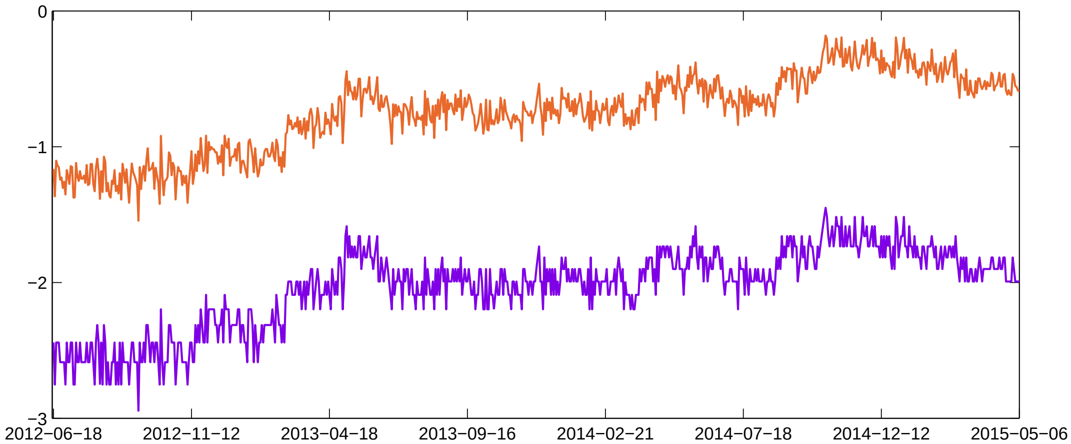
$$\hat{\beta}_{\text{LP}} = -0.282, \quad \hat{\beta}_{\text{LD}} = 0.334, \quad \hat{\beta}_{\text{IP}} = 0.062, \quad \hat{\beta}_{\text{GP}} = -0.170.$$

Thus, it would appear that the government parties, CP and LD, move together in the long-run, although movements in the LD poll share are only about 1/3 of those in the CP poll share. It also seems that the IP poll share moves in the same direction as CP, but again with a coefficient less than one. On the other hand, the poll shares of both opposition parties, LP and GP, move in the opposite direction of the government parties, as expected.

Another relevant interpretation of the model can be obtained from the permanent-transitory (PT) decomposition of Gonzalo and Granger (1995) applied to the FCVAR model. For any matrix Q, we define $Q_\perp$ such that $Q_\perp' Q = Q' Q_\perp = 0$. Then, according to the PT decomposition, X_t may be decomposed into a transitory (stationary) part, $\beta' X_t$, and a permanent part, $W_t = \alpha_\perp' X_t$, using the identity $\beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha(\beta'\alpha)^{-1}\beta' = I_p$, which implies that

$$X_t = (\beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha(\beta'\alpha)^{-1}\beta')X_t = A_1 W_t + A_2 \beta' X_t. \tag{19}$$

Figure 4: Time series plot of estimated common stochastic trend and UKIP data series



Note: Orange line is estimated common stochastic trend ($\hat{W}_t$) and purple line is UKIP (IP_t).

Here, W_t is the common permanent component of X_t, or the common stochastic trend(s). In the case of poll shares, W_t is interpreted as the long-run dominant theme(s) of public opinion, in the sense that information that does not affect W_t will not have a permanent effect on the poll shares given in X_t. On the other hand, $\beta' X_t$ is the transitory or stationary component of X_t, and is interpreted as information that does not have a permanent effect on the poll shares in X_t.

We first calculate the estimated common stochastic trend, $\hat{W}_t = \hat{\alpha}'_\perp X_t$, which is plotted in Figure 4 together with the UKIP data series. Interestingly, the common stochastic trend does not appear to reflect the traditional right-left political spectrum, but rather seems to follow the UKIP series very closely. Thus, we next test the hypothesis that the common stochastic trend is in fact $W_t = \mathrm{IP}_t$, i.e. that $\alpha_\perp = [\ 0 \quad 0 \quad 0 \quad 1 \quad 0\ ]'$. The corresponding mirror hypothesis on α is that the fourth row is equal to zero, so that the test is seen to have four degrees of freedom. The LR statistic is 9.753 with a P value of 0.045 in the asymptotic χ^2_4-distribution. This P value is formally less than 5%, although not by much. Thus, one could make a case for either rejecting the null hypothesis or not. In any case, re-estimating the model with the restrictions imposed (results not reported) yields very similar results to those presented in Table 8 for the unrestricted model, and we shall continue with the unrestricted model.

Whether there is an exact match between the common stochastic trend (W_t) and the UKIP vote share (IP_t) or not, i.e. whether imposing the restriction that $W_t = \mathrm{IP}_t$ or not, the interpretation of the common stochastic trend seen in Figure 4 is that the main theme of political debate in the UK during this time period from June 18th, 2012, to May 6th, 2015, has revolved around the independence question in relation to the European Union, at least in terms of long-run movements of poll shares. We proceed to calculate the coefficient on W_t in (19), which yields

$$\hat{A}_1 = [\ 0.076 \quad -0.269 \quad 0.228 \quad 1.230 \quad -0.446\ ]'$$

for the unrestricted model (i.e., without imposing $W_t = \mathrm{IP}_t$). For the restricted model (imposing $W_t = \text{IP}_t$), the results are qualitatively similar.

Interpreting W_t as a measure of the strength of Euro-skepticism in public opinion, the estimated coefficients in $\hat{A}_1$ suggest that, as Euro-skepticism gains ground and W_t increases, this leads to a large increase in popularity of the UKIP. However, when W_t increases, the government parties (CP and LD) also increase in popularity, whereas the opposition parties (LP and GP) decrease in popularity.[8]

5.2 Predicting vote shares of the election

The UKIP and Green Party poll series exhibit an important caveat, which is that unlike the three major political parties, the UKIP and the Green Party are not stated explicitly as choices in the survey question posed to the poll participants. In the forecasting analysis we do not include the Green Party because doing so would further reduce the sample size and the increased dimension of the model tends to cause problems in the multi-dimensional numerical optimization required for the FCVAR models with either two fractional parameters or with the level parameter. Thus, in the forecasting analysis we consider either three parties (CP_t, LP_t, and LD_t) or four parties (including also IP_t).

We next present predictions of all models in the portfolio leading into the 56$^{\text{th}}$ UK general election held on May 7$^{\text{th}}$, 2015. Specifically, Tables 9 (three parties) and 10 (four parties) show the opinion poll and the election vote share predictions of each model one month, one week and one day preceding the election day, as well as the final election vote share outcome. The predicted values represent forecasts of the poll standings which can be viewed as predicted vote shares for each political party.

Relative to the poll predictions on May 6$^{\text{th}}$, 2015, which underestimated the vote share of the Conservative Party and overestimated the vote shares for Labour and the Liberal Democrats, the forecasts in both Table 9 and Table 10 show that all models in the portfolio predicted a fall in the vote share for the Liberal Democrats, in agreement with the realized vote share. The three-party forecasts in Table 9 generated on the day before the election (Panel C) show that (i) all models predicted vote shares very similar to the poll predictions for the Conservative Party and the Labour Party (with the exception of the ARMA prediction for the Labour Party), (ii) the two FCVAR models with a level parameter and the CVAR model predicted vote shares closest to the election outcomes for the Labour Party, and (iii) the multivariate models predicted vote shares closest to the election outcomes for the Liberal Democrats. The four-party forecasts in Table 10 generated on the day before the election (Panel C) show that (i) variants of the FCVAR model predicted vote shares that are closest to the election outcome for the Conservative Party, (ii) with the exception of the ARMA model all models show similar predictions for the Labour Party, (iii) the multivariate models predicted vote shares closest to the election outcomes for the Liberal Democrats, and (iv) with the exception of the CVAR model all models show similar predictions for the UKIP.

As shown in Tables 9 and 10, the opinion polls severely underestimated the Conservative Party vote shares in the election and similarly overestimated the Labour Party vote shares. Since all statistical models considered here rely solely on the opinion poll series for information, it is not surprising that for political parties for which the election result strongly

[8] Recall that the poll shares are logit transformed. Because of this nonlinear transformation, absolute magnitudes of the coefficients are difficult to interpret, and, in particular, there is no requirement that the coefficients sum to zero.

Table 9: Vote share prediction and results for the 2015 UK general election: three parties

		Conservative	Labour	Liberal Democrats
Panel A: 1 month before election				
Poll		33.00	35.00	8.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	33.41	34.59	7.56
	$\text{FCVAR}_{d,b,\mu}$	33.38	33.76	7.66
	$\text{FCVAR}_{d=b,\rho}$	33.15	34.65	7.19
	$\text{FCVAR}_{d=b,\mu}$	33.22	33.67	7.51
	CVAR_{ρ}	32.68	32.91	7.23
	$\text{ARFIMA}(0, d, 0)$	34.15	35.07	8.38
	$\text{ARFIMA}(p, d, q)$	34.31	34.69	8.13
	$\text{ARMA}(p, q)$	34.63	39.45	9.30
Panel B: 1 week before election				
Poll		34.00	35.00	8.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	33.81	34.65	8.00
	$\text{FCVAR}_{d,b,\mu}$	33.72	34.35	8.10
	$\text{FCVAR}_{d=b,\rho}$	33.64	34.58	7.69
	$\text{FCVAR}_{d=b,\mu}$	33.64	34.24	8.04
	CVAR_{ρ}	33.57	33.79	7.70
	$\text{ARFIMA}(0, d, 0)$	34.04	34.79	8.43
	$\text{ARFIMA}(p, d, q)$	34.07	34.59	8.38
	$\text{ARMA}(p, q)$	34.61	39.36	9.27
Panel C: 1 day before election				
Poll		34.00	34.00	10.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	33.95	34.02	8.68
	$\text{FCVAR}_{d,b,\mu}$	33.89	33.77	8.90
	$\text{FCVAR}_{d=b,\rho}$	33.98	34.05	8.53
	$\text{FCVAR}_{d=b,\mu}$	33.83	33.77	8.90
	CVAR_{ρ}	33.81	33.66	8.65
	$\text{ARFIMA}(0, d, 0)$	33.89	34.11	9.23
	$\text{ARFIMA}(p, d, q)$	33.86	33.98	9.04
	$\text{ARMA}(p, q)$	34.16	37.07	9.80
2015 election result		36.80	30.50	7.90

Notes: The table shows the opinion poll and the election vote share predictions of each model one month (Panel A), one week (Panel B), and one day (Panel C) preceding the election day. The last row shows the election vote share outcomes. Results are for three political parties using the full data set spanning the entire political cycle from May 14$^{\text{th}}$, 2010 to May 6$^{\text{th}}$, 2015. For all three series, the ARFIMA(p, d, q) model specifies $(p, q) = (1, 0)$ and the ARMA(p, q) model specifies $(p, q) = (0, 1)$.

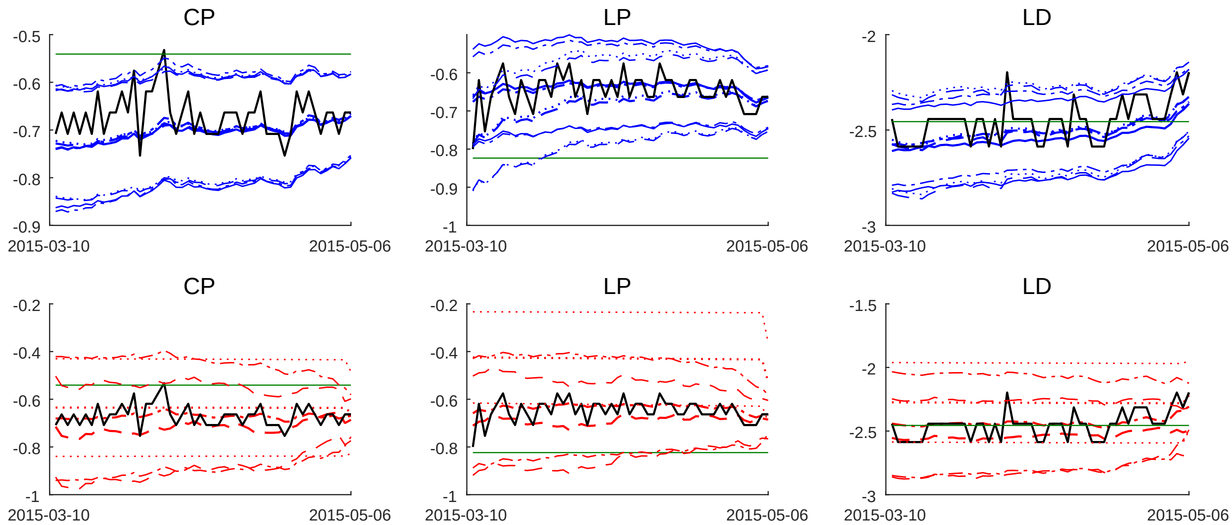
Table 10: Vote share prediction and results for the 2015 UK general election: four parties

		Conservative	Labour	Liberal Democrats	UKIP
Panel A: 1 month before election					
Poll		33.00	35.00	8.00	14.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	34.97	34.24	8.10	12.27
	$\text{FCVAR}_{d,b,\mu}$	34.97	34.24	8.10	12.27
	$\text{FCVAR}_{d=b,\rho}$	33.41	34.20	7.54	14.01
	$\text{FCVAR}_{d=b,\mu}$	34.97	34.24	8.10	12.27
	CVAR_{ρ}	32.99	33.57	6.81	14.64
	$\text{ARFIMA}(0,d,0)$	33.00	35.67	8.01	12.75
	$\text{ARFIMA}(p,d,q)$	33.00	35.67	7.93	12.99
	$\text{ARMA}(p,q)$	32.59	38.81	8.84	11.26
Panel B: 1 week before election					
Poll		34.00	35.00	8.00	12.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	33.71	34.81	8.03	12.97
	$\text{FCVAR}_{d,b,\mu}$	34.46	34.60	8.34	12.45
	$\text{FCVAR}_{d=b,\rho}$	34.37	34.71	7.92	12.82
	$\text{FCVAR}_{d=b,\mu}$	34.46	34.60	8.34	12.45
	CVAR_{ρ}	32.84	34.83	7.58	14.13
	$\text{ARFIMA}(0,d,0)$	33.30	35.18	8.21	12.53
	$\text{ARFIMA}(p,d,q)$	33.30	35.18	8.24	12.53
	$\text{ARMA}(p,q)$	32.60	38.68	8.82	11.31
Panel C: 1 day before election					
Poll		34.00	34.00	10.00	12.00
Forecasts	$\text{FCVAR}_{d,b,\rho}$	33.82	34.33	8.58	12.58
	$\text{FCVAR}_{d,b,\mu}$	34.28	34.07	8.86	12.51
	$\text{FCVAR}_{d=b,\rho}$	34.15	34.26	8.48	12.53
	$\text{FCVAR}_{d=b,\mu}$	34.28	34.07	8.86	12.51
	CVAR_{ρ}	32.97	33.86	7.22	14.63
	$\text{ARFIMA}(0,d,0)$	33.58	34.19	8.98	12.20
	$\text{ARFIMA}(p,d,q)$	33.58	34.19	8.86	12.21
	$\text{ARMA}(p,q)$	32.90	36.57	9.43	11.56
2015 election result		36.80	30.50	7.90	12.70

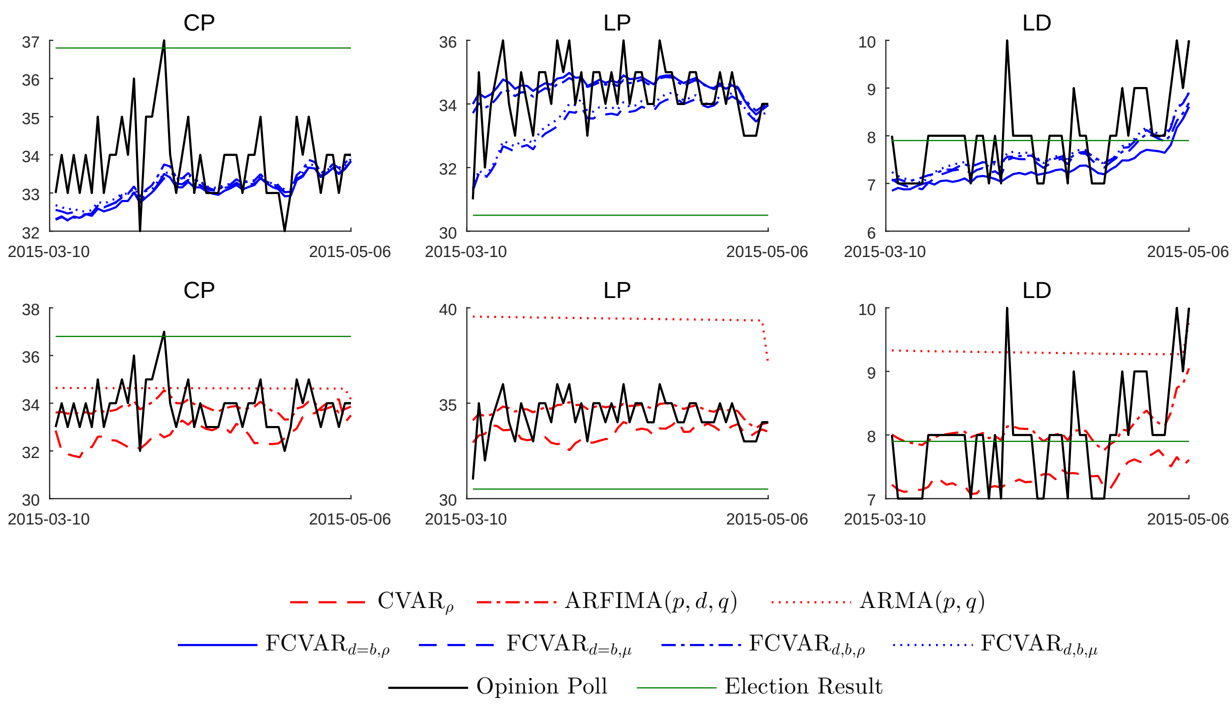
Notes: The table shows the opinion poll and the election vote share predictions of each model one month (Panel A), one week (Panel B), and one day (Panel C) preceding the election day. The last row shows the election vote share outcomes. Results are for four political parties using the subsample of the data set spanning the second half of the political cycle from April 16$^{\text{th}}$, 2012 to May 6$^{\text{th}}$, 2015. For the CP and LP series, the ARFIMA(p,d,q) model specifies $(p,q)=(0,0)$ and the ARMA(p,q) model specifies $(p,q)=(0,1)$. For the LD and IP series, the ARFIMA(p,d,q) model specifies $(p,q)=(1,0)$ and the ARMA(p,q) model specifies $(p,q)=(0,1)$.

Figure 5: Forecasts over 50 polling days leading into the election: three parties

(a) Forecasts of logit transformed series



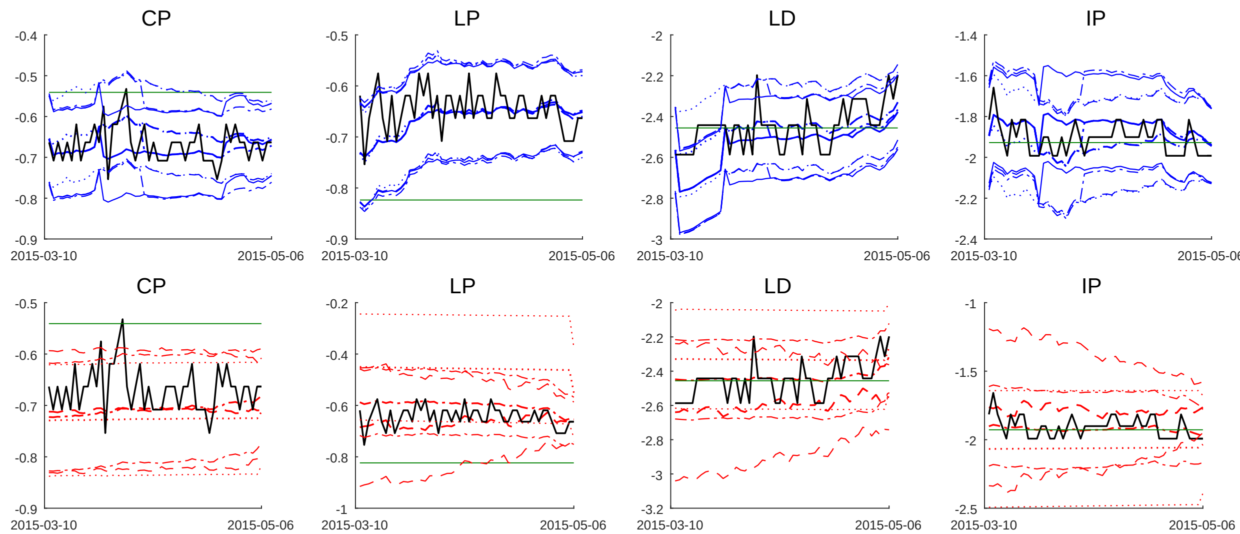
(b) Forecasts of vote shares in percentage



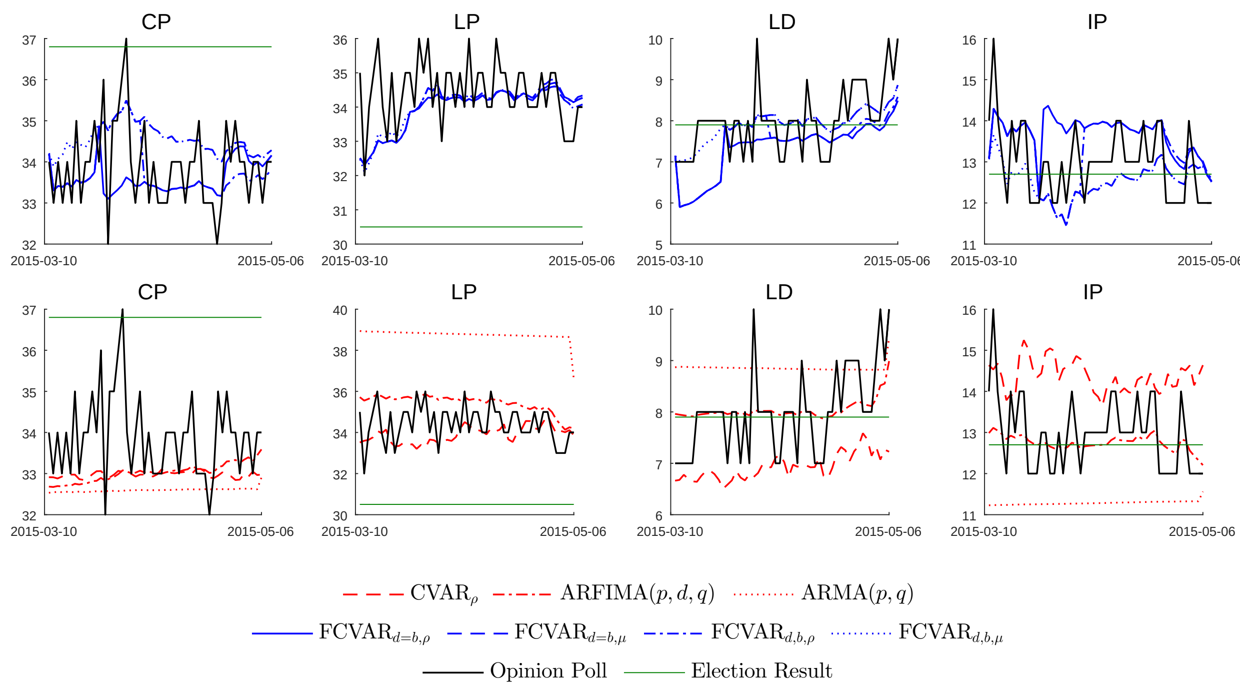
Notes: Each subfigure shows the evolution of forecasts by different models variants for the May 7$^{\text{th}}$, 2015 general election. Panel (a) shows forecasts of logit transformed series, where 90% confidence bands are shown using with slightly thinner lines, and Panel (b) shows forecasts of vote shares in percentage. The forecasts are calculated starting with the data available 50 polling days prior to the election and continue to May 6$^{\text{th}}$, 2015, the day before the election. The results use the full data set spanning the entire political cycle (May 14$^{\text{th}}$, 2010 to May 6$^{\text{th}}$, 2015).

Figure 6: Forecasts over 50 polling days leading into the election: four parties

(a) Forecasts of logit transformed series



(b) Forecasts of vote shares in percentage



Notes: Each subfigure shows the evolution of forecasts by different models variants for the May 7$^{\text{th}}$, 2015 general election. Panel (a) shows forecasts of logit transformed series, where 90% confidence bands are shown using slightly thinner lines, and Panel (b) shows forecasts of vote shares in percentage. The forecasts are calculated starting with the data available 50 polling days prior to the election and continue to May 6$^{\text{th}}$, 2015, the day before the election. The results use the subsample of the data set spanning the second half of the political cycle (April 16$^{\text{th}}$, 2012 to May 6$^{\text{th}}$, 2015).

deviated from the opinion poll prediction, the statistical models find it difficult to predict the high realized vote shares for the Conservative Party and low realized vote shares for the Labour Party. It is in this context that the evolution of the predictions serves to complement opinion polls, since as the most recent poll data became available, forecasters would naturally obtain forecasts for the election.

To illustrate how the analysis presented in this paper complements the industry standard of using the latest opinion poll as an indicator of future political support, we examine the evolution of the model predictions as the observed data approaches the election day on May 7$^{\text{th}}$, 2015, which allows the analyst to monitor the dynamics of the opinion poll and the model predictions. Figures 5 and 6 show forecasts by all models over 50 polling days leading into the election, for the three-party and the four-party analysis, respectively. That is, at time h polling days prior to the election, each figure shows the h-step ahead predictions from each model. Panel (a) of each figure shows forecasts of the logit transformed series, where 90% confidence bands are shown using slightly thinner lines, and Panel (b) of each figure shows forecasts of vote shares in percentage. With these plots, we can analyze the dynamics of model forecasts leading into a particular date of interest, in this case the election day, and compare both across models and with the actual daily poll series.

For the three-party analysis in Figure 5, the evolution of forecasts for all FCVAR model variants show an upward trend in the support for the Conservative Party leading into the election, and a downward trend for the Labour Party for one week leading into the election. These two trends project the correct direction for the realized vote shares in the election. For the Liberal Democrats, the strong upward trends in the predicted vote share, as shown by nearly all models, tend to follow the opinion poll and is therefore suspect to exhibit the same tendency as political polls, in that they tend to over-represent support for smaller political parties when compared to the election vote share outcomes and the representation in government.

For all models, the confidence bands in Figure 5(a) become narrower as the election approaches, which is expected because the forecast horizon shortens. For all the multivariate models, the election outcomes for both the Conservative Party and the Labour Party lie outside the confidence bands for most of the time period considered in the figure. For the univariate models, the confidence bands are wider and include the Conservative Party and Labour Party election outcomes for a large part of the sampling period (except the ARMA model for LP). For the Liberal Democrats the confidence bands of all models include the election outcome throughout the time period in the figure.

In the four-party analysis in Figure 6, the realized vote share for the UKIP, which was ranked third by the political opinion polls, was very close to that prescribed by the opinion poll. As a result, most models in the portfolio appear to perform well. However, the evolution of the FCVAR model predictions appear to converge somewhat more closely to the realized vote share for the UKIP. In particular, the CVAR model overshoots the election result for the UKIP. For both the three-party and the four-party analyses, we find that the FCVAR model predictions are on average closer to the realized vote shares than their analogs from the CVAR model.

The confidence bands in Figure 6(a) are broadly in agreement with those from Figure 5(a) for the Conservatives, Labour, and Liberal Democrats, with the exception that the univariate forecast bands now also exclude the election outcomes for the Conservatives and Labour.

For the UKIP we find that the election result is within the confidence bands for all models throughout the time period considered in the figure. However, noting the different scaling of the second axis for the IP plots for fractional and non-fractional models, respectively, it is apparent that the overprediction of the election result by the CVAR model is also accompanied by a somewhat wider confidence band compared with the FCVAR model variants.

In general, the forecasting results from this empirical application show how modeling time series of political opinion polls using the FCVAR model, which is strongly favored by the model forecast comparisons, can complement the industry standard of basing predictions solely on the most recent opinion poll (e.g., the poll standings on the day preceding the election day) and provide a more informative assessment of the current state of public opinion. Specifically, in the case of the 2015 UK general election, the FCVAR model forecasts provide additional information on party support compared with the hung government prediction of the opinion poll.

6 Concluding remarks

This paper has examined the forecasting performance of the fractionally cointegrated vector autoregressive (FCVAR) model of Johansen (2008) and Johansen and Nielsen (2012) relative to a portfolio of competing models at several forecast horizons. The model was applied in the context of predicting political support in the form of opinion polls; a very relevant topic in the context of forecasting. The analysis used a novel data set of daily polling of political support in the United Kingdom over the period 2010–2015. The analysis has shown how statistical modeling of time series of public opinion polls can improve forecasting public opinion beyond the most recent poll date. This complements the industry standard for measuring the current state of political support through opinion polling, and contributes to decision making processes that rely on poll evidence as inputs.

Specifically, the forecasting analysis has shown that the FCVAR model delivers valuable gains in predicting political support. The accuracy of both short, medium, and long term forecasts generated by the FCVAR model is better than all multivariate and univariate models in the portfolio. Indeed, overall, the four variants of the FCVAR model are the top performing models. Not only do they perform better relative to the other models, but the forecasting performance of all FCVAR model variants are within close range of each other. When compared to both the fractional benchmark model (ARFIMA) and the multivariate benchmark model (CVAR), the FCVAR model significantly outperforms the benchmark at all forecast horizons and the gains are more pronounced at the longer forecast horizons, where the root mean squared forecast error is up to 15% lower than the fractional benchmark and 20% lower than the multivariate benchmark. Overall, the evidence provides strong support for the application of the FCVAR model relative to both univariate fractional models and multivariate non-fractional models. Fractional cointegration substantially improves forecast accuracy, and the gains become more pronounced at longer forecast horizons.

More generally, it is of interest to consider how generalizable our results are to other data sets. The arguments in Section 2 certainly suggest that all polling data may be best modeled as fractional time series. This in turn suggests that the FCVAR model would be applicable to all such data, and that the FCVAR model would produce superior forecasts for such data. In fact, the arguments in Section 2 suggest that our results may be generalizable to all data that have an aggregation structure, such as, in political science, any polling data, voting data, government support data, partisan indicators, etc. A thorough investigation of whether this is in fact the case is beyond the scope of this paper, and we leave this interesting issue for future research. However, we do note that the very general applicability of fractional integration models is in line with the literature, see e.g. Box-Steffensmeier and Tomlinson (2000) and Lebo et al. (2000).

In an empirical application to the 2015 UK general election, we have discussed the FCVAR model estimation results, with interpretations of both the estimated cointegrating relations and estimated common stochastic trend. It appears that the latter can be interpreted as a measure of Euro-skepticism, rather than an indicator of the more traditional left-right political spectrum, reflecting public opinion and debate in the sampling period which was to a great extent focused on the European Union question. The forecasts generated by the FCVAR model leading into the election appear to provide a more informative assessment of the current state of public opinion on electoral support than the hung government prediction of the opinion polls. Specifically, when three political parties are modeled over the full election cycle, the FCVAR model predicts the correct direction for the realized vote shares in the election for the Conservative and Labour parties. When four political parties are modeled over the shorter available data set spanning the second half of the election cycle, the predictions of the FCVAR models are closer to the realized vote shares than their analogs from the CVAR model, and in particular the FCVAR models appear to converge more closely to the realized vote share for the UKIP.

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