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A fractionally cointegrated VAR model with deterministic trends and application to commodity futures markets

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Abstract

We apply the fractionally cointegrated vector autoregressive (FCVAR) model to analyze the relationship between spot and futures prices in five commodity markets (aluminium, copper, lead, nickel, and zinc). To this end, we first extend the FCVAR model to accommodate deterministic trends in the levels of the processes. The methodological contribution is to provide representation theory for the FCVAR model with deterministic trends, where we show that the presence of the deterministic trend in the process induces both restricted and unrestricted constant terms in the vector error correction model. The consequences for the cointegration rank test are also briefly discussed. In our empirical application we use the data from Figuerola-Ferretti and Gonzalo (2010), who conduct a similar analysis using the usual (non-fractional) cointegrated VAR model. The main conclusion from the empirical analysis is that, when using the FCVAR model, there is more support for the cointegration vector $(1,-1)'$ in the long-run equilibrium relationship between spot and futures prices, and hence less evidence of long-run backwardation, compared to the results from the non-fractional model. Specifically, we reject the hypothesis that the cointegration vector is $(1,-1)$ using standard likelihood ratio tests only for the lead and nickel markets.

JEL codes: C32, G14.
Keywords: backwardation, contango, deterministic trend, fractional cointegration, futures markets, vector error correction model.

1 Introduction

A large empirical literature is aimed at assessing to what extent futures markets are efficient. Letting s_t and f_t denote log-spot and log-futures prices, respectively, this has traditionally involved assessing whether the parameter β_2 is equal to unity in relationships of the type $s_t = \beta_2 f_t + \rho$. When the pair $X_t = (s_t, f_t)$ is an integrated time series (of order one), there is a large literature focusing on investigating this one-for-one relationship using cointegration methods. In that framework, letting β denote the cointegration vector, the issue is to assess a version of long-run efficiency, i.e. whether (i) cointegration exists such that $\beta' X_t$ is a stationary relation, and (ii) the cointegration vector is $\beta = (1,-1)'$.

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Despite the widespread acceptance of efficient markets in theory, the cointegration vector $\beta = (1, -1)'$ that it postulates has been difficult to verify in empirical work, see, e.g., Chowdhury (1991), Brenner and Kroner (1995), and Kellard, Newbold, Rayner, and Ennew (1999) for some early evidence, and Chow et al. (2000) for a survey. In particular, although the presence of unit roots in both spot and futures prices is generally accepted under standard no-arbitrage conditions, most of the early empirical evidence based on the usual I(1) vs I(0) paradigm has rejected the one-for-one cointegration hypothesis, see e.g. Chow et al. (2000) and Westerlund and Narayan (2013) for detailed discussions and reviews of the empirical evidence. Recently, Figuerola-Ferretti and Gonzalo (2010) (FG hereafter) develop an equilibrium model with finite elasticity of supply of arbitrage services and endogenously modeled convenience yields, and show how this model implies cointegration between spot and futures prices, although the cointegration vector β need not be equal to $(1, -1)'$. When the slope of the cointegration vector $\beta_2 > 1$ ($\beta_2 < 1$) the market is said to be under long-run backwardation (in contango). Generally, backwardation (contango) exists when prices decline (increase) with time to delivery, so that spot prices are greater (lower) than futures prices (Routledge, Seppi, and Spatt, 2000), e.g. due to convenience yields, storage costs, etc.

In recent literature, there has been much research on the potential presence of fractional integration (or long memory) in the equilibrium relation between spot and futures prices, i.e. in $\beta' X_t$ or $s_t - f_t$. This implies fractional cointegration between futures and spot prices in the sense that, although the prices themselves are I(1), $\beta' X_t$ is fractionally integrated of a lower order between zero and one. This generalizes the usual notion of cointegration where $\beta' X_t$ would be I(0). Specifically, with respect to futures markets, recent studies have found evidence of fractional integration in the forward or futures premium; see, e.g., Baillie and Bollerslev (1994), Lien and Tse (1999), Maynard and Phillips (2001), Kellard and Sarantis (2008), and Coakley, Dollery, and Kellard (2011).

One of the latest developments in the (fractional) cointegration literature is the fractionally cointegrated vector autoregressive (FCVAR) model of Johansen (2008) and Johansen and Nielsen (2012). This model generalizes Johansen's (1995) cointegrated vector autoregressive (CVAR) model to allow for fractionally integrated time series that cointegrate to a lower (fractional) order. Asymptotic theory for estimation and inference in the FCVAR model was developed recently in a series of papers by Johansen and Nielsen (2010, 2012, 2015). They prove asymptotic distribution results for the maximum likelihood estimators and for the likelihood ratio test for cointegration rank. Furthermore, Nielsen and Popiel (2015) provide a accompanying Matlab programs for calculation of estimators and test statistics, and MacKinnon and Nielsen (2014) provide computer programs for calculation of P values and critical values for the cointegration rank tests.

The FCVAR model has many advantages when estimating a system of fractional time series variables that are potentially cointegrated. The flexibility of the model permits one to determine the cointegrating rank, or number of equilibrium relations, via statistical tests and to jointly estimate the adjustment coefficients and the cointegrating relations, while accounting for the short-run dynamics. Each of these features will typically be relevant to the research question in empirical work. For example, in our empirical application, the cointegrating rank is the number of long-run equilibria that exist between the spot and futures prices, while the cointegrating relations themselves are the linear combinations of these variables that form a stationary equilibrium, which we can investigate for the presence of backwardation or contango. The only potential limitation of the FCVAR framework in relation to empirical application to asset prices is that the analysis in Johansen (2008) and Johansen and Nielsen (2010, 2012, 2015) allows at most a non-zero mean (a so-called restricted constant term), but no deterministic trend in the observed variables.

This paper aims to make two distinct contributions. The first is a methodological contribution to allow for deterministic linear time trends in the observed variables in the FCVAR model, i.e. a drift in the price processes in our application. We provide the necessary representation theory for the model with deterministic trend and derive the relevant error correction (regression-type) equation to be estimated. In the fractional case, the presence of deterministic trends in the process induces both so-called restricted and unrestricted constant terms in the error correction model. The consequences for the cointegration rank (trace) test are also briefly discussed.

Our second contribution is to apply the FCVAR model to re-analyze the empirical application in FG, who apply the usual non-fractional CVAR model. To justify this application of the FCVAR model from economic theory, we first suggest a variation of the equilibrium model developed by FG from which it follows that spot and futures prices are fractionally cointegrated. We thus apply the FCVAR model to the dataset from FG which consists of daily observations from the London Metal Exchange on spot and futures prices in five commodity markets for non-ferrous metals (aluminium, copper, lead, nickel, and zinc) from January 1989 to October 2006. In all markets the spot and futures prices are cointegrated, and for all markets except copper the fractional integration parameter is highly significant, showing that the usual non-fractional model is not appropriate.

The first main finding in our empirical application is that, when allowing for fractional integration in the long-run equilibrium relations, fewer lags appear to be needed in the autoregressive formulation compared to the non-fractional model, further stressing the usefulness of the fractional model. Secondly, compared to the results from the non-fractional model, there is more evidence in favor of cointegration vector $\beta = (1, -1)'$. That is, there is more support for stationarity of the spread, $s_t - f_t$, and hence less support for long-run backwardation or contango, compared to the analysis in FG based on the CVAR model. Specifically, we reject the hypothesis that the cointegration vector is $\beta = (1, -1)'$ using standard likelihood ratio tests only for the lead and nickel markets, whereas FG reject the hypothsis for all markets except copper.

The remainder of the paper is organized as follows. Section 2 presents the FCVAR model and the representation theory to allow for deterministic time trends. Section 3 presents the equilibrium model in FG and the extension that results in fractional cointegration. In Section 4 we describe the data and empirical results and Section 5 concludes.

2 The FCVAR model with a deterministic linear trend

In our empirical analysis we use the FCVAR model of Johansen (2008) and Johansen and Nielsen (2010, 2012, 2015). This model is a generalization of Johansen's (1995) CVAR model to allow for fractional processes of order d that cointegrate to order $d - b$. However, the development of the model in Johansen (2008) and Johansen and Nielsen (2010, 2012, 2015) does not accommodate drift in prices, i.e. deterministic linear trends in the observed variables. In this section we develop an extension of the model to allow such linear time trends.

2.1 Fractional differencing and fractional integration

The fractional (or fractionally integrated) time series models are based on the fractional difference operator,

$$\Delta^d X_t = \sum_{n=0}^{\infty} \pi_n(-d) X_{t-n}, \tag{1}$$

where the fractional coefficients $\pi_n(u)$ are defined in terms of the binomial expansion $(1-z)^{-u} = \sum_{n=0}^{\infty} \pi_n(u) z^n$, i.e.,

$$\pi_n(u) = \frac{u(u+1)\cdots(u+n-1)}{n!} \leq c n^{u-1} \tag{2}$$

for some $c < \infty$ that does not depend on n or u. Many other details and intermediate results regarding this expansion and the fractional coefficients are given in, e.g., Johansen and Nielsen (2015, Appendix A). Calculating fractional differences can be computationally very demanding, but efficient calculation using the fast Fourier transform, which we apply in our estimation, is discussed in Jensen and Nielsen (2014).

Given the definition of the fractional difference operator in (1), a time series X_t is said to be fractional of order d, denoted $X_t \in \mathrm{I}(d)$, if its d'th difference is fractional of order zero, i.e. if $\Delta^d X_t \in \mathrm{I}(0)$. The I(0) property can be defined in the frequency domain as having spectral density that is finite and non-zero near the origin or in terms of the linear representation coefficients if the sum of these is non-zero and finite, see, e.g., Johansen and Nielsen (2012). For example, the stationary and invertible ARMA model is I(0).

2.2 The FCVAR model without deterministic terms

The FCVAR model is most easily derived starting from the well-known non-fractional CVAR model. To that end, let $Y_t, t = 1, \ldots, T$, be a p-dimensional I(1) time series. Then the CVAR model for Y_t is

$$\Delta Y_t = \alpha\beta' Y_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta Y_{t-i} + \varepsilon_t = \alpha\beta' L Y_t + \sum_{i=1}^{k} \Gamma_i \Delta L^i Y_t + \varepsilon_t, \tag{3}$$

where, as usual, ε_t is p-dimensional independent and identically distributed with mean zero and covariance matrix Ω. On the right-hand side of (3) the lag and difference operators, Δ and $L = 1-\Delta$, are written explicitly, and the FCVAR model is derived by replacing these by their fractional counterparts, Δ^b and $L_b = 1 - \Delta^b$, respectively. We then obtain

$$\Delta^b Y_t = \alpha\beta' L_b Y_t + \sum_{i=1}^{k} \Gamma_i \Delta^b L_b^i Y_t + \varepsilon_t, \tag{4}$$

and applying this model to $Y_t = \Delta^{d-b} X_t$ we obtain the FCVAR model,

$$\Delta^d X_t = \alpha\beta' L_b \Delta^{d-b} X_t + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t. \tag{5}$$

Some of the parameters in the FCVAR model are well-known parameters from the CVAR model and these have the usual interpretations also in the FCVAR model. Most importantly are the long-run parameters α and β, which are $p \times r$ matrices with $0 \le r \le p$. The columns of β constitute the cointegrating vectors such that $\beta' X_t$ are the stationary linear combinations of the variables in the system, i.e. the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables. The short-run dynamics of the variables are governed by the parameters Γ_i in the autoregressive augmentation.

The FCVAR model has two additional parameters compared with the CVAR model, namely the fractional parameters d and b. First, d denotes the fractional integration order of the observable time series. Presumably, most financial assets would have $d = 1$ based on standard no-arbitrage arguments, so we assume in our empirical study that $d = 1$, which is also in accordance with Assumptions B.3 and C.3 in the economic model below. The consequence is that we consider $d = 1$ to be fixed and known, and therefore not estimated as would usually be the case. Next, b determines the degree of fractional cointegration, i.e. the reduction in fractional integration order of $\beta' X_t$ compared to X_t itself. We do not fix this parameter but rather estimate it jointly with the remaining parameters. There are two relevant ranges for b, namely $(0, 1/2]$, in which case the equilibrium errors are fractional of order greater than 1/2, and are thus non-stationary although mean reverting, and $(1/2, 1]$, in which case the equilibrium errors are fractional of order less than 1/2 and are thus stationary. Finally note that when $d = b = 1$ the FCVAR model reduces to the CVAR model, which is therefore nested in the FCVAR model as a special case.

The FCVAR model has several advantages and features that it shares with the CVAR model. It (i) admits statistical tests to determine how many long-run equilibrium relations exist, (ii) enables simultaneous modeling of the long-run equilibria, the adjustment responses to deviations from the equilibria, and the short-run dynamics of the system, and (iii) enables evaluation of model fit, i.e. whether the assumptions underlying the asymptotic distribution theory are likely satisfied, by examining the model residuals using, for instance, tests for serial correlation.

2.3 Introducing a linear trend in the FCVAR model

In many empirical applications, including our application to spot and futures prices, and following our Assumption C.3 in the economic equilibrium model below, we want to accommodate a deterministic linear trend in the variables. For convenience in the mathematical derivations we use

$$\pi_t(1) = 1_{\{t\geq 0\}} \text{ and } \pi_t(2) = (t+1)1_{\{t\geq 0\}} \tag{6}$$

as the constant (mean) and linear trend terms, respectively, where $\pi_t(\cdot)$ is defined in (2) and $1_{\{A\}}$ denotes the indicator function of the event A. These simple $\pi_t(\cdot)$ coefficients are preferred instead of the usual $(1, t)$ because of the property $\Delta^b \pi_t(a) = \pi_t(a-b)$ which is not shared by $(1, t)$. However, assuming the data started at some finite time in the past, $(\pi_t(1), \pi_t(2))$ could be replaced by $(1, t)$ in what follows at the cost of a small asymptotically negligible approximation error.

We thus assume that X_t is generated by the unobserved components model

$$X_t = \tau_1 \pi_t(1) + \tau_2 \pi_t(2) + X_t^0, \tag{7}$$

where X_t^0 is the zero-mean FCVAR in (5). The deterministic terms in (6) are therefore defined such that the parameters τ_1 and τ_2 in (7) allow for a linear deterministic trend in X_t. However, the trend is not empirically warranted in the equilibrium relation $\beta' X_t$, and consequently we impose the restriction $\beta'\tau_2 = 0$, see also (19) below. Recall that, when deriving the representation theory, we are discussing the data generating process, and hence all parameters are interpreted as true values.

The representation theory for the zero-mean FCVAR model (5) is given in Johansen (2008) and Johansen and Nielsen (2012, Theorem 2). Under the conditions imposed there, X_t^0 has solution

$$X_t^0 = C\Delta_+^{-d}\varepsilon_t + \Delta_+^{-(d-b)} Y_t + \phi_t, \quad t = 1, \ldots, T, \tag{8}$$

when $d \geq 1/2$, where the subscript "+" on Δ denotes that the summation in the fractional difference operator (2) is truncated at $n = t-1$, Y_t is an I(0) process with zero mean, ϕ_t is a term that depends only on the initial values of X_t^0, and the matrix C is given by

$$C = \beta_\perp(\alpha_\perp' \Gamma \beta_\perp)^{-1}\alpha_\perp', \quad \Gamma = I_p - \sum_{i=1}^{k} \Gamma_i, \tag{9}$$

with $\beta_\perp$ defined such that $\beta_\perp'\beta = \beta'\beta_\perp = 0$ and similarly for $\alpha_\perp$. When X_t^0 is stationary, i.e. $d < 1/2$, the representation is instead

$$X_t^0 = C\Delta^{-d}\varepsilon_t + \Delta^{-(d-b)} Y_t, \quad t = 1, \ldots, T. \tag{10}$$

In terms of the representations (8) and (10), we can now define the restriction $\beta'\tau_2 = 0$ such that the trend parameter τ_2 is proportional to $\beta_\perp$, that is we can model $\tau_2 = C\theta$ for θ freely varying.

The purpose of this section is to derive the vector error correction model that corresponds to the unobserved components representation (7), i.e. the implied FCVAR model. To that end, we apply

$$\Pi(L) = I_p\Delta^d - \alpha\beta' L_b \Delta^{d-b} - \sum_{i=1}^{k} \Gamma_i L_b^i \Delta^d \tag{11}$$

to X_t in (7), where we find that the stochastic term is $\Pi(L)X_t^0 = \varepsilon_t$ for $t \geq 1$, see (5). For the deterministic terms we find the following result.

Theorem 1. *Let $\Pi(L)$ be given by (11) and define the deterministic coefficients $\pi_t(\cdot)$ as in (2). Then*

$$\begin{aligned}
\Pi(L)(\tau_1\pi_t(1) + \tau_2\pi_t(2)) = &- \alpha\beta'\tau_1 L_b \Delta^{d-b}\pi_t(1)1_{\{d-b\leq 1/2\}} &(12)\\
&+ \Gamma\tau_1\pi_t(1-d)1_{\{d\leq 1/2\}} + \Gamma\tau_2\pi_t(2-d)1_{\{d\leq 3/2\}} &(13)\\
&+ \sum_{j=1}^{\max\{k,[(1/2-d)/b]\}} \binom{k}{j}\Psi_j\tau_1\pi_t(1-d-jb) &(14)\\
&+ \sum_{j=1}^{\max\{k,[(3/2-d)/b]\}} \binom{k}{j}\Psi_j\tau_2\pi_t(2-d-jb) + \zeta_t &(15)
\end{aligned}$$

for $t \geq 1$, where ζ_t has finite information in the sense that $\sum_{t=1}^T \zeta_t^2 < \infty$ and $[\cdot]$ denotes the integer part of the argument.

Proof. We apply $\Pi(L)$ in (11) to $\tau_1\pi_t(1) + \tau_2\pi_t(2)$ and find, using $\beta'\tau_2 = 0$, that

$$\begin{aligned}
\Pi(L)(\tau_1\pi_t(1) + \tau_2\pi_t(2)) &= - \alpha\beta'\tau_1 L_b \Delta^{d-b}\pi_t(1) + \Gamma\tau_1\pi_t(1-d) + \Gamma\tau_2\pi_t(2-d)\\
&\quad + \sum_{i=1}^{k}\Gamma_i(I_p - L_b^i)(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d))\\
&= - \alpha\beta'\tau_1 L_b \Delta^{d-b}\pi_t(1) + \Gamma\tau_1\pi_t(1-d) + \Gamma\tau_2\pi_t(2-d)\\
&\quad + \sum_{i=1}^{k}\Gamma_i(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d))\\
&\quad - \sum_{i=1}^{k}\Gamma_i(1-\Delta^b)^i(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d))\\
&= - \alpha\beta'\tau_1 L_b \Delta^{d-b}\pi_t(1) + \Gamma\tau_1\pi_t(1-d) + \Gamma\tau_2\pi_t(2-d)\\
&\quad - \sum_{i=1}^{k}\Gamma_i\sum_{j=1}^{i}\binom{i}{j}(-1)^j\Delta^{jb}(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d)),
\end{aligned}$$

where the last equality follows from the binomial theorem applied to $(1-\Delta^b)^i$. Defining the parameters $\Psi_j = (-1)^{j+1}\sum_{i=1}^{j}\Gamma_i$ and reversing the summations over i and j we obtain

$$\begin{aligned}
- \sum_{i=1}^{k}\Gamma_i\sum_{j=1}^{i}\binom{i}{j}(-1)^j\Delta^{jb}(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d)) &= \sum_{j=1}^{k}\binom{k}{j}\Psi_j\Delta^{jb}(\tau_1\pi_t(1-d) + \tau_2\pi_t(2-d))\\
&= \sum_{j=1}^{k}\binom{k}{j}\Psi_j(\tau_1\pi_t(1-d-jb) + \tau_2\pi_t(2-d-jb)).
\end{aligned}$$

Finally, we use the fact that all $\pi_t(a)$ terms with argument $a < 1/2$ have finite information because $\sum_{t=1}^T \pi_t(a)^2 \leq c\sum_{t=1}^T t^{2a-2} \leq c$. These become part of ζ_t, and the result follows. ☐

From (12)–(15) in Theorem 1 it is clear that the simple deterministic trend introduced in the model for the observed variables via the unobserved components formulation (7) gives rise to many deterministic trend-type terms in the vector error correction model. The term on the right-hand side of (12) is the so-called restricted constant term in the terminology of Johansen (1995), while (13)–(15) are so-called unrestricted deterministic terms. In the non-fractional CVAR model with $d = b = 1$ these are such that the arguments in the $\pi_t(\cdot)$ coefficients are always integers, in which case the many terms in (13)–(15) can be combined and the associated parameters are freely varying. However, in the present fractional setup, there are many "trends" introduced in the form of $\pi_t(\cdot)$ coefficients with different arguments, and these do not necessarily combine into simple expressions. Moreover, these new terms in the vector error correction model have parameters that are complicated nonlinear functions of the underlying model parameters, and they are therefore not really "unrestricted".

Note that the indicator functions in (12) and (13) and the upper summation limits in (14) and (15) imply that the deterministic terms may not even be present. This generalizes the well-known results that one cannot consistently estimate the mean of a random walk or unit root process (which has $d = 1$). In the present model, this situation occurs if the integration order, d, of the variables and/or the integration order, $d - b$ of the equilibrium errors are too high to enable estimation of mean and trend terms.

However, in applications it will often be the case that $d = 1$, as for example in the case of asset prices based on a no-arbitrage argument. Furthermore, it will often be assumed that $b > 1/2$ such that the equilibrium errors are stationary, i.e. have fractional integration order $d-b = 1-b < 1/2$. In this special case Theorem 1 simplifies in a nice way, which is stated as a corollary.

Corollary 1. *Under the assumptions of Theorem 1 with $d = 1$ and $b > 1/2$ we find*

$$\Pi(L)(\tau_1\pi_t(1) + \tau_2\pi_t(2)) = -\alpha L_b \Delta^{1-b}\rho'\pi_t(1) + \xi\pi_t(1) + \zeta_t \tag{16}$$

for $t \geq 1$, where ζ_t has finite information and the parameters $\rho' = \beta'\tau_1$ and $\xi = \Gamma\tau_2$ are freely varying.

Proof. The result follows from Theorem 1 by setting $d = 1$, $b > 1/2$, and using that $\pi_t(0) = 0$ for $t \geq 0$. □

The result in Corollary 1 implies a relatively simple vector error correction equation for X_t, which is obtained by application of $\Pi(L)$ to X_t and using (5) and (16). Then,

$$\Delta X_t = \alpha L_b \Delta^{1-b}(\beta' X_t - \rho'\pi_t(1)) + \sum_{i=1}^{k} \Gamma_i \Delta L_b^i X_t + \xi\pi_t(1) + \varepsilon_t, \tag{17}$$

where the negligible term ζ_t in (16) is ignored (or absorbed into ε_t). Here, ρ is interpreted as the mean of the stationary linear combinations, $\beta' X_t$, and ξ gives rise to the linear deterministic trend in the levels of the variables. In the terminology of Johansen (1995), (17) contains both a restricted constant, $\rho'\pi_t(1)$, and an unrestricted constant, $\xi\pi_t(1)$.

To interpret the model and the representation results obtained, we examine the behavior of X_t in different directions. This will also be useful in a discussion of the asymptotic distribution of the test for cointegration rank. Combining (7) and (8), the process X_t is given by, under the assumptions of Corollary 1,

$$X_t = \tau_1\pi_t(1) + C\theta\pi_t(2) + C\sum_{t=1}^{T} \varepsilon_t + \Delta_+^{-(1-b)} Y_t + \phi_t, \quad t = 1, \ldots, T. \tag{18}$$

In the cointegrating directions, β, we find that

$$\beta' X_t = \beta'(\tau_1\pi_t(1) + C\theta\pi_t(2) + X_t^0) = \rho'\pi_t(1) + \beta'\Delta_+^{-(1-b)} Y_t + \beta'\phi_t, \tag{19}$$

which is I$(1-b)$ with a non-zero mean given by ρ' (apart from the asymptotically negligible term arising from the initial values component ϕ_t).

In the trend direction given by $\bar{\tau}_2 = \tau_2(\tau_2'\tau_2)^{-1}$, we find that

$$T^{-1}\bar{\tau}_2' X_{[Tu]} = T^{-1}\bar{\tau}_2'\tau_1\pi_{[Tu]}(1) + T^{-1}\pi_{[Tu]}(2) + T^{-1}\bar{\tau}_2' X^0_{[Tu]} = u + o_P(1)$$

indeed generates the trend. The quantity relevant to the asymptotic theory is

$$T^{-b}\bar{\tau}_2'\Delta^{1-b} X_{[Tu]} = \frac{1}{\Gamma(1+b)} u^b + o_P(1), \tag{20}$$

where $\Gamma(\cdot)$ denotes the Γ function.[1]

Finally, let γ be orthogonal to $(\beta', \tau_2')'$ so that γ denotes the remaining, non-stationary directions. In these directions we obtain

$$\gamma' X_t = \gamma'(\tau_1\pi_t(1) + \tau_2\pi_t(2) + X_t^0) = \gamma'\tau_1\pi_t(1) + \gamma' X_t^0$$

and it is seen that the last term on the right-hand side dominates. In this case, the relevant term for the asymptotic theory is

$$\begin{aligned} T^{1/2-b}\gamma'\Delta^{1-b} X_{[Tu]} &= T^{1/2-b}\gamma'\tau_1\pi_{[Tu]}(b) + T^{1/2-b}\gamma'\Delta^{1-b} X^0_{[Tu]} \\ &= o(1) + T^{1/2-b}\gamma'\Delta^{1-b} X^0_{[Tu]} \Rightarrow W_{b-1}(u), \end{aligned} \tag{21}$$

where "$\Rightarrow$" denotes weak convergence and $W_{b-1}(u) = \Gamma(b)^{-1}\int_0^u (u-s)^{b-1} \mathsf{d}W(s)$ is fractional Brownian motion (fBM) of type II, see e.g. Johansen and Nielsen (2012, Equation (6)), and $W(s)$ is p-dimensional Brownian motion generated by ε_t.

Based on the above discussion and interpretation of the process X_t in the different directions β, $\bar{\tau}_2$, and γ, and with some inspiration from the non-fractional CVAR model, see Johansen (1995, Chapter 6), and the FCVAR model without trend, see Johansen and Nielsen (2012, Section 5), we make the following conjectures regarding the asymptotic distribution of the likelihood ratio (LR) test for cointegration rank (the so-called trace test). When $b < 1/2$, this will be $\chi^2((p-r)^2)$ as in Johansen and Nielsen (2012, Theorem 11(ii)). When $b > 1/2$, the asymptotic distribution will have the usual form,

$$\operatorname{tr}\left\{ \left(\int_0^1 F(u) \mathsf{d}B(u)' \right)' \left(\int_0^1 F(u)F(u)' \mathsf{d}u \right)^{-1} \left(\int_0^1 F(u) \mathsf{d}B(u) \right) \right\}, \tag{22}$$

where $B(u)$ is $(p-r)$-dimensional standard Brownian motion. In the basic case without deterministic terms, $F(u) = B_{b-1}(u)$ is $(p-r)$-dimensional standard fBM, see Johansen and Nielsen (2012, Theorem 11(i)). However, inspired by the analysis of the CVAR model, we make the following observations:

a) The trend direction (20) dominates the non-stationary directions (21), such that, because the trend direction is one-dimensional, the last fBM in $F(u)$ is replaced by u^b.
b) Because an unrestricted constant is included in the vector error correction model (17), the process $F(u)$ should be demeaned.
c) When a restricted constant is in the model, the process $F(u)$ should be extended by u^{b-1}, see Johansen and Nielsen (2012, Theorem 11(iv)).

[1] This notation is unfortunate, since Γ also denotes the parameter $I_p - \sum_{i=1}^k \Gamma_i$, see (9). However, the two uses are both very standard in the literature, and because the two are used in different places and play very different roles in the analysis, the notation hopefully shouldn't cause any confusion.

Thus, we conjecture that the asymptotic distribution of the LR test for cointegration rank is given by (22) with

$$F_i(u) = B_{i,b-1}(u) - \int_0^1 B_{i,b-1}(u)\mathsf{d}u,\ i = 1, ..., p-r-1,$$
$$F_{p-r}(u) = u^b - \int_0^1 u^b \mathsf{d}u = u^b - 1/(b+1),$$
$$F_{p-r+1}(u) = u^{b-1} - \int_0^1 u^{b-1}\mathsf{d}u = u^{b-1} - 1/b.$$

Of course, as in the cases considered by Johansen and Nielsen (2012), this asymptotic distribution depends on the unknown (true value of the) scalar parameter b, so it will need to be simulated on a case-by-case basis, which is what we do in our empirical application.

2.4 Other estimation and inference results for the FCVAR model

The asymptotic theory of estimation and inference for the model is discussed in Johansen and Nielsen (2012), and practical implementation as well as Matlab computer programs for the calculation of estimators and test statistics are provided in Nielsen and Popiel (2015). The latter programs are also applicable to our model with linear trends. Specifically, maximum likelihood estimation of (17) can be performed as follows. For fixed b, (17) is estimated by reduced rank regression of ΔX_t on $L_b\Delta^{1-b}(X_t', -\pi_t(1))'$ corrected for $\{\Delta L_b^i X_t\}_{i=1}^k$ and $\pi_t(1)$. This results in a profile likelihood which is a function only of b that can be maximized numerically.

Note that the fractional difference operator in (1) is defined in terms of an infinite series. However, any observed sample has only a finite number of observations, thus prohibiting calculation of fractional differences as defined. A popular simplifying assumption in the literature, which would allow calculation of the fractional differences, is that X_t were zero before the start of the sample. However, this would hardly ever be the case in practice, and we certainly cannot reasonably make such an assumption in our empirical application. Johansen and Nielsen (2015) use higher-order expansions in a simpler model to analyze the bias introduced by making such an assumption to allow calculation of the fractional differences, and they suggest several ways to alleviate the bias. In particular, they show, following ideas in Johansen and Nielsen (2010, 2012), that the bias can be alleviated by splitting the observed sample into initial values to be conditioned upon and observations to include in the likelihood. This is therefore similar to the well-known way in which, in estimation of AR(k) models, a sample is divided into k initial values and $T-k$ observations to reduce conditional maximum likelihood estimation to least squares regression. In our empirical analysis we will apply maximum likelihood inference conditional on initial values.

The asymptotic distribution theory derived in Johansen and Nielsen (2012) shows that the maximum likelihood estimator of $(b, \alpha, \Gamma_1, \ldots, \Gamma_k)$ is asymptotically normal, while the maximum likelihood estimator of (β, ρ) is asymptotically mixed normal when the true value $b_0 > 1/2$ and asymptotically normal when $b_0 < 1/2$. The important implication of these results is that asymptotic χ^2-inference can be conducted on the parameters $(b, \rho, \alpha, \beta, \Gamma_1, \ldots, \Gamma_k)$ using ordinary LR tests.

We will test a number of interesting hypotheses on the model parameters in our empirical analysis, specifically on b and β. In this context, it is important to recall that β itself is not identified; only the span of β is identified. Hence we can not interpret or conduct tests on the absolute value of parameters in β, but only on the relative value of these. The general theory of hypothesis testing on β for the CVAR model (Johansen, 1995, Chapter 7) carries over almost unchanged to the FCVAR model. In particular, the degrees of freedom is equal to the number of overidentifying restrictions under the null. Although counting the degrees of freedom in tests on β

is non-standard because of the normalization required to separately identify α and β, this is done in the same way for the FCVAR model as for the CVAR model. In particular, the main hypothesis of interest on β can be formulated as

$$\beta = H\varphi, \tag{23}$$

where the known $p \times s$ matrix H specifies the restriction(s) and φ is an $s \times r$ matrix of freely varying parameters. In this case the same restriction is imposed on each cointegrating relation, as is relevant in our empirical analysis. The degrees of freedom of the test is given by $\text{df} = (p - s)r$.

Using the framework in (23) with one cointegration vector, i.e. $r = 1$, the hypothesis $\mathscr{H}_\beta : \beta = (1, -1)'$ can be formulated with

$$\underset{p \times s}{H} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where $(p, s, r) = (2, 1, 1)$ and φ is a freely varying scalar. The degrees of freedom for the test of $\mathscr{H}_\beta$ is thus $\text{df} = (p - s)r = (2 - 1)1 = 1$.

3 Economic equilibrium model

The economic model for the dynamics of spot and futures prices that will provide the foundation for our empirical analysis is a variation of the equilibrium model developed by FG which in turn builds on Garbade and Silber (1983) and a long tradition of models known as "cost-of-carry" models going back to Kaldor (1939). We first briefly review the FG model by presenting the two cases of their model separately: (i) infinite elasticity of supply of arbitrage services and (ii) finite elasticity of supply of arbitrage services. In the third subsection we then discuss a simple variation of their model that will establish a natural connection to the FCVAR model described in Section 2 above.

3.1 FG equilibrium model of spot and futures prices with infinite elasticity of supply of arbitrage services

The following set of standard market conditions are collectively referred to as Assumption A.

A.1 No taxes or transaction costs.
A.2 No limitations on borrowing.
A.3 No costs other than financing a futures position (short or long) and storage costs.
A.4 No limitations on short sale in the spot market.

Let the log-spot price of a commodity in period t be denoted s_t and the contemporaneous log-futures price for a one-period-ahead futures contract be denoted f_t. For that same period t, let r_t and c_t denote the continuously compounded interest rate and storage cost, respectively. The time series behavior of these variables is described in the following conditions, collectively referred to as Assumption B.

B.1 $r_t = \bar{r} + u_{rt}$, where $\bar{r}$ denotes the mean of r_t and u_{rt} denotes an I(0) process with mean zero and finite positive variance.
B.2 $c_t = \bar{c} + u_{ct}$, where $\bar{c}$ denotes the mean of c_t and u_{ct} denotes an I(0) process with mean zero and finite positive variance.
B.3 Δs_t is an I(0) process with mean zero and finite positive variance.

Under Assumption A, no-arbitrage equilibrium conditions imply that the cost of the futures position should be the same as the spot price plus interest and storage cost, i.e.,

$$f_t = s_t + r_t + c_t. \tag{24}$$

Imposing Assumption B on (24) we obtain

$$f_t - s_t = \bar{r} + \bar{c} + u_{rt} + u_{ct}, \tag{25}$$

which implies that s_t and f_t are both I(1) and cointegrate to I(0) with cointegration vector $(1, -1)$.

3.2 FG equilibrium model of spot and futures prices with finite elasticity of supply of arbitrage services

The next step in the development of the equilibrium model of FG is to assume finite elasticity of arbitrage services. This could reflect the existence of factors such as basis risk, convenience yields, constraints on storage space and other factors that make arbitrage transactions risky. Convenience yield, in particular, is the benefit associated with storing the commodity instead of holding the futures contract (Kaldor, 1939) and is the key element in the FG model. Generally, by the definition of Brennan and Schwartz (1985), convenience yield is "the flow of services that accrues to an owner of the physical commodity but not to an owner of a contract for future delivery of the commodity". Accordingly, backwardation can now be given an economic interpretation, defined by FG as "the present value of the marginal convenience yield of the commodity inventory". When this is negative, the market is said to be in contango.

Let the convenience yield be denoted by y_t. Then the no-arbitrage condition (24) is modified to

$$f_t + y_t = s_t + r_t + c_t. \tag{26}$$

In the literature, the convenience yield is generally characterized as a (linear or nonlinear) function of s_t, f_t, and possibly other variables as well. FG approximate y_t by a linear combination of s_t and f_t, i.e. $y_t = \gamma_1 s_t - \gamma_2 f_t$ with $\gamma_i \in (0, 1)$ for $i = 1, 2$. Imposing Assumption B on (26) then implies the equilibrium condition

$$s_t = \beta_2 f_t + \rho + u_{rt} + u_{ct}, \tag{27}$$

where β_2 and ρ are simple functions of the model parameters. In particular, β_2 can take three different values (with the interpretations assuming a small enough value of ρ):

(i) $\beta_2 > 1$: there is long-run backwardation ($s_t > f_t$).
(ii) $\beta_2 < 1$: there is long-run contango ($s_t < f_t$).
(iii) $\beta_2 = 1$: there is neither backwardation nor contango in the long run.

Item (iii) is therefore seen to be a special case of the economic equilibrium model, which, in general, admits the (empirically warranted) theoretical possibility of having a cointegration parameter β_2 different from unity as in items (i) and (ii). It is exactly the hypothesis (iii) that we will test in the empirical application.

3.3 Fractionally cointegrated equilibrium model

We now propose a simple variation of the FG model described in the previous subsection, which will link this theoretical framework to the FCVAR model. It is seen from the above analysis that the I(0) term, $u_{rt} + u_{ct}$, in the equilibrium (cointegrating) relationship (27) stems from Assumptions B.1 and B.2, where interest rates and storage costs are assumed to be I(0). While storage costs are basically unobserved, interest rates are observed and are typically not found to be I(0). In addition, Assumption B.3 implies that spot prices are drift-less (Δs_t has mean zero), which may not be realistic from an empirical point of view. For example, it does not appear obvious from Figure 1, which presents the data series analyzed in the empirical section below, whether the series considered in our empirical analysis are drift-less, and hence it seems prudent to allow a possible drift rather than to rule it out a priori.

To obtain a model with both fractional cointegration and drift, we therefore replace Assumption B by the following conditions, collectively referred to as Assumption C.

C.1 $r_t = \bar{r} + v_{rt}$, where $\bar{r}$ denotes the mean of r_t and v_{rt} denotes an I$(1-b)$ process with $b > 1/2$, mean zero, and finite positive variance.

C.2 $c_t = \bar{c} + v_{ct}$, where $\bar{c}$ denotes the mean of c_t and v_{ct} denotes an I$(1-b)$ process with $b > 1/2$, mean zero, and finite positive variance.

C.3 Δs_t is an I(0) process with mean μ and finite positive variance.

Here, Assumptions C.1 and C.2 generalize B.1 and B.2 to fractionally integrated interest rates and storage costs, and Assumption C.3 allows a drift in spot prices when $\mu \neq 0$.

To simplify notation we assume that interest rates and storage costs have the same order of fractional integration, i.e. that both v_{rt} and v_{ct} are I$(1-b)$. If the b parameters for the two processes were in fact different, then the b parameter in the following analysis would be replaced simply by the minimum of the two. The assumption that $b > 1/2$ ensures that the processes v_{rt} and v_{ct} are stationary, since then $1 - b < 1/2$. The latter assumption is not critical, nor even necessary for the economic equilibrium model, but the condition is useful in the FCVAR model with a linear trend (see Corollary 1). If, instead, $b < 1/2$, then v_{rt} would not be stationary and in that case we would define $\bar{r}$ simply as a constant, rather than interpret it as the mean of r_t, and v_{rt} would denote an I$(1-b)$ process initialized at zero. Similarly for v_{ct}.

When we impose Assumption C on (26) instead of Assumption B, we no longer obtain (27), but instead obtain the equilibrium condition

$$s_t = \beta_2 f_t + \rho + v_{rt} + v_{ct}. \tag{28}$$

Note that the no-arbitrage argument leading to (24) implies that the drifts in Δs_t and Δf_t, i.e. the linear trends in s_t and f_t, must be such that the trend disappears in (28). In other words, in the terminology of the FCVAR model with a linear trend, it follows from the economic equilibrium model that $\beta' \tau_2 = 0$.

In summary, replacing Assumption B in the FG model with Assumption C implies the same cointegration vector. However, the equilibrium condition differs from that in the FG model in that the long-run equilibrium errors are fractionally integrated of order $1 - b$ rather than I(0). It follows that s_t and f_t are fractionally cointegrated such that the FCVAR model is directly applicable to this economic model.

4 Data and empirical results

4.1 Model selection

As a preliminary step before we can estimate the FCVAR model and test our hypotheses of interest, we have to make some model selection choices. First, for all our results we apply estimation conditional on $N = 21$ initial values corresponding to one month of trading days. For robustness, we also computed results (unreported) with $N = 5$ and $N = 63$, and these were similar to those reported.

Second, we have to specify the lag length, k, in the vector error correction model (17). We apply several different statistics to carefully select the lag length, namely the Bayesian Information Criterion (BIC), the LR test statistic for significance of Γ_k, and univariate Ljung-Box Q tests (with $h = 10$ lags) for each of the two residual series, in each case based on the model that includes all the deterministic components considered and has full rank $r = p$. In addition, we examined the unrestricted estimates of b and β_2 which, when the lag length is misspecified, will sometimes be very far from what should be expected. In particular, due to a non-identification issue in the

FCVAR model with misspecified lag length, it is sometimes found that, e.g., $\hat{b} = 0.05$ or similar, see Johansen and Nielsen (2010, Section 2.3) and Carlini and Santucci de Magistris (2014) for a theoretical discussion of this phenomenon. For each commodity, we first use the BIC as a rough guide to the lag length, and starting from there we find the nearest lag length which satisfies the criteria (i) Γ_k is significant based on the LR test, (ii) the unrestricted estimates of b and β_2 are reasonable (very widely defined), and (iii) the Ljung-Box Q tests for serial correlation in the two residual series do not show signs of misspecification.

Third, we select the deterministic components and the cointegrating rank, r. We maintain the hypothesis that the restricted constant, $\rho\pi_t(1)$, is present based on the theoretical framework in Section 3. Therefore, the selection of deterministic components comes down to the absence or presence of the unrestricted constant, $\xi\pi_t(1)$, i.e. the trend component. Because the limit distribution of the test of cointegrating rank depends on the presence or absence of the trend as well as on the actual cointegrating rank, we have to simultaneously decide the cointegration rank and whether or not to include the trend. The rank and deterministic terms can be determined jointly in the same way as for the CVAR model; see the detailed discussion in Johansen (1995, pp. 170–174). In our model with dimension $p = 2$, we first note that the models with rank $r = 0$ are always rejected regardless of the presence or absence of the trend. Given this fact, the method reduces to selecting the model with rank $r = 1$ and both constant terms if the model with rank $r = 1$ and only a restricted constant is rejected, and selecting the model with rank $r = 1$ and only a restricted constant term if it is not rejected.

4.2 Data description

To facilitate comparison with the analysis in FG, which is based on the usual non-fractional CVAR model, we use the same data set as in their empirical analysis. This data set includes daily observations (business days only) from the London Metal Exchange on spot and 15-month futures prices[2] for aluminium, copper, lead, nickel, and zinc for the period from January 1989 to October 2006 and the sample period has approximately 4484 observations. The data from the London Metal Exchange has the advantage that there are simultaneous spot and futures prices, for fixed maturities, on every business day. For details, see FG.

The (logarithmic) data is shown in Figure 1. From casual inspection of the graphs in Figure 1, it is not clear whether the log-prices have a non-zero drift or not. In this respect the graphs are inconclusive, and for that reason it seems prudent to allow for at least the possibility of a drift in the analysis rather than to rule it out a priori. Furthermore, inspection of the graphs does not appear to reveal any obvious patterns of behavior between s_t and f_t that would be informative with respect to possible backwardation or contango.

4.3 Futures vs forward contracts

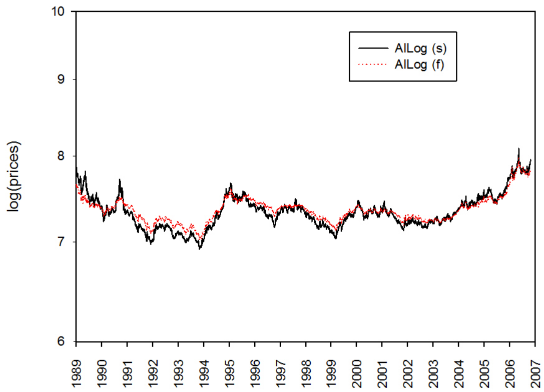
[3] It is relevant at this point to distinguish between forwards and futures. In more complicated multi-period models, instead of the one-period model discussed in Section 3 above, futures contracts are normally marked-to-market which forward contracts are not. That is, the profit or loss from a forward contract is realized at maturity, whereas the profit or loss made on the price change in a futures contract is settled at the end of each trading day by the brokerage house with whom the account is held. This generates cash-flow and consequent interest payments in intermediate periods and has implications for optimal futures pricing and hedging, see e.g. Hodrick (1987), Baillie and

[2] As pointed out by an anonymous referee, although FG describe their data as forward prices (e.g. their p. 100), the historical data from the London Metal Exchange in fact appear to be futures prices. Implications of this difference are discussed below in Section 4.3.

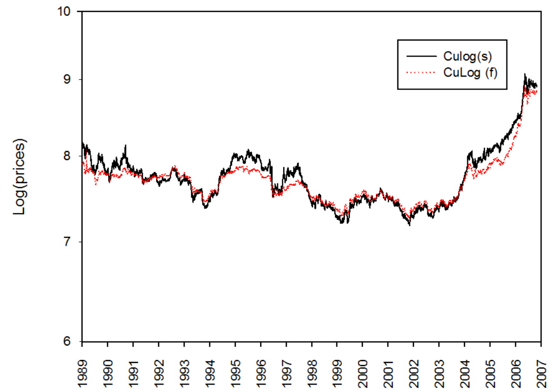
[3] We thank an anonymous referee for comments that essentially developed into this subsection.

Figure 1: Daily spot and futures log-prices

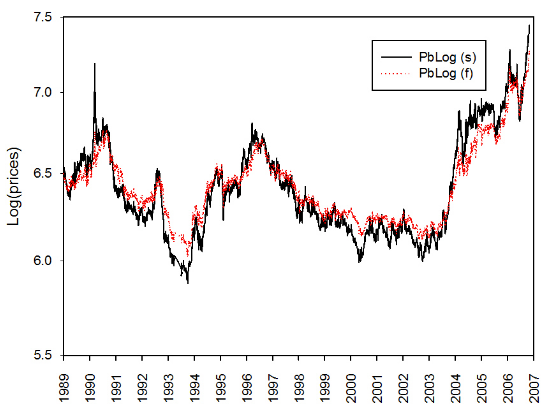
(a) aluminium



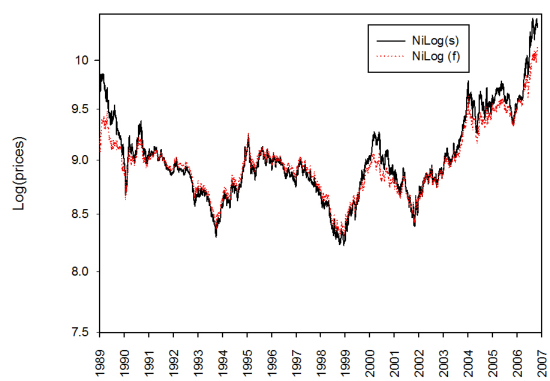
(b) copper



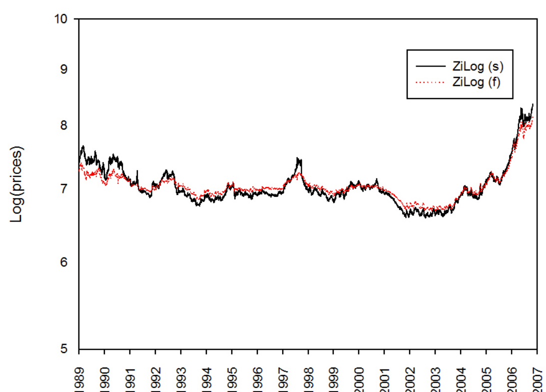
(c) lead



(d) nickel



(e) zinc



Myers (1991), and Chow et al. (2000).

The implications of the mark-to-market feature of futures contracts for the cointegration properties and cointegrating relation between spot and futures prices are discussed in, e.g., Chow et al. (2000, Section 4.4.2), who show that marking-to-market implies an additional non-stochastic term in the intercept of the cointegrating relation, i.e., in (27) or (28). Indeed, Chow et al. (2000, p. 216) argue that in empirical work "For practical purposes, therefore, it is customary to assume that forward and futures prices are equivalent."

A related implication is the possibility that the interest rate should be explicitly included in the cointegrating relation and hence in the model. For example, Baillie and Myers (1991, eqn. (2)) arrive at essentially the same equilibrium condition as our eqn. (24), and they argue that if the interest rate, r_t, is I(1) then it should be included in the model and hence included explicitly in the cointegrating relation. On the other hand, FG make the explicit assumption that the interest rate is I(0), see Assumption B.1, and hence forms part of the error term in the cointegrating relation (27). In our analysis, and in our economic equilibrium model in Section 3.3, we attempt to strike a balance between these two conflicting views by assuming that the interest rate is fractionally integrated of order $1 - b$, see Assumption C.1, leading to the cointegrating relation (28). Thus, in our analysis, if the economic model is otherwise true, but the interest rate is in fact I(1), then cointegration between s_t and f_t should be rejected. The same conclusion is reached by Baillie and Myers (1991, p. 113) in their setup. However, Assumption C.1 of our model allows the generality of fractional integration of the interest rate and hence is able to accommodate both views, i.e. that the interest rate is I(1) and I(0), as special cases.

Thus, we continue with the analysis of the pair (s_t, f_t) and leave the interest rate as specified in Assumption C.1 of the economic model. Nonetheless, to further investigate the implication that the interest rate should be included in the model, and as a robustness analysis, we also briefly consider at the end of Section 4.4 estimation of trivariate systems that include the interest rate as an additional variable.

4.4 Empirical results

The results of our empirical analysis are presented in Tables 1–6 with one table for each of the five metals (two for zinc). The tables are laid out in the same way and each have up to six panels. The first panel presents the LR tests for cointegration rank, as well as critical values for the 1%, 5%, and 10% significance level. Here, $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms, and $\mathscr{M}_r^*$ denotes the model with rank r but only a restricted constant. The selected model is highlighted in bold. In the next two panels the unrestricted estimation results are shown for the model selected in the first panel. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual ($i = 1, 2$). The fourth panel is a subtable with the results of the tests of the three hypotheses of main interest,

$$\mathscr{H}_b : b = 1, \tag{49}$$

$$\mathscr{H}_\beta : \beta = (1, -1)', \tag{50}$$

$$\mathscr{H}_b \cap \mathscr{H}_\beta : b = 1 \text{ and } \beta = (1, -1)', \tag{51}$$

where P values in bold denote non-rejected hypotheses. The latter are then imposed in the restricted model which is presented in the final two panels, except for lead and nickel where all hypotheses are rejected.

In terms of model specification, we first find that, allowing for the possibility of fractional cointegration, fewer lags are needed to adequately model the data. In FG, using the non-fractional

Table 1: FCVAR results for aluminium

Rank tests:

r	LR($\mathscr{M}_r \vert \mathscr{M}_p$)	CV 1%	CV 5%	CV 10%	LR($\mathscr{M}_r^* \vert \mathscr{M}_p^*$)	CV 1%	CV 5%	CV 10%
0	27.080	13.277	9.488	7.779	24.921	13.277	9.488	7.779
1	4.319	9.505	5.816	4.642	**0.085**	10.885	7.259	5.821

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} -0.036 \\ 0.015 \end{bmatrix} z_t + \sum_{i=1}^{5}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + \hat{\varepsilon}_t \tag{29}$$

$$\hat{b} = \underset{(0.043)}{0.738}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.655)}{7.722}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.712)}{7.132}, \quad \log(\mathscr{L}) = 30433.383$$

Equilibrium relation:

$$s_t = -0.930 + 1.121 f_t + z_t \tag{30}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	22.632	3.721	29.204
P value	0.000	**0.054**	0.000

Restricted model:

$$\Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} = \Delta^{1-\hat{b}}\begin{bmatrix} -0.030 \\ 0.013 \end{bmatrix} z_t + \sum_{i=1}^{5}\Gamma_i L_{\hat{b}}^i \Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} + \hat{\varepsilon}_t \tag{31}$$

$$\hat{b} = \underset{(0.040)}{0.739}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.653)}{7.751}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.704)}{7.223}, \quad \log(\mathscr{L}) = 30431.523$$

Equilibrium relation:

$$s_t = -0.039 + f_t + z_t \tag{32}$$

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests, where P values in bold denote non-rejected hypotheses that are imposed in the restricted model, which is presented in the final two panels. The sample size is $T = 4487$.

CVAR model they select $k = 17, 14, 15, 18$, and 16 lags for aluminium, copper, lead, nickel, and zinc, respectively, based on the Akaike Information Criterion (AIC). However, using our lag selection method described above, we select $k = 5, 4, 4, 4$, and 1 lags for the five metals. Thus, allowing for b to be fractional we can select a much smaller number of lags while maintaining white noise residuals, although part of the difference can likely be attributed to FG's use of the AIC for lag length selection. For cointegration rank, all five metals clearly have $r = 1$ as expected from theory, and also in accordance with the CVAR analysis in FG.

The first set of results are for aluminium and are presented in Table 1. The selection of deterministic terms shows that the trend is not needed for aluminium, so the model with only a restricted constant term is estimated. We first note that the Ljung-Box serial correlation tests for the residuals show no signs of model misspecification. The unrestricted point estimate of the fractional parameter is $\hat{b} = 0.738$ with a standard error of 0.043. The hypothesis $\mathscr{H}_b$ in (49), i.e.

Table 2: FCVAR results for copper

Rank tests:

r	LR($\mathscr{M}_r\vert\mathscr{M}_p$)	CV 1%	CV 5%	CV 10%	LR($\mathscr{M}_r^*\vert\mathscr{M}_p^*$)	CV 1%	CV 5%	CV 10%
0	21.503	24.267	19.147	16.835	17.197	24.067	19.325	16.975
1	8.263	9.297	6.043	4.664	**4.128**	12.380	8.732	7.033

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} -0.001 \\ 0.005 \end{bmatrix} z_t + \sum_{i=1}^{4}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + \hat{\varepsilon}_t \tag{33}$$

$$\hat{b} = \underset{(0.094)}{0.950}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.565)}{8.652}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.938)}{4.179}, \quad \log(\mathscr{L}) = 27554.320$$

Equilibrium relation:

$$s_t = 0.335 + 0.957 f_t + z_t \tag{34}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	0.197	0.061	0.198
P value	**0.657**	**0.805**	**0.906**

Restricted model:

$$\Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} = \Delta^{1-\hat{b}}\begin{bmatrix} -0.001 \\ 0.004 \end{bmatrix} z_t + \sum_{i=1}^{4}\Gamma_i L_{\hat{b}}^i \Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} + \hat{\varepsilon}_t \tag{35}$$

$$\hat{b} = 1, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.523)}{9.093}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.923)}{4.476}, \quad \log(\mathscr{L}) = 27554.221$$

Equilibrium relation:

$$s_t = 0.012 + f_t + z_t \tag{36}$$

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests, where P values in bold denote non-rejected hypotheses that are imposed in the restricted model, which is presented in the final two panels. The sample size is $T = 4496$.

that $b = 1$, is formally tested in the hypothesis tests subtable, where it is strongly rejected with an LR statistic of 22.632 and a P value of 0.000. Because the estimate of the fractional parameter b is greater than one-half it follows that $\beta' X_t$ is a stationary process with long memory, i.e. with integration order $1 - b$ in the range $(0, 1/2)$. More generally, because the fractional parameter b is significantly different from one, this lends support to the FCVAR model specification over the non-fractional CVAR model that has $b = 1$ imposed.

The cointegration parameter β_2 is estimated at $\hat{\beta}_2 = 1.121$, which suggests some long-run backwardation. However, from the hypothesis tests subtable, the hypothesis $\mathscr{H}_\beta$ in (50) that the cointegration vector is in fact $\beta = (1, -1)'$ cannot be rejected at the 5% level with an LR statistic of 3.721 and a P value of 0.054. The final test in the hypothesis tests subtable is the joint test of $\mathscr{H}_b \cap \mathscr{H}_\beta$ in (51), which is strongly rejected. The implication of the non-rejection of the hypothesis $\mathscr{H}_\beta$ is that the long-run equilibrium is $\beta = (1, -1)'$ such that there is neither backwardation nor

Table 3: FCVAR results for lead

Rank tests:

r	LR($\mathscr{M}_r\|\mathscr{M}_p$)	CV 1%	CV 5%	CV 10%	LR($\mathscr{M}_r^*\|\mathscr{M}_p^*$)	CV 1%	CV 5%	CV 10%
0	48.221	20.286	16.038	13.893	48.757	20.228	15.868	13.734
1	0.946	9.443	6.111	4.668	**0.920**	10.811	7.337	5.673

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} -0.007 \\ 0.045 \end{bmatrix} z_t + \sum_{i=1}^{4}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + \hat{\varepsilon}_t \tag{37}$$

$$\hat{b} = \underset{(0.046)}{0.739}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.315)}{11.554}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.060)}{17.667}, \quad \log(\mathscr{L}) = 25662.232$$

Equilibrium relation:

$$s_t = -1.076 + 1.162 f_t + z_t \tag{38}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	16.390	5.917	27.912
P value	0.000	0.015	0.000

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests. The sample size is $T = 4428$.

contango in the long run.

To complete the empirical results for aluminium, the last two panels of the table show the final restricted model, including serial correlation tests for the residuals that show no signs of serial correlation, thus indicating that the model is well specified. As expected (because the imposed hypothesis is not rejected in the unrestricted model), the results in the restricted model are very similar to those in the unrestricted model.

Next, Table 2 shows the results for copper. Again the model with only a restricted constant is selected and the Ljung-Box serial correlation tests show no signs of model misspecification. The unrestricted estimation results show $\hat{b} = 0.950$, and in fact copper is the only metal for which $\mathscr{H}_b : b = 1$ is not rejected. It therefore appears that a CVAR is in fact adequate to model copper spot and futures prices. Furthermore, the hypothesis $\mathscr{H}_\beta$ and the joint hypothesis $\mathscr{H}_b \cap \mathscr{H}_\beta$, i.e. the hypothesis that $b = 1$ and $\beta = (1, -1)'$ are both supported by the data with P values of 0.805 and 0.906, respectively. It follows that for copper, as was found for aluminium, there is neither backwardation nor contango in the long run. The results of the restricted model, i.e. imposing $\mathscr{H}_b \cap \mathscr{H}_\beta$, are in agreement with those of the unrestricted model and the Ljung-Box tests again show no signs of misspecification.

For lead and nickel, with results presented in Tables 3 and 4, we first note that again we select the model with only a restricted constant term, that there are no signs of model misspecification based on the Ljung-Box tests, and that $\hat{b}$ is significantly different from one. Furthermore, we find from both the unrestricted $\hat{\beta}$ estimates and from the hypothesis tests that these markets are in long-run backwardation. The unrestricted point estimates are $\hat{\beta}_2 = 1.162$ and $\hat{\beta}_2 = 1.244$ for lead

Table 4: FCVAR results for nickel

Rank tests:

r	LR($\mathscr{M}_r \mid \mathscr{M}_p$)	CV 1%	CV 5%	CV 10%	LR($\mathscr{M}_r^* \mid \mathscr{M}_p^*$)	CV 1%	CV 5%	CV 10%
0	41.899	13.277	9.488	7.779	39.153	13.277	9.488	7.779
1	8.199	9.070	6.066	4.640	**0.531**	10.095	6.606	5.149

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} -0.059 \\ 0.035 \end{bmatrix} z_t + \sum_{i=1}^{4}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + \hat{\varepsilon}_t \tag{39}$$

$$\hat{b} = \underset{(0.065)}{0.636}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.610)}{8.183}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.924)}{4.452}, \quad \log(\mathscr{L}) = 25441.641$$

Equilibrium relation:

$$s_t = -2.206 + 1.244 f_t + z_t \tag{40}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	15.972	11.532	30.710
P value	0.000	0.000	0.000

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests. The sample size is $T = 4484$.

and nickel, respectively. Since all hypotheses are rejected for these two metals, we do not show any restricted estimation results.

Finally, for zinc the model selection results are inconclusive with respect to the absence or presence of the unrestricted constant term. Specifically, the model with only a restricted constant term is rejected at 10% level and its unrestricted estimate of α_1 has the wrong sign such that spot prices do not move towards the equilibrium. We therefore present two sets of results for zinc: with a restricted constant only (denoted zinc (a) and shown in Table 5) and with both a restricted and an unrestricted constant term (denoted zinc (b) and shown in Table 6).

The results from the two specifications for zinc are quite similar. We first note that the estimate of the fractional parameter b is less than 1/2 in both cases, suggesting that $\beta' X_t$ is not actually a stationary process, although it is mean reverting. That is, there is a weaker form of fractional cointegration for zinc than for the other metals where b was estimated to be greater than 1/2. In terms of the equilibrium relation we find that, although the unrestricted estimates of the cointegration parameter appear slightly different at $\hat{\beta}_2 = 0.964$ and $\hat{\beta}_2 = 1.117$, respectively, the hypothesis $\mathscr{H}_\beta$ is not rejected under either specification (P values of 0.830 and 0.506, respectively). It follows that there is neither backwardation nor contango in the long run for zinc regardless of which of the two model specifications are applied.

To summarize our empirical results using the new FCVAR framework, we have found that the fractional parameter b is significantly different from one in all markets except copper, suggesting that the FCVAR model is more appropriate for modeling these data than the non-fractional CVAR model. For the cointegration vector, we have found that it is significantly different from $\beta = (1, -1)'$

Table 5: FCVAR results for zinc (a)

Rank tests:

r	LR($\mathscr{M}_r \mid \mathscr{M}_p$)	CV 1%	CV 5%	CV 10%	LR($\mathscr{M}_r^* \mid \mathscr{M}_p^*$)	CV 1%	CV 5%	CV 10%
0	29.344	13.277	9.488	7.779	28.204	13.277	9.488	7.779
1	5.620	6.635	3.841	2.706	**3.730**	6.635	3.841	2.706

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} 0.015 \\ 0.173 \end{bmatrix} z_t + \sum_{i=1}^{1}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + \hat{\varepsilon}_t \tag{41}$$

$$\hat{b} = \underset{(0.041)}{0.339}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.047)}{18.499}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.205)}{13.343}, \quad \log(\mathscr{L}) = 27810.399$$

Equilibrium relation:

$$s_t = -0.028 + 0.964 f_t + z_t \tag{42}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	32.724	0.046	42.826
P value	0.000	**0.830**	0.000

Restricted model:

$$\Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} = \Delta^{1-\hat{b}}\begin{bmatrix} 0.009 \\ 0.163 \end{bmatrix} z_t + \sum_{i=1}^{1}\Gamma_i L_{\hat{b}}^i \Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} + \hat{\varepsilon}_t \tag{43}$$

$$\hat{b} = \underset{(0.042)}{0.349}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.047)}{18.448}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.209)}{13.251}, \quad \log(\mathscr{L}) = 27810.376$$

Equilibrium relation:

$$s_t = -0.240 + f_t + z_t \tag{44}$$

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests, where P values in bold denote non-rejected hypotheses that are imposed in the restricted model, which is presented in the final two panels. The sample size is $T = 4493$.

only for lead and nickel, and for both these metal markets the point estimates suggest that there is long-run backwardation. For the remaining metal markets, i.e. aluminium, copper, and zinc, we did not reject the hypothesis $\mathscr{H}_\beta : \beta = (1, -1)'$ and conclude that for these markets there is neither backwardation nor contango in the long run.

In contrast, the findings in FG based on the non-fractional CVAR model are that all markets are in long-run backwardation, with the exception of copper for which they do not reject $\beta = (1, -1)'$. Moreover, with the exception of nickel, the unrestricted point estimates of the cointegration parameter β_2 are all greater in magnitude in FG than in our results. Overall, our analysis using the newly developed FCVAR model therefore gives much more support to the cointegration vector $\beta = (1, -1)'$ in the long-run equilibrium relationship between spot and futures prices compared with the analysis in FG using the CVAR model.

Finally, as discussed in Section 4.3, to further investigate the implication that the interest rate

Table 6: FCVAR results for zinc (b)

Rank tests:

r	$\text{LR}(\mathscr{M}_r \vert \mathscr{M}_p)$	CV 1%	CV 5%	CV 10%	$\text{LR}(\mathscr{M}_r^* \vert \mathscr{M}_p^*)$	CV 1%	CV 5%	CV 10%
0	29.344	13.277	9.488	7.779	28.204	13.277	9.488	7.779
1	**5.620**	6.635	3.841	2.706	3.730	6.635	3.841	2.706

Unrestricted model:

$$\Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) = L_{\hat{b}}\Delta^{1-\hat{b}}\begin{bmatrix} -0.003 \\ 0.102 \end{bmatrix} z_t + \sum_{i=1}^{1}\Gamma_i L_{\hat{b}}^i \Delta\left(\begin{bmatrix} s_t \\ f_t \end{bmatrix}\right) + 10^{-3}\begin{bmatrix} 0.197 \\ 0.436 \end{bmatrix} + \hat{\varepsilon}_t \tag{45}$$

$$\hat{b} = \underset{(0.057)}{0.441}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.045)}{18.594}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.185)}{13.736}, \quad \log(\mathscr{L}) = 27811.441$$

Equilibrium relation:

$$s_t = -0.690 + 1.117 f_t + z_t \tag{46}$$

Hypothesis tests:

	$\mathscr{H}_b$	$\mathscr{H}_\beta$	$\mathscr{H}_b \cap \mathscr{H}_\beta$
df	1	1	2
LR	33.707	0.443	43.956
P value	0.000	**0.506**	0.000

Restricted model:

$$\Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} = \Delta^{1-\hat{b}}\begin{bmatrix} 0.009 \\ 0.135 \end{bmatrix} z_t + \sum_{i=1}^{1}\Gamma_i L_{\hat{b}}^i \Delta\begin{bmatrix} s_t \\ f_t \end{bmatrix} + 10^{-3}\begin{bmatrix} 0.203 \\ 0.317 \end{bmatrix} + \hat{\varepsilon}_t \tag{47}$$

$$\hat{b} = \underset{(0.046)}{0.385}, \quad Q_{\hat{\varepsilon}_1}(10) = \underset{(0.046)}{18.551}, \quad Q_{\hat{\varepsilon}_2}(10) = \underset{(0.182)}{13.806}, \quad \log(\mathscr{L}) = 27811.219$$

Equilibrium relation:

$$s_t = -0.020 + f_t + z_t \tag{48}$$

Notes: The first panel of the table shows FCVAR cointegration rank tests with the selected model highlighted in bold, where $\mathscr{M}_r$ denotes the model with rank r including both restricted and unrestricted constant terms and an asterisk denotes models with only a restricted constant. The next two panels show the unrestricted estimation results for the selected model. Standard errors are in parentheses below $\hat{b}$ and P values are in parentheses below $Q_{\hat{\varepsilon}_i}$, which is the Ljung-Box Q test for serial correlation in the i'th residual. In the fourth panel are the results of the hypothesis tests, where P values in bold denote non-rejected hypotheses that are imposed in the restricted model, which is presented in the final two panels. The sample size is $T = 4493$.

should be included in the model, and as a robustness analysis, we also briefly consider estimation of trivariate systems that include the interest rate as an additional variable. That is, albeit without the full model selection procedure, we re-estimated (not reported) all the above FCVAR models as trivariate systems including r_t as an additional variable given by the 3-month U.S. Treasury bill rate. In these systems, we first tested the hypothesis that the adjustment coefficient in the equation for r_t is zero, i.e. weak exogeneity of the interest rate, and this was not rejected in any of the models (P values between 0.189 and 0.977). We also tested the hypothesis that the coefficient on r_t in the cointegrating relation is zero, and this was also not rejected in any of the models (P values between 0.050 and 0.721). We further tested the joint hypothesis, which was not rejected in any of the models either (P values between 0.121 and 0.881). Similar conclusions have been found in other empirical studies that include interest rates; for a discussion, see Chow et al. (2000, Section 5) and the references therein. At least for the data we analyze, it thus appears reasonable to leave the interest rate as specified in Assumption C.1 of the economic model in Section 3.3, with the detailed results discussed earlier in this section.

5 Concluding remarks

In this paper we have applied the recently developed fractionally cointegrated vector autoregressive (FCVAR) model to the dynamics of spot and futures prices for five different commodities (aluminium, copper, lead, nickel, and zinc). To apply the model we first developed an extension of the FCVAR model to accommodate drift in the prices, i.e. a linear deterministic time trend in the data. Specifically, we provide a representation theory for the extended model showing that the trend gives rise to both restricted and unrestricted constant terms in the vector error correction model. We also briefly discussed the consequences for the distribution of the LR trace test for cointegration rank.

The empirical model has economic foundation using a variation of the economic equilibrium model of FG that is able to generate both fractional cointegration and to capture the existence of backwardation and contango in the long-run equilibrium relationship of spot and futures prices. In our empirical analysis we found that spot and futures prices for all metals are cointegrated, and—with the exception of copper—the cointegration is of the fractional type. Our first finding is that, when allowing for fractional integration in the long-run equilibrium relations, fewer lags appear to be needed in the autoregressive formulation. Furthermore, compared to the results from the non-fractional model in FG, there is more evidence in favor of the cointegration vector $\beta = (1, -1)$ and hence less evidence of long-run backwardation or contango. Specifically, we reject the hypothesis that the cointegration vector is $(1, -1)$ using standard likelihood ratio tests only for the lead and nickel markets.

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