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An Alternative Explanation of the Chance of Casting a Pivotal Vote

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Abstract: This paper is about a model of uncertainty in voting that allows for a schedule of people`s preferences for one party over another, that gives rise to a chance of casting a pivotal vote which is small but not, as often supposed, infinitesimal, that is not inconsistent with evidence about the chance of casting a pivotal vote and that preserves a role for self-interest, along with a duty to vote, in the decision whether to vote or abstain.

JEL Classification: D72

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A huge discrepancy has emerged between estimates of one's chance of casting a pivotal vote a) from statistical evidence about past elections and b) from theory about the variability of voter preference and behaviour. The discrepancy matters because the chance of casting a pivotal vote determines the relative importance of duty and self-interest in voting, because theory is required to identify bias in the outcome of an election and because one cannot place much faith in models that do not fit the facts.

The chance of casting a pivotal vote is the probability that, all by itself, a person`s decision to vote rather than to abstain swings the outcome of an election from a win by the party the person opposes to a win by the party the person supports. The importance of the chance of casting a pivotal vote is that no rational, self-interested person would ever vote if the outcome of the election would be the same regardless of whether or not he votes.[1]

The focus of this paper is the development of an alternative to the usual model of

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[1]What is being called the chance of casting a pivotal vote is one's probability of swinging the outcome of an election by not abstaining rather than by switching one's vote from one party to another. The chance of casting a pivotal vote is the same for everybody in one sense, but different in another. It is the same in the sense that if I believe my chance to be one-in-ten thousand, I would be illogical not to suppose that your chance is one-in-ten thousand as well. On the other hand, possessing different information about the election, you and I may have different estimates of our common chance of casting a pivotal vote.

The chance of an additional vote being pivotal is virtually independent of whether the original number of voters is even or odd. Let N be the original number of voters. When N is *even* and when an additional person would vote left if he votes at all, the extra person's vote can only swing the outcome of the election from right to left if a) the left party would win exactly half (N/2) of the votes in the event that the additional person abstains and b) the left party would have *lost* the coin-toss to break the tie. Otherwise, the additional person's vote cannot be pivotal because either the right party would win regardless or the left party would win regardless. When N is *odd* and when an additional person would vote left if he votes at all, the vote of the extra person can only swing the outcome of the election if a) but for that person's vote, the left party would have won exactly (N-1)/2 votes, losing the election by exactly one vote, and b) the left party would *win* the coin-toss to break the tie created by the extra vote for the left party. As long as N is not too small, the probabilities of the two events categorized as (a) - a tied vote when N is even or a loss by one vote when N is odd - are the virtually same, and the outcomes of the coin-toss in (b) are both equal to ½. Nor does it make any difference to the chance of casting a pivotal vote whether the additional person would have voted left or right. The chance of a vote being pivotal is the same. That being so, there is no harm in assuming from here on that N is even.

uncertainty in elections from which the chance of casting a pivotal vote is computed. The alternative broadly consistent - as the usual model is not - with evidence about the outcomes of past elections. The usual model is based upon *person-by-person* randomization. The proposed alternative is based upon *nation-wide* randomization of a schedule of voters valuations from highest for one party to highest to the other.

The standard representation, in Riker and Ordeshook (1968), of one's decision whether to vote or abstain is that one votes if and only if

$$\pi B + D > C \tag{1}$$

where B is one's personal benefit if one's preferred party wins the election
D is the value one places upon voting as a duty to the rest of the community.
C is one's cost of voting
and π is the probability of casting a pivotal vote.

The three questions are suggested by this representation of the decision to vote or abstain: How large is π? How is π determined? Is there more to self-interest than the chance of casting a pivotal vote? About the first question, the paper has nothing to say except as preface to what follows. Existing estimates of π based upon the record of past elections are presumed to be more or less correct. The main focus of this paper is on the second question, the design of a model of nation-wide randomization that is broadly consistent with the evidence and that predicts who votes and who abstains. Having developed the model, it is natural to consider what, if anything, might be left out of the equation, whether the chance of casting a pivotal vote may be combined with other self-interested motives for voting rather than abstaining.

The paper begins with a comparison of estimates of the chance of casting a pivotal vote, one based upon historical evidence and the other based upon a model of person-by-person randomization. In a constituency of 100,000 people, some historical evidence suggests that the chance of casting a pivotal vote is about 1 in 10,000, while a comparable example of person-by-person randomization suggests a chance of about 1 in 800 billion! The historical evidence is, to put it mildly, casual, but a discrepancy that large can only be attributed to the difference in methods of computation. It is argued that estimation based upon historical evidence is more or less right, while estimation based upon person-by-person randomization is seriously wrong. There follows a description and critique of the model of nation-wide randomization. The final section of the paper is a discussion of self-interest not reflected in the chance of casting a pivotal vote.

The Discrepancy

Difficulties in estimating the chance of casting a pivotal vote are dramatically illustrated in a comparison of estimates based upon the historical record and upon the model of person-by-person randomization that is commonly employed in the literature of majority rule voting. As applied to a constituency of about 100,000 voters, one method yields a probability of casting a pivotal vote of about 1 in ten thousand which is small but not small enough to be ignored altogether. The other yields a probability of casting a pivotal vote of about 1 in 800 billion!

The historical estimate is from Blais (2000, page 64). It is a rough and ready computation that differs somewhat in its parameters from the analytical estimate based upon person-to-person randomization to follow, but this differences in parameters is swamped by difference in method. An historical estimate looks at the record of past elections, observes the distribution of the percentage of votes of one of two competing parties (somehow fudging for the presence of third parties), computes the probability of a tie and divide that in half to get the probability that an additional voter can swing the outcome of the vote from one party to the other (when ties are broken by the flip of a coin). Blais observed that, among all of the 4,626 elections in Canadian Federal constituencies between 1945 and 1997, only one was tied, suggesting as a very crude first approximation that the probability of casting a pivotal vote is about 1 in 9,252. Blais also noted that in only 5 out of the 295 constituencies in the 1997 election was the margin of victory less or equal to 100. Assuming the probability of a tied vote to be one hundredth of the probability of a margin of victory of 100, a person's chance of finding himself in a constituency where, but for his vote, the election would be tied becomes (1/100)(5/295) and one's chance of casting a pivotal vote would be (1/2)(1/100)(5/295), equal to 0.0000847, or 1 in 11,800. Averaging the two numbers - 9,252 and 11,800 - we can say that the chance of casting a pivotal vote in a Canadian constituency is in the neighborhood of one-in-ten thousand.

Blais' computation can be modified to account for additional information. In measuring the probability of a pivotal vote in the US House of Representatives and in US state legislatures - estimated as 1 in 89,000 and as 1 in 15,000 respectively - Mulligan and Hunter (2003) distinguish between the chance of a person`s vote being pivotal in a randomly-chosen *state* (essentially Blais' computation) and the chance of a randomly-chosen person's vote being pivotal dependent on the population of the state where he resides. A more accurate estimation of the chance of casting a pivotal vote is constructed from a function connecting the incidence of votes within a given margin (Blais' margin was 100) and the width of the margin. Gelman, King and Boscardin (1998) employ information from opinion polls to estimate both the mean and the variance of the expected vote for one of two competing parties, where, the greater the deviation of the mean from 50% and the smaller the variance, the smaller is a person's chance of casting a pivotal vote. Gelman, Silver and Edlin (2008) estimate a person's chance of casting the pivotal vote for the President of the United States to be about 1 in 60 million, where the chance must be computed as the product of the probability of one's vote being pivotal within one's state and the probability that one's state is pivotal within the nation.

The actual number of voters in each election is not formally part of the computation, but that number is not too far off 100,000, and Blais' estimated probability of a tie would have been very different had the number of voters in each constituency been very much smaller or very much larger. If the cost and trouble of casting one's ballot is, say, \$20, it becomes individually advantageous to vote rather than to abstain when one's benefit of a win by the party one votes for is in excess of \$200,000. That is surely more than most people would induce most people to vote on selfish grounds alone. On the other hand, a millionaire who expects the party he favours to levy an income tax at a rate of, say, 3% less than the rate that would be imposed by the other party, values a win by his preferred party \$30,000 per year which amounts to more than \$200,000 over a ten year period. The probability of casting a pivotal vote is small, but not so small as to deter every single voter or remove the chance of casting a pivotal vote from the list of considerations, to be discussed later on, in determining whether one votes or abstains.

The expected benefit, πB, by a win for one`s preferred party may be influential without being decisive. For example, if C = \$20, B = \$30,000 and
$\pi = 1/10{,}000$, the expected benefit from voting becomes \$3, or 15% of the cost, leaving the remaining 85% to be covered, if at all, by a sense of duty or some similar consideration. Thus, the question of whether the chance of casting a pivotal vote is infinitesimal or merely small is really about the roles of duty and self-interest in the decision whether to vote or abstain. *Small* leaves some room for self-interest. *Infinitesimal* leaves none.

The probability is large enough to make small voting pacts effective. If a person's chance of casting a pivotal vote is 1 in 10,000, then 100 like-minded people , none of whom would be inclined to vote otherwise, might pledge one another to vote, raising their combined chance of being pivotal to 1 in 100, and lowering each person's required benefit to make voting advantageous from \$200,000 to \$2,000.

Historical estimation provides a rough idea of the chance that a vote turns out to be pivotal, but there is some question as to whether outcomes in past elections are really predictive of the outcome in an election today. Regardless of the history of elections, the probability of a voter being pivotal may be relatively high in some elections and relatively low in others, high when an election is expected to be close, and low when one party is almost certain to win. Most importantly, the probability of a voter being pivotal is observed rather than derived from assumptions about voters' behaviour. The historical model is not really a model at all, for it has nothing to say about why the chance of a tied election is what it is inferred to be. It is a black box generating numbers. It is most useful as a check on other models to be discussed below, raising suspicion whenever outcomes are too far from what the common sense model would suggest.

The analytic estimate with person-to-person randomization is based upon an analogy between voting and sampling.[2] Ignoring abstentions, imagine all voters lined up at the ballot box, and voting one by one. Before each person votes, the angel of chance assigns that person a

[2] The first such models I know of are those of Beck(1975) and of Good and Mayer (1975).

preference, for the left party with a probability of p and for the right party with a probability (1 - p). With a total of N votes cast, the number of votes for the left party is analogous to the number of blue balls in N drawings from an urn containing blue and red balls in proportions p and (1 - p). The distribution of the number of blue balls is binomial with mean pN and standard deviation $[p(1 - p)N]^{1/2}$. Suppose for convenience that N is an even number.

The election is tied when exactly N/2 blue balls are drawn. The probability, T, of a tie is "the probability that the first N/2 balls are blue and the remaining N/2 balls are red" times "the number of ways to place N/2 balls in N slots". Specifically,

$$T = \{(p)^{N/2}(1 - p)^{N/2}\}\{N!/[(N/2)!(N/2)!]\} \qquad (2)$$

which, when simplified by Stirling's formula [3], reduces to

$$T = \{2/\Pi N\}^{1/2}[(2p)^{N/2}(2(1- p))^{N/2}] \qquad (3)$$

where Π is the ratio of the circumference to the diameter of a circle. The probability that any person's vote turning out to be pivotal is half the probability of a tie.

As the binomial distribution is bell shaped, the probability of any particular number of votes for the left party, n, depends on its distance from the mean, pN, highest when n is the mean, and steadily lower the farther from the mean n happens to be. The probability that n = N/2 is greatest when p = ½.

To estimate the probability of a tied vote, T, in accordance with equation (3), it is necessary to know the total population of voters, N, and the probability, p, that a randomly chosen person votes for the left party. To produce an estimate comparable to the historical estimate above, assume there to be 100,000 voters, a round figure roughly comparable to the number of voters in a Canadian constituency. It is less clear what to choose as the value of p. For

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[3] Stirling's formula is an approximation of [N!] by $(2\,\Pi N)^{½}(N/e)^N$

From Stirling's formula, it follows that

$$\alpha = N!/[(N/2)!(N/2)!] = \{(2\Pi N)^{½}(N/e)^N\}/\{(2\Pi N/2)^{½}(N/2e)^{N/2}\}\{(2\Pi N/2)^{½}(N/2e)^{N/2}\}$$

$$= [\{(2\Pi N)^{½}\}/\{(2\Pi N/2)^{½}\}\{(2\Pi N/2)^{½}\}][\{(N/e)^N\}/\{(N/2e)^{N/2}\}\{(N/2e)^{N/2}\}]$$

$$= [(2/\Pi N)^{½}][2^N]$$

from which equation (3) follows immediately.

this example, the value of p is set equal to .49 which is a good deal closer to a fifty-fifty split than is observed in most elections.[4] On these assumptions, the chance of a tied vote in accordance with equation (3) turns out to be $(.5)10^{-11}$ amounting to one chance in two hundred billion so that the chance of an extra vote being pivotal is 1 in 400 billion. Thus, with a cost of voting of $20, the required benefit from a win for one's preferred party sufficient to make voting advantageous rises from $200 thousand as estimated above to $8 trillion, or approximately five times the national income of Canada. It is calculations like this that have led many observers to treat the chance of casting a pivotal vote as infinitesimal and irrelevant.[5]

Some features of person-to-person randomization should be noted. Hidden within person-to-person randomization is a special assumption about sampling. Imagine a population of exactly one million voters with no abstentions, where exactly half a million people vote left and where the other half vote right. Clearly, on these assumptions, the outcome is a tie, and any additional voter must be pivotal despite the fact that the probability of a randomly-chosen person voting for the left party remains equal to ½. With p = ½ , the probability of a tied election is 100% in this case, as compared with a quarter of a percent in the calculation based upon person-by-person randomization.[6] The discrepancy arises from the distinction between sampling with and without replacement. A ball picked randomly from an urn may or may not be returned to the urn before the next ball is drawn. Person-by-person randomization employs sampling with replacement. Voting may be more like sampling without replacement. Enormous in this example, the difference between estimated probabilities of a tie would be less pronounced when the proportion of population that votes is significantly less than 1.

Second, as discussed so far, person-to-person randomization is of *interests*. Everybody knows what is best for himself, but some people would be made better off with a win by the left party while other people would be made better off with a win by the right. Alternatively, randomization may be of *opinions.* One of two parties may be better for everybody, but people may differ in their judgments about which party that is. Everybody gains the same fixed amount B when the better party wins, but a proportion, p, of eligible voters thinks the left party is better and a proportion (1 - p) thinks the right party is better. If people are correct on average - in the sense that p > ½ if and only if the left party is preferable - and as long as the value of p is not common knowledge, the probability of the better party winning the election is an increasing

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[4]For tables of the chance of a tied vote, T, dependent on alternative values of N and p, see Beck (1975, 75) and Chamberlain and Rothschild (1981, 153).

[5]In larger elections, such as that for the President of the United States, the chances of casting a pivotal vote would be significantly smaller, but the discrepancy between estimates would remain.

[6]Chamberlain and Rothschild (1981, 153)

function of the number of people who vote rather than abstain. The socially optimal proportion of voters in the population becomes that for which a person's cost of voting is just equal to the expected gain to society as a whole from the chance of swinging the election advantageously.

It is also possible that some people know what is best, others do not, and everybody knows who is who. In such situations, public decision-making is best left to the experts rather than to the ballot box, or voting may confined to people whose task it would be to identify experts.

Third, recognition of the huge gap between the estimates of the chance of casting a pivotal vote when the probability, p, of voting for the left party differs, however slightly, from 50%, has provoked attempts to close the gap by modifying the assumptions of the person-to-person model. Chamberlain and Rothschild (1981) assume instead that p itself is chosen from a probability distribution of possible values between 0 and 1. There are two angels of chance, the first choosing p for the entire population, and the second choosing each person's vote for the given value of p. Chamberlain and Rothschild show that, if the distribution of p is uniform (and in some other cases as well), the probability of a tie is proportional to 1/N. The probability of a tie is still very small, but not as small as in person-by-person randomization where p is fixed and differs from ½ by as little as 1%. Chamberlain and Rothschild's model is in effect a cross between the historical model and person-to-person randomization, replacing the black box in the historical model with a mechanism that could generate similar outcomes.

The chance of casting a pivotal vote by can be increased by lowering the number of people who choose to vote. Palfrey and Rosenthal (1985) assume all eligible voters to adopt a mixed strategy, choosing to vote or abstain by the flip of a weighted coin. Left-party weights would differ from right-party weights, but both weights, and the corresponding numbers of people voting for both parties, would be small enough to make voting advantageous in accordance with equation (1) with D = 0. Typically, though not invariably, estimated numbers of voters are very, very small.[7] Using a variant of this model, Myerson (1998) shows that, from a population of three million eligible voters of whom one million favour the left party and two

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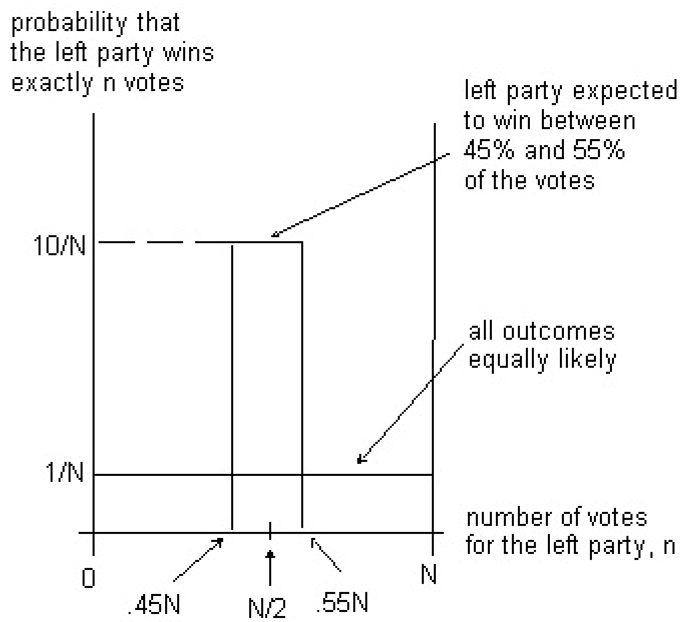
[7] Multiple equilibria are rare but not impossible. With random voting and with equal numbers of eligible voters to the left and to the right, one's chance of casting a pivotal vote is relatively high when almost everybody votes and when almost everybody abstains, for the chance of an even split is the same among a given number of abstainers as among a given number of voters. In the former case, the number of abstentions is small enough that the number of voters for each party approaches the number of eligible voters who favour it. In the latter case, the number of abstentions is large enough to contract the total number of votes cast to the point where each voter has a significant impact on the outcome of the election. The former case has no counterpart when numbers of eligible voters supporting each party are not the same. See Palfrey and Rosenthal (1985) Figure 1.

million favour the right party, an average of as few as 32 supporters of each party can be expected to vote rather than abstain. Ledyard (1984) shrinks the number of voters by allowing people's cost of voting to vary from 0 to some maximal amount.

Fourth and most importantly, since the average probability of the left party winning exactly n out of a total of N votes cannot be other than 1/N, the chance of a tie between a left party and a right party can only be less than 1/N if it is less than the average of all other outcomes. In the preceding examples, N was set at 100,000, so that the average chance had to be 1/100,000. In the example of historical estimation, the chance of a tied vote was significantly larger, one out of 20,000. In person-by person randomization, it was very, very much smaller, one out of 400 billion, equivalent to one 4 millionth of the average of all possible outcomes. This requires some explanation, especially as one would expect the probability of a tied vote to exceed probabilities at the extremes where one of the two parties gets no votes at all.

Figure 1: Two Patterns of the Chances of Alternative Outcomes of an Election between a Left Party and a Right Party

(Drawing not to scale)



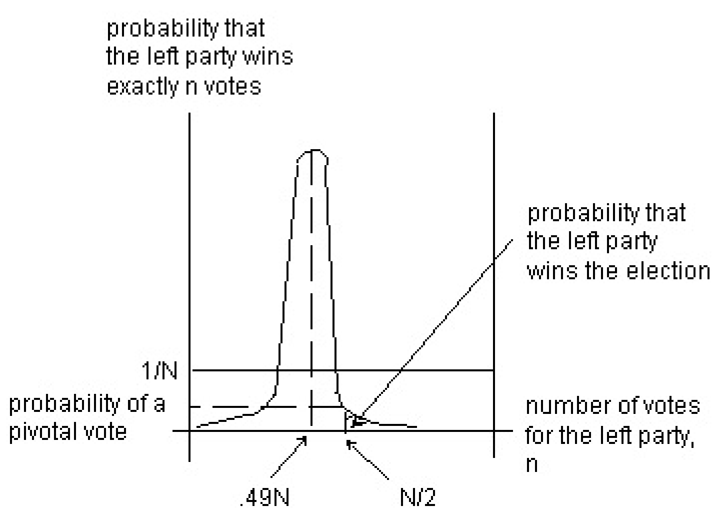
For an election between a left party and a right party, two out of a great many patterns of outcomes are illustrated in Figure 1 with the number of votes for the left party, denoted as n, on the horizontal axis and the probability of exactly n votes for the left party on the horizontal axis. [The horizontal axis is shown as continuous, but it is really a set of N + 1 points because there is no such thing as a fraction of a vote.]

When all outcomes are equally likely, the probability of a tied vote is 1/N as shown by the height of the low horizontal line extending all the way from 0 to N. Another possibility, represented by the post, is that there is seen to be no chance whatsoever of either party winning less than 45% of the votes, but that all outcomes from 45% left and 55% right to 45% right to 55% left are seen as equally likely, raising the chance of a tied vote from 1/N to 10/N. For N = 100,000, the latter possibility is similar to Blais calculation for Canadian Federal elections.

The compressed distribution of outcomes in Figure 1 is drawn in such a way that both parties have equal chances of winning the election. That need not be so. The compressed distribution can be shifted, right or left, changing the parties' chances of winning the election without altering the chance of a tie. For example, the minimum and maximum proportions of the vote going to the left party, might be altered from .45N and .55N, as shown in the figure, to .49N and .59N, implying that the left party has a 9 out of 10 chance of winning the election despite the fact that the chance of a tied vote remains at 10/N. Of course, the distribution cannot be shifted too far in either direction. A rightward shift to the range between .55N and .65N or a leftward shift to the range between .35N and .45N removes all chance of a pivotal vote because the left party is sure to win in one case and the right party is sure to win in the other.

The only way for the probability of a tie to be significantly less than 1/N is for the weight of probability to be concentrated on one side of the 50% mark as illustrated in Figure 2. In the computation for person-by-person randomization with 100,000 voters and with each person given a 49% chance of voting for the left party, the distribution of votes for the left party is binomial with a mean of 49,000 and a standard deviation of about 150, so that a tie, with 50,000 votes for each party, is 1,000 away from the mean, equivalent to about 6 standard deviations.

Figure 2: The Probability Distribution of Numbers of Votes for the Left Party when Each Voter Has a 49% Chance of Voting for the Left Party
(Drawing Not to Scale)



That being so, the bell-shaped curve in Figure 2 becomes more like a needle than a bell, and it should be no surprise that the chance of a tied vote is infinitesimal.

A direct consequence of these assumptions is that there is virtually no uncertainty about the outcome of the election. The probability of the left party winning the election, as illustrated as the shaded area in the Figure 2, is the chance of a variable turning out to be about six standard deviations from its mean, equal to about one in a billion. What this means is that a person whose uncertainty about the outcome of an election is as postulated in person-by-person randomization must believe not just that the chance of casting a pivotal vote is 1 in 200 billion, as estimated above, but that the left party has no more than a one-in-a-billion chance of winning the election. Such a person would be willing to bet $10 million against 1¢ that the left party would lose. Nobody in the realm of politics is ever that certain about anything.

The culprit, in my opinion, is the assumption in person-by-person randomization that people's votes are uncorrelated. On the opposite assumption that votes are perfectly correlated, a 49% chance that any given person votes left means that the right party must win the election and a range of chances varying between, say, 45% and 55% supplies both parties with equal chances and that any person's chance of casting a pivotal vote is 10/N as illustrated in figure 1. In nation-wide randomization, to be discussed below, correlation is combined with random shifts in all voters in valuations of a win for the party preferred.

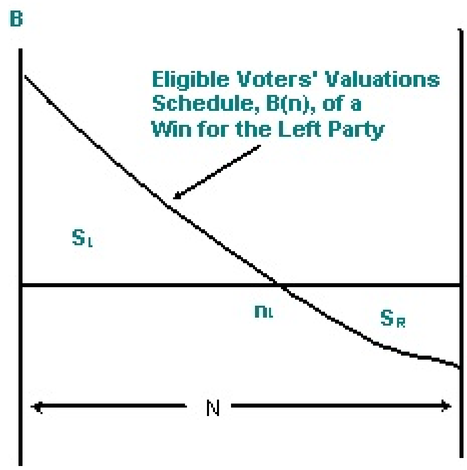
The Alternative

Nation-wide randomization begins by supposing not just that people favour the left party or the right party as the case may be, but that people place different monetary values on their preferences and that a schedule of voters' valuations may be identified. Each person knows for certain his own, positive or negative, value of a win for the left party, but nobody has more than a vague idea of the schedule of voters' valuations in the electorate as a whole. A person's value of a win for the left party is designated as B. If Joan places a value of $4,000 on a win for the left party, then, for Joan, B = 4,000. If Charles places a value of $10,000 on a win for the right party, then, for Charles, B = - 10,000. Along the true but unobservable schedule, people are ordered from left to right in accordance with their valuations, B, of a win for the left party. In an electorate with N people eligible to vote, B(1) is the dollar value of a win for the left party to the person with the highest such valuation, B(2) is the next highest valuation, and so on, until B(N) which is the lowest of all valuations and which must be negative if the right party is to capture any votes at all.

When the income of person n is $y^L(n)$ in the event of a win for the left party and $y^R(n)$ in the event of a win for the right party, then B(n) becomes

$$B(n) = y^L(n) - y^R(n) \qquad (4)$$

Figure 3: The Distribution of Voters' Valuations of a Win for the Left Party



A voters' valuations schedule is illustrated in figure 3 with benefits B(n) on the vertical axis and with the N eligible voters lined up appropriately on the horizontal axis. For convenience, the voters' valuations schedule is drawn continuously, but the schedule is really confined to integral values of n from 0 to N. If everybody votes and as long as all votes are cast selfishly - for the left party when $B(n) > 0$ and for the right party when $B(n) < 0$ - then the left party wins whenever $n_L > N/2$ and the right party would win whenever $n_L < N/2$. The areas designated as S_L and S_R respectively are the total of the valuations by all left-supporters of a win for the left party and the total of the valuations by all right-supporters of a win for the right party. Specifically,

$$S_L = \text{sum of all } B(n) \text{ for which } B(n) > 0 \tag{5a}$$

and

$$S_R = \text{sum of the absolute values of all } B(n) \text{ for which } B(n) < 0 \tag{5b}$$

As the figure is drawn, $n_L > N/2$ and $S_L > S_R$ signifying that the left party wins the election as long as everybody votes and that a win for the left party is best for the nation as a whole. A different postulated shape of the voters' valuation curve - with the same value of n_L but flatter to the left of n_L and steeper to the right - could create a discrepancy between number of votes and aggregate benefits.[8]

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[8]Note that $S_L - S_R = \sum_{n=1}^{N} y^L(n) - \sum_{n=1}^{N} y^R(n)$ which is the differ

national income as it would be if the left party wins and the national income if the right party wins.

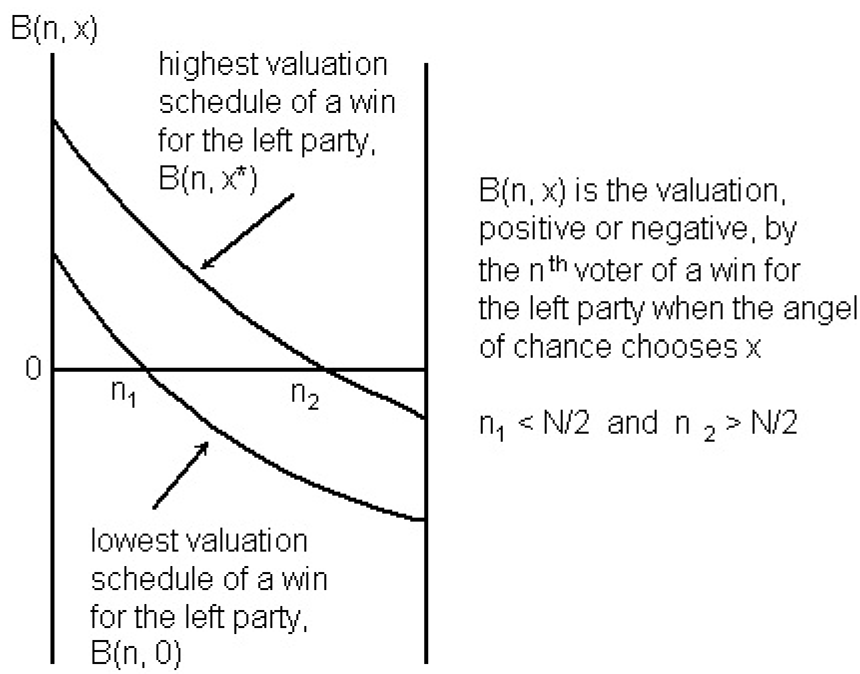
Uncertainty about whether or not one's vote is pivotal can only arise from each voter's uncertainty about the electorate as a whole. In person-to-person randomization, that uncertainty is created by designating each and every voter, one by one, as a left-supporter or a right-supporter, in accordance with the flip of a weighted coin. In nation-wide randomization, that uncertainty is about the exact location of a voters' valuations schedule. Each person looks upon the political preferences of the electorate as a whole as a voters' valuations schedule selected by the angel of chance from a set of feasible schedules within a range from highest to lowest as illustrated in figure 4.

Selection by the angel of chance of one out of a range of possible voters' valuations schedules can be represented as the choice of a single parameter x. Imagine a "basic" schedule B(n) such as is illustrated in figure 1 that is shifted up or down in accordance with a random variable x selected by the angel of chance, converting the valuation schedule of a win for the left party from B(n) to
B(n, x) where

$$B(n, x) = B(n) + x \tag{6}$$

where x varies from 0 to some maximal value, x*. An increase in x pushes the schedule up, raising all valuations of a win for the left party and lowering all valuations of a win for the right party accordingly. It is difficult to say *a priori* what the distribution of the random variable, x, might be, but, in the interest of simplicity, it is assumed here to be uniform, equally likely to take on any value from a minimum of 0 to a maximum of x*.

Figure 4: Highest and Lowest Voters' Valuations Schedules



On these assumptions, the voters' valuations schedule is equally likely to lie anywhere between the highest schedule, B(n, x*), generating, for each person n, the largest possible valuation of a win for the left party (or the smallest possible valuation of a win for the right party), and the lowest schedule, B(n, 0), generating, for each person n, the largest possible valuation of a win for the right party.

It is important to emphasize what voters do and do not know. With fixed rankings on the voters' valuations schedule, voters could infer from their own dollar values of B which curve the angel of chance must have selected and would know for certain whether or not the votes of the rest of the electorate are tied. But voters are presumed not to know their rankings. A person with a high value of B does not know whether that is because his B is relatively high or because the community's x is high. He may be especially partial to the left party, or the entire community may have become so. For example, if B(2,172) = 4,000 on the lowest voters' valuations schedule, if B(10,956) = 4,000 on the highest voters' valuations schedule, and if Joan knows her own value of a win for the left party to be $4,000, Joan could infer that she is between the 2,172th and 10,956th left-leaning person in the entire electorate, but she would have no idea where between these limits her ranking lies. Choice by the angel of chance of a voters' valuations schedule is no more than a rationalization of the general idea that voters have some imperfect perception about the preferences of the rest of the electorate.

To keep the model as simple as possible, a common stylized uncertainty is imposed. Everybody is alike in their perceptions of i) the shape of the voters' valuations schedule, ii) the location of the highest and lowest schedules and iii) the distribution of x, seen as uniform between 0 and x*. A bell-shaped distribution of x would be more realistic but less tractable. Voters may in practice have different pictures in their minds about how the rest of the electorate behaves, but that possibility is being assumed away to ensure that everybody's estimate of π is the same.

On these assumptions, intervention by the angel of chance creates two distinct electoral probabilities, the probability, π, of any person's vote being pivotal and the probability, P, of the left party winning the election. With reference to figure 4 and assuming all numbers of left supporters between n_1 and n_2 to be equally likely, the probability, π, of a person's vote becoming pivotal depends upon the *width* of the band between n_1 and n_2 and the probability, P, of a win for the left party depends upon the *location* of the band, whether it is mainly to the right or mainly to the left of the center point, N/2.

Since all schedules within the band between n_2 and n_1 are equally likely, the chance of a tie must be $1/[n_2 - n_1]$, and a voter's chance of being pivotal must be

$$\pi = 1/[2\,(n_2 - n_1)] \qquad (7)$$

A slightly more complicated derivation of this formula is redundant in the present context where nobody abstains because voting is costless, but becomes useful for generalizing the formula once a cost of voting is introduced. Since there are N votes in total, the average number of votes for both parties together must be N/2, and the party with more than N/2 votes wins the election. It follows at once that the range of possible outcomes of the election is the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average. Since the largest possible number of votes for the left party is n_2 and the largest possible number of votes for the right party is $N - n_1$, the range of possible electoral outcomes must be

$$\{n_2 - N/2\} + \{(N - n_1) - N/2\} = n_2 - n_1 \tag{8}$$

from which equation (7) follows at once.

The probability, P, of a win for the left party is the ratio of “the largest number of votes for the left party in excess of the average” and “ the sum of the largest numbers of votes for both parties in excess of the average”.

$$P = [n_2 - N/2]/[n_2 - n_1] \tag{9}$$

which, as figure 4 is drawn, is somewhat less than ½.

Probabilities of a win for the left party and of casting a pivotal vote can be adjusted to take account of people’s decisions to vote or abstain once a cost of voting is introduced. With costly voting and as long as people are strictly self-interested, the decision to vote or abstain is in accordance with equation (1) above with D set equal to 0. With a common cost of voting, C, and with a common probability, π, of casting a pivotal vote, a person votes for the left party when

$$B > C/\pi \tag{10a}$$

for the right party when

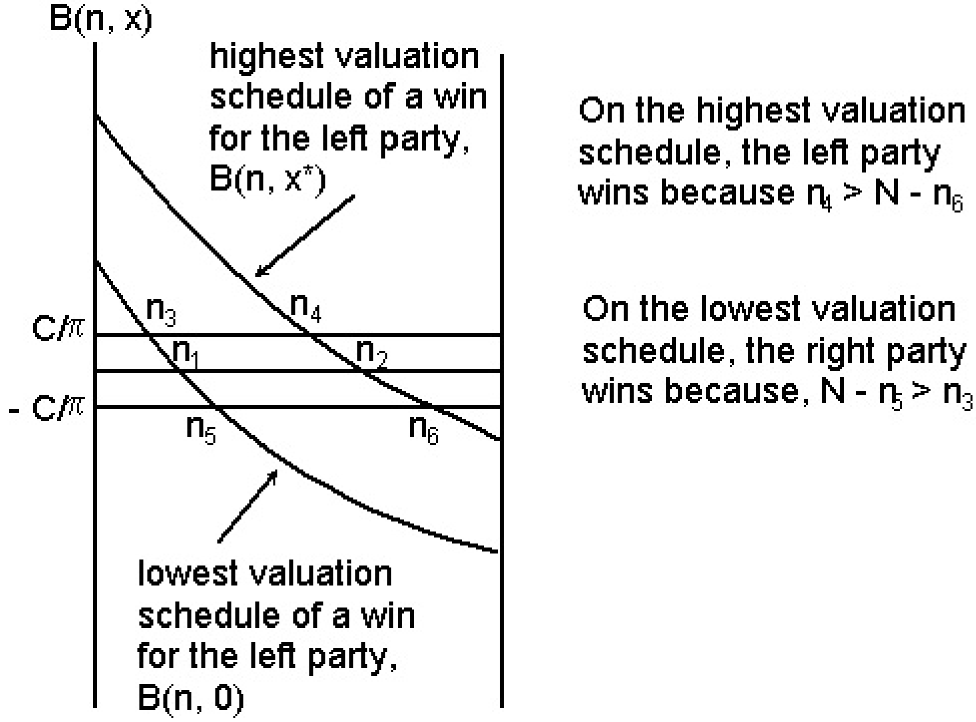
$$- B > C/\pi \tag{10b}$$

and abstains otherwise, i.e. when

$$C/\pi > |B| \tag{10c}$$

Like market prices, π is at once a signpost for each and every voter and a characteristic of the community of voters as a whole. A person votes or abstains in accordance with the electorate’s value of π, but the electorate’s value of π depends on what voters choose to do.

Figure 5: Uncertainty about Other's Preferences, the Cost of Voting and the Chance of Casting a Pivotal Vote



Numbers of voters and abstainers are illustrated in figure 5, a reproduction of figure 4 with the addition of two horizontal lines at distances C/π above and below the horizontal axis. In accordance with equation (10), one votes for the left party if one's value of B is above the higher line, one votes for the right party if one's value of B is below the lower line, and one abstains in between.

For any given π and for any given maximal and minimal voters' valuation schedules, the number of votes for each party and the number of abstentions can be inferred from equation (10). If the highest valuation schedule (with $x = x^*$) is chosen, then n_4 people vote for the left party, $N - n_6$ people vote for the right party, $n_6 - n_4$ people abstain and the left party wins the election. If the lowest valuation schedule (with $x = 0$) is chosen, then n_3 people vote for the left party, $N - n_5$ people vote for the right party, $n_5 - n_3$ people abstain and the right party wins the election. A situation could arise where one party is so much preferred to the other that it wins the election regardless of which valuation schedule is chosen. Were that so, no voter could ever be pivotal and the outcome of the election would depend upon the mobilization of each party's supporters, a matter not discussed in this paper. Assume for the present that that is not so. Assume the left party wins when the highest valuation schedule is chosen, and the right party wins when the lowest valuation schedule is chosen. The assumption is that

$$n_4 > N - n_6 \quad \text{and} \quad n_3 < N - n_5 \tag{11a}$$

yielding a unique value of the probability of casting a pivotal vote.

The derivation of the probability of casting a pivotal vote is similar to what it was with costless voting, but with one important exception. It remains true that the range of possible outcomes of the election is the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average, but the sum of the votes for both parties is reduced by the number of abstentions and the required number of votes to win the election is reduced accordingly.

On the highest voters' valuations schedule where the left party is destined to win the election, the number of votes cast is $n_4 + (N - n_6)$, the average is half that, and the number of votes for the left party in excess of the average is $n_4 - \{n_4 + (N - n_6)\}/2 = ½(n_4 + n_6 - N)$. On the lowest voters' valuations schedule where the right party is destined to win the election, the number of votes cast is $n_3 + (N - n_5)$, the average is half that, and the number of votes for the right party in excess of the average is $(N - n_5) - \{n_3 + (N - n_5)\}/2 = ½(N - n_5 - n_3)$. Thus, "the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average" becomes

$$½(n_4 + n_6 - N) + ½(N - n_5 - n_3) = ½(n_4 + n_6 - n_5 - n_3) \qquad (12)$$

so that the probability of a tied vote becomes $2/(n_4 + n_6 - n_5 - n_3)$ and the probability of casting a pivotal vote becomes

$$\pi = 1/(n_4 + n_6 - n_5 - n_3) \qquad (13)$$

which reduces to equation (7) when $C = 0$, so that $n_4 = n_6 = n_2$ and $n_5 = n_3 = n_1$.

Table 1: Numbers of Votes and Abstentions
(On the assumption that each party has some chance of winning the election.)

	highest valuation schedule	lowest valuation schedule
abstentions	$n_6 - n_4$	$n_5 - n_3$
votes for the left party	n_4	n_3
votes for the right party	$N - n_6$	$N - n_5$
total votes	$N - n_6 + n_4$	$N - n_5 + n_3$
votes for the winning party in excess of the average	$½(n_4 + n_6 - N) > 0$	$½ (N - n_5 - n_3) > 0$

The probability, P, that the left party wins the election must be the ratio of the maximal win for the left party to the full range between the maximal win for the left and the maximal win for the right, i.e.

$$P = [n_4 + n_6 - N] / [n_4 + n_6 - n_3 - n_5] \qquad (14)$$

Information on votes, abstentions and the chance of casting pivotal vote is summarized in table 1.

Equation (13) completes the equilibrium in figure 5. The critical numbers in the table - n_3, n_4, n_5 and n_6 - are dependent upon the value of π, but π itself is dependent on these numbers. Note the family resemblance between historical estimation and nation-wide randomization in the determination of the probability of casting a pivotal vote. A common feature of these models is that if one party might win by up to Q votes and the other party might win by up to R votes and if all outcomes in between are equally likely, the probability of a person's vote being pivotal must be 1/2(Q + R). Nation-wide randomization can be thought of as a rationalization of estimation based upon the historical record. As long as the highest and lowest voters' valuations schedules are parallel, one's chance of casting a pivotal vote is just the inverse of the horizontal distance between the two schedules regardless of the size of C/π.

Note finally the independence of P and π. The highest and lowest voters' valuations schedules may both shift upward increasing the left party's chance of winning the election without at the same time changing anybody's chance of casting a pivotal vote. That is because P depends on the average height of the two curves while π depends on the distance between them. Person-by-person randomization has a very different implication. There, a change in the left party's chance of winning the election would normally be accompanied by a massive change in the probability of casting a decisive vote.

A Numerical Example

Figure 6 is a reconstruction of figure 5 with numbers of voters and abstainers computed by the imposition of a specific voters' valuations schedule. The postulated schedule is

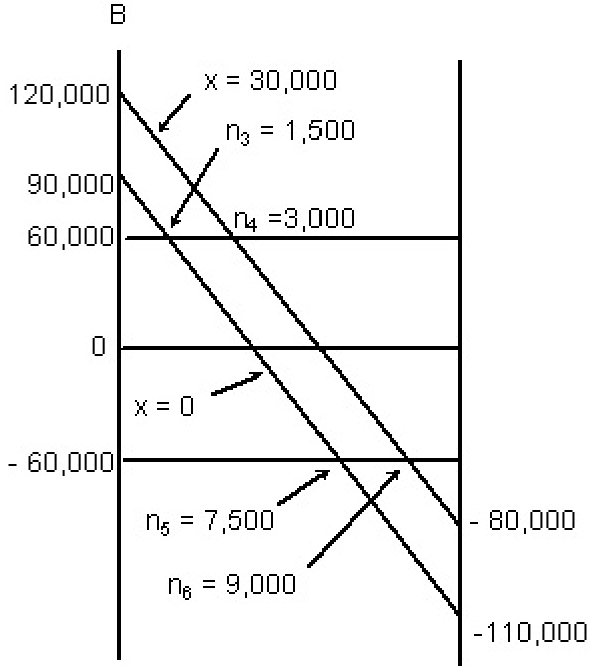
$$B(n, x) = 90{,}000 - 20n + x \qquad (15)$$

where n is the ordering of people from left to right, where the angel of chance picks x from a minimum of 0 to a maximum of 30,000 and where the number of eligible voters, N, is 10,000.

As shown on figure 6, the minimal schedule, for which x = 0, is

$$B = 90{,}000 - 20n \qquad (16a)$$

Figure 6: Votes and Abstentions when Voting is Costly



which is a downward-sloping straight line beginning at B = 90,000 when n = 0 at the left hand vertical axis, ending at B = - 110,000 when n = 10,000 at the right hand vertical axis, and cutting the horizontal axis where B = 0 at n = n_1 = 4,500. The maximal schedule, for which x = 30,000 is

$$B = 90{,}000 - 20n + 30{,}000 \qquad (16b)$$

beginning at B = 120,000 when n = 0, ending at B = -80,000 when n = 10,000 and crossing the horizontal axis at n = n_2 = 6,000.

As minimal and maximal schedules are parallel straight lines, one's chance of casting a pivotal vote is, in accordance with equation (7) and (13), the inverse of twice the horizontal distance between them,

$$\pi = 1/2(n_2 - n_1) = 1/\{2(6{,}000 - 4{,}500)\} = 1/3{,}000 \qquad (16c)$$

Numbers of votes for the left and right parties depend on the cost of voting, C, and the choice of x by th angel of chance. Suppose once again that C = 20, so that C/π = 20/{1/3,000} = 60,000. Connecting B with C/ π in accordance with equation (10) above, values of n_3, n_4, n_5 and n_6 can be derived from the equation

$$90{,}000 - 20n + x = z \qquad (16d)$$

where $n = n_3$ when $x = 0$ and $z = 60{,}000$
$n = n_4$ when $x = 30{,}000$ and $z = 60{,}000$
$n = n_5$ when $x = 0$ and $z = - 60{,}000$
and $n = n_6$ when $x = 30{,}000$ and $z = - 60{,}000$

The computed values turn out to be

$$n_3 = 1{,}500,\ n_4 = 3{,}000,\ n_5 = 7{,}500,\ n_6 = 9{,}000 \text{ and } \pi = 1/3{,}000 \qquad (17)$$

Out of a total population of 10,000, 4,000 people choose to vote and 6,000 people choose to abstain. That these numbers are independent of x is a consequence of the postulated linear form of the voters' valuations schedules.

The information in figure 6 can be looked upon from two points of view. It is, on the one hand, a picture of how the collectivity behaves, showing everybody's probability of being pivotal, the range of possible pluralities for each
party, the number of abstentions, and the valuations of a win for one's preferred party required to induce a person to vote. It is, on the other hand, a guide for deciding whether to vote or to abstain, supplying the critical value of π in equation (1). These points of view are consistent as long as nobody can infer from his own value of B what valuation schedule the angel of chance has selected.

Artificial as it is, this example suggests an important principle about who votes and who abstains. In so far as voting is from self-interest rather than from a sense of duty, it is the extremists who vote and the moderates who abstain. One votes if and only if the value of a win for one's preferred party exceeds 60,000. Everybody else abstains. Symmetries in the example prevent this consideration from influencing the outcome of the election, but, as will be discussed below, there is no guarantee of such symmetry in other, more realistic situations.

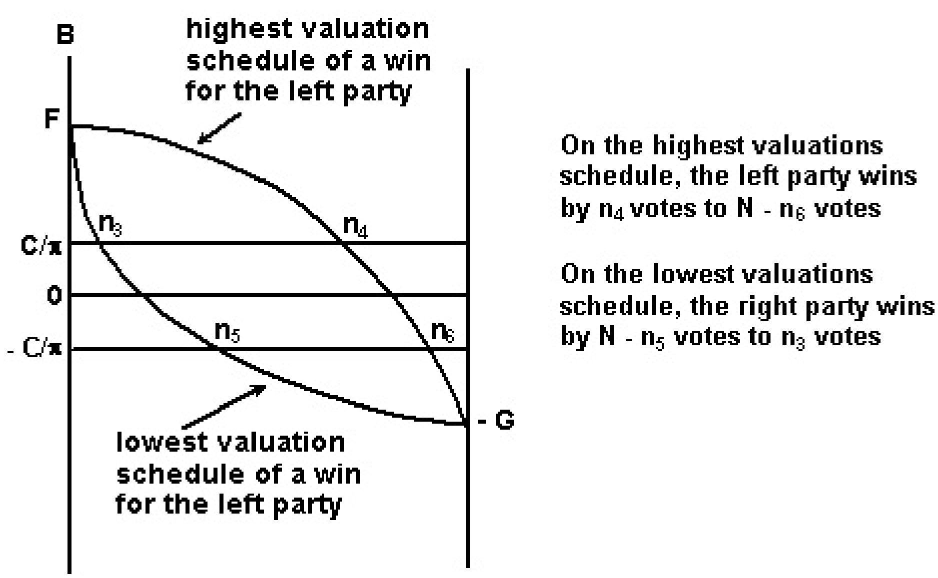
Qualifications and Exceptions:

The numerical example serves to establish that there *may* be an equilibrium – with some people eligible voters voting for the left party, others voting for the right party and still others choosing to abstain - in circumstances where each voter knows his own preference but has only a rough idea about the distribution of preferences in the rest of the population, and, above all, where those who vote do so out of pure self interest-rather than from a sense of duty. The parameters of the example were chosen to generate such an outcome. The example does not establish that there *must* be an equilibrium or that the outcome is necessarily a reflection of the will of the electorate.

A Sure Winner: The locations of maximal and minimal valuation schedules in figure 6 were chosen to ensure that both parties have some chance of winning the election. The maximal schedule delivers a win the left party, the minimal valuation schedule delivers a win for the right party and some schedule in between delivers a tie. That need not always be so. Regardless of what value of x is chosen by the angel of chance, significantly higher schedules than those in equation (15) above would deliver a sure win to the left party, and significantly lower schedules would deliver a sure win to the right. This is, however, a consequence of the postulated uniform distribution of x. A normal distribution would have yielded some probability of winning to both parties. Though it is probably rare for a political party to have absolutely no chance of winning in a two-party race, the possibility cannot be ruled out altogether.

Evidence about the Angel of Chance: Nation-wide randomization requires people to know their own values of B without at the same time having more than a general idea of the preferences of the electorate as a whole. Nobody must know which voters' valuations schedule the angel of choice has selected. But if the highest and lowest schedules are linear as in figure 6, such information cannot be completely concealed. In particular, a person who observes his own B to be 120,000 (where the highest schedule cuts the left-hand axis) cannot help knowing the highest schedule has been chosen, and a person who observes his own B to be - 110,000 cannot help but knowing that the lowest schedule has been chosen. In both cases, the person must know for certain which party wins the election. Anybody who observes his value of B to lie between 120,000 and 90,000 - the intersection of the highest and lowest schedules with the left-hand vertical axis - must know that the angel of chance has selected a voters' valuations schedule within a restricted portion of the full range of feasible schedules bordering on the highest possible schedule, and anybody who observes his value of B to lie between - 80,000 and - 110,000 must know that the scheduled is confined in the opposite direction.

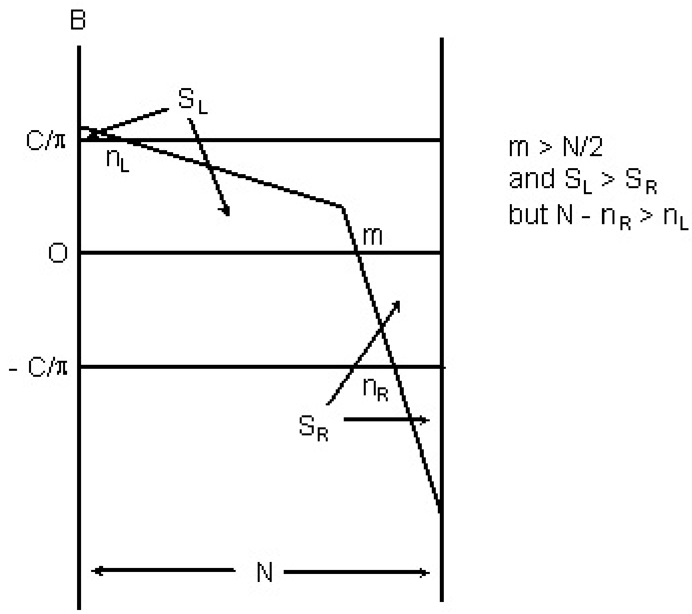
Figure 7: Valuation Schedules with Common Maximal and Minimal Values



This difficulty is more apparent than real. A way out is to change the postulated shape of the voters' valuations schedule as shown in figure 7 with all possible schedules intersecting at both vertical axes. The pattern of schedules in figure 7 serves to suppress everybody's knowledge about which schedule the angel of chance has chosen by bunching all schedules together at the maximal and minimal values of B (shown in the figure as F and - G), so that neither the person with the highest B nor the person with the lowest B nor anybody in between can have any idea whatsoever which among the set of feasible voters' valuations schedules the angel of chance has chosen. Values of the five relevant variables, n_3, n_4, n_5, n_6 and π, would be determined simultaneously from equations (13) and whatever modification is required to equation (15) to account for the postulated change in the shape of the voters' valuations schedule.

The important consideration here is that imposition of common upper and lower limits to the voters' valuations schedule is a device imposed to make uncertainty precise. In practice, voters would have no more than a vague idea of the location of the true schedule on election day, with no firm boundary between what may and what may not happen.

Figure 8: A Majority of the Population Prefers the Left Party to the Right Party and the Left Party Supplies the Larger Aggregate Surplus, but the Right Party Wins the Election



<u>Bias in Favour of Small Groups with Strong Preferences</u>: A situation can easily arise where a majority of the population prefers one party, but where the other party wins the election because its supporters place higher values on a win for their preferred party and are therefore less likely to abstain. The electoral triumph of a small group with strong preferences is illustrated in figure 8 where the voters' valuations schedule is flatter to the left than to the right. Think of B(n) in figure 8 as the true schedule selected by the angel of chance, and suppose the highest and lowest

schedules are sufficiently above and below the true schedule that both left and right parties are seen as having some chance of winning the election, allowing everybody some probability, π, of casting a pivotal vote.

If voting were costless, m people would vote for the left party where, as the figure is drawn, $m > N/2$, signifying that the left party wins the election. But when voting is costly, n_L people vote for the left party, $N - n_R$ people vote for the right party, but $(N - n_R) > n_L$, signifying that the right party wins the election. This is not an implausible outcome. Rich people may well place the greater dollar value on a win for their preferred political party and have a greater incentive than poor people to vote rather than abstain. With a voters' valuations schedule as in figure 8, a majority of the population prefers the left party to the right, but a "decisive minority" with a sufficiently strong preference for the right party enables the right party to win the election.[9]

Social Welfare: Parallel to the possible discrepancy between numbers of supporters of each party and numbers of votes cast is a second possible discrepancy between the outcome of the election and social welfare. As illustrated in figure 3 above, the net gain (or loss as the case may be) to society as a whole from a win by the left party is $S_L - S_R$, the difference between the gain to left-party supporters from a win by the left party and the gain to right-party supporters from a win by the right party. As the figure is drawn, the party favoured by a majority of the population supplies the larger new gain as well. That need not always be so. As illustrated in figure 8, the voters' valuations schedules could be tweaked so that
$S_L < S_R$ despite the fact that $m > N/2$. A relatively indifferent majority of the population may prefer that shopping on Sunday be allowed, while a passionate minority prefers that shopping on Sunday be forbidden. The dollar value of the potential loss to the minority may well exceed the dollar value of the potential gain to the majority.

Surplus has so far been graduated in dollars to ensure that B and C/π are commensurate. Nevertheless, if one party tends to favour the rich while the other tends to favour the poor and if the disputed public policy is to narrow the distribution of income, then it may be of some interest to measure surplus in utils as well. For any given voters' valuations curve, the surpluses, S_L and S_R, measured in dollars as illustrated in figure 8, are

$$S_L = \int_0^v B(n)dn \qquad S_R = \int_v^N - B(n)dn \tag{16}$$

where v is the value of n such that $B(v) = 0$. With a common utility of income function, the

[9] "..for large electorates, no matter how great the proportion of the electorate that prefers a particular outcome, that outcome is likely to lose the election if the opposing outcome is preferred by an expected majority of voters with a sufficiently large incentive to vote", Campbell (1999, 1203).

surpluses, S_L and S_R, can be transformed from dollars to utils. Expressed in utils, the surpluses become

$$U_L = \int_0^v [u(y^L(n) - u^R(y^R(n))]dn \qquad U_R = \int_v^N [u(y^R(n) - u(y^L(n))]dn \tag{17}$$

where $y^L(n)$ and $y^R(n)$ are the incomes of person n depending on which party wins the election. Equation (17) is constructed on the assumption that everybody`s utility of income function is the same. Otherwise, each person would have his own version of equation (17).

Self-interest and Duty Combined

The picture of nation-wide randomization in the preceding section is of a community of self-interested people, of dyed-in-the-wool economic men, who vote or abstain depending on whether expected benefit from voting is greater than cost. There is another possibility. People may be moved by a duty to vote regardless of whether πB is greater than C. Voting from a sense of duty influences how people vote, and not just whether they vote or abstain.

A sense of duty can affect voting in two distinct ways. On the one hand, the cost of voting becomes like a fixed cost that must be borne regardless of which party one votes for, with no additional cost of choosing among political parties. Once the cost of getting to the ballot box is borne, there is no extra cost in choosing between the left party and the right party, and a community of dutiful voters finds itself in a situation like that illustrated in figure 4 where everybody for whom $B > 0$ votes for the left party, everybody for whom $B < 0$ votes for the right party. The left party's chance of winning the election becomes P as in equation (9) rather than equation (14), and the bias in favour of small groups with strong preferences, as described in figure 8 , disappears. Voting is more reflective of the preferences of the majority when it is dutiful than when it is strictly self-interested.

On the other hand, a duty to vote might extend from not abstaining to the criterion for choosing between political parties. One may vote on utilitarian grounds for whichever party is expected to yield the larger surplus, as represented by the larger of S_L and S_R in equation (16) or by the larger of U_L and U_R in equation (17).[10]

When some people vote selfishly and others vote dutifully, the bias in favour of extremists is diminished but not eliminated altogether.

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[10]Literature on the content of a duty to vote is reviewed in Usher (2011a).

Self-interested Voting Reconsidered

Ignoring for the moment all sense of duty, a person's decision to vote or abstain is, as shown in equation (1), a balancing of his benefit from a win by the party he prefers, weighted by the chance as he sees it that his vote is pivotal, against the cost of voting that he bears for sure. The chance of casting a pivotal vote is commonly believed to be infinitesimal, very much smaller than would be required to make voting advantageous when more than a tiny proportion of the electorate chooses to vote rather than to abstain.[11] Several lines of argument point in the opposite direction, suggesting that the role of self-interested voting may have been dismissed too quickly: It is argued throughout this paper that the probability of casting a pivotal vote is grossly under-estimated by reliance upon a model of person-by-person randomization, and that a model of nation-wide randomization generates more plausible estimates commensurate experience in past elections. But there may be advantages to voting over and above the chance of casting a pivotal vote. Voting may be fun. Voting may be expressive as well as instrumental. A vote that is not pivotal may still influence public policy.[12] Votes today may be influential in elections to come. These possibilities will be considered in turn.

That voting may not be personally advantageous was recognized by Downs (1957) and by Tullock (1967). Tullock suggested people's willingness to vote rather than to abstain might be explained by the possibility that voting is not costly at all. Voting for a political party might be like supporting a local football team or voting for the American Idol or congregating in Times Square on New Year's Eve. Incorporating this possibility transforms equation (1) into

$$\pi B + D + E > C \qquad (18)$$

where E is the value one attaches to participation in voting as a public event.

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[11] "....instrumental voters, if rational, are predicted not to vote because the cost of voting in terms of time exceeds the expected benefit of voting based on the insignificant liklihood of one's vote being decisive.", (Hillman, 2010, page 403). "Instrumental" in this context refers to the material benefit to the voter that a win by one's preferred party would convey. ".. One problem with this line of thought is that there is no explanation of why most people vote at all." (Glaser, 1987, page 257).

[12]Throughout this paper and in much of the literature on pivotal voting, it is assumed implicitly that π refers to the chance of a unique event. That is not always so. In Canada, for example, one's vote has a certain chance of being pivotal in the election of the Member of Parliament for one's constituency, together with a much smaller chance of being pivotal in the determination of the governing party. In other countries, voting for a list of offices is contained on one and the same ballot.

People may vote "expressively", seeking to state their preferences and concerns regardless of the outcome of the election. "A person may support Gary Hart for the same reason he buys a Cuisinart: to show his friends that he likes fresh ideas, is part of a new generation, or that he is not yet solidly middle class". (Glaser, 1987, 259). The notion of expressive voting may be more or less broadly defined. Fiorina (1976, 395) associates expressive voting with "the utility ...of satisfying... one's party allegiance". Glaser (1987) confines it to signaling preference for its own sake. Hillman (2010) categorizes all motives for voting as either "material" or "expressive" utility, the latter including altruism, duty and delusion. Expressive voting may or may not be socially-advantageous depending on what is being expressed. One may express good will toward one's fellow citizens by voting for a party expected to provide essential social services to the poor. One may express allegiance to the upper classes, the One True God, the master race, or the glorious leader. What could be more expressive than *The Triumph of the Will*? There is also something peculiar about expressing opinions and attitudes through a secret ballot, but, as Brennan and Hamlin (1998) point out, expressive and instrumental motives for voting reinforce one another in a person's decision to vote rather than to abstain.

Looked upon so far as an unambiguous choice between this and that, voting is sometimes looked upon instead as a basis for compromise between extremes. Stigler (1972) proposed a rudimentary model of this sort. Alesina and Rosenthal (1995) have worked out a much more elaborate model of compromise within legislature and between legislatures and the Presidency. To see what is at stake, think of politics as the choice of v on a left right scale between 0 and 1 where the benefit to every left-leaning person is s, where the benefit to every right-leaning person is 1-s and where so little is known about people's preferences that, prior to the election, there is thought to be an equal chance of any number, n, of left-leaning voters from 0 to N. In the model of politics as majority rule voting between this and that, the left party chooses 1 as its platform, the right party chooses 0 as its platform, the left party wins if and only if $n > N/2$, a person's probability of being pivotal is 1/2N and, since the gain from a win by the party one favours is 1, everybody's expected gain from voting rather than abstaining is also 1/2N.

Alternatively, in the model of politics as compromise, the choice of s depends upon the proportion of left-leaning voters so that s is set equal to n/N regardless of whether or not $n > N/2$. Now imagine an additional person who would vote for the left party if he votes at all, but is choosing whether to vote or abstain. If this person votes, the proportion of left votes increases from n/N to + 1)/(N + 1) raising the left party's proportion of votes by $(n + 1)/(N + 1) - n/N$ which decreases steadily with n from approximately 1/N when $n = 0$, to 0 when $n = N$ and raising x accordingly. The increase in s is the average of the two, equal to 1/2N which, mirable dictu, is exactly the same as the expected increase in s in accordance with an up-or-down majority rule vote. The choice is between a tiny chance of a big gain or a certainty of a small gain. In the cases to follow, the balance of advantage might go either way.

The central idea is that is that the paradox of not voting is not rendered irrelevant by the identification of advantages of voting over and above the chance that one's vote is pivotal. One person's vote is nevertheless a small part of the whole and can be expected to have no more than a tiny influence on whatever nation-wide voting procures.

Contrary to what has been assumed so far, a person's vote may be pivotal not just in today's election but in elections to come by helping to communicate to the rest of the electorate that the party one votes for is supported by a significant share of the population.[13] Support for a political party may be built up gradually. Equation (1) above is a static simplification. Both B and π are enlarged when the effect of today's vote on future elections is taken into account. The British Labour party hung on for generations before finally attaining office.

Estimation of the chance of casting a pivotal vote has been based so far on the assumption that parties' platforms are fixed. Ignored altogether is the parties' response to voters' behaviour. If turnout is known to be large among any group of people, all political parties slant policies to capture that group's vote. If politics is exclusively about the choice of a point on a one-dimensional continuum and with only two competing parties, both parties adopt as their platforms the first preference of the median voter, the median of those who actually vote rather than of those who are entitled to do so but may abstain instead. If politics is about several issues at once, an increase in any group's propensity to vote might be expected to slant both parties' platforms in that groups favour, but without eliminating differences in parties' platforms altogether. Parties may balance competing interests in accordance with the probabilistic voting theorem (Austin-Smith, 1987), or predatory majorities may emerge to appropriate disproportionate benefits for their members. In either case, propensities to vote matter. In the extreme, a group with no turnout whatsoever would never be awarded a share of the spoils. Recognizing this, the voter acquires an incentive to vote over and above the incentive from the chance of casting a pivotal vote. Influence of a person's vote on policies and platforms is a large chance of a tiny gain, as distinct from the tiny chance of a large gain in the standard picture of pivotal voting. In Canada, the Cooperative Commonwealth Federation never won a federal election but is said to have provoked the ruling Liberal party to adopt many social programs. Nobody believes Canada's Green Party to have any chance of winning a Federal election, but its supporters may be content with the role of the party in keeping protection of the environment on the national agenda.

These considerations are complements rather than substitutes. All by itself, the chance of casting a pivotal vote may be insufficient to account for voting from self-interest alone, but, as one consideration among others, it may on occasion be sufficient to tip the balance.

Conclusion

Relying upon a model of person-by-person randomization, some authors have been

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[13]On voting as signaling of preference, see Meirowitz and Shotts (2008).

inclined to write off the probability of casting a pivotal vote as altogether too small to induce anybody to vote rather than abstain. Nation-wide randomization raises that probability enough to justify its inclusion among other considerations. The chance of casting a pivotal vote has been rescued from total irrelevance, though it remains unlikely that a majority of the electorate would be inclined to vote rather than abstain on the strength of self-interest alone.

As a description of uncertainty in voting, nation-wide randomization captures aspects of voter preference that person-by-person randomization abstracts away.

- Political preferences are monetized. People are not simply partitioned as left-supporters or right supporters, but are assigned dollar values of a win for one party over the other, giving rise to a schedule of voters' valuations from extreme left to extreme right.

- Uncertainty in the outcome of elections - without which nobody's vote could be pivotal - is generated by swings in the entire scale of left-right preferences rather than, as in person-by-person randomization, by the random assignment of people to the left or to the right.

- Uncertainty in the outcome of an election is disconnected from the chance of casting a pivotal vote. By contrast, as long as each voter's chance of supporting the left party is other than exactly 50%, person-by-person randomization makes it virtually certain which party will win the election.

Unwillingness to vote becomes less troublesome in one respect but more so in another. It becomes less troublesome than with person-by-person randomization in that microscopically low probabilities of casting a pivotal vote are eliminated within the range of outcomes where both competing parties have some chance, however small, of winning the election. In a sense, the model of nation-wide randomization inherits the variability of estimation from the historical record.

Unwillingness to vote becomes more troublesome in that nation-wide randomization may give rise to systematic bias - as illustrated in figure 8 - where the party most likely to win the election when everybody votes becomes likely to lose when a a significant portion of the electorate abstains because the expected benefit of voting, πB, falls short of the cost of voting, C. Such bias is particularly likely to emerge when one party is relatively favourable to the rich and the other is relatively favourable to the poor. Portrayal of such bias is, alas, a strength of rather than a weakness of nation-wide randomization because the bias can in practice arise. Recognition by citizens of a duty to vote is required to align the outcome of voting with the interests of the nation as a whole.

The central proposition in economics is that the outcome in markets with well-established property rights and where everybody does what is best for himself alone is not just determinate, but best for the community as a whole. The proposition is subject to many well-known qualifications, and "best for the community" must be understood in a very special sense. Even so, the proposition is at first sight implausible. Only familiarity dulls our surprise that it is in fact true. The central question in the economics of voting is whether there is an analogous proposition when property rights in markets are replaced by voting rights in elections. A strong case can be made that there is not. The outcome when everybody chooses to vote or to abstain from self-interest alone is not the best for the community as a whole. To sustain majority rule voting, there must be a willingness of politicians to compromise[14] and a recognition among a sufficient proportion of citizens of a duty to vote.[15] At issue in this paper is whether duty displaces self-interest completely. The claim is that it does not.

[14]Literature on bargaining and voting is discussed in Usher (2011a)

[15] The nature of the duty to vote and the role of social pressure in reinforcing such duty is discussed in Gerber, Green and Larimer (2008). The content of the duty to vote is discussed in Usher (2011b)

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