Queen's Economics Department Working Paper No. 1170

On Beckmann's Dispersed "Interaction City"

John Hartwick
Queen's University

Andrei Bazhanov

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

7-2007

On Beckmann's Dispersed "Interaction City"

Andrei V. Bazhanov[†,‡], John M. Hartwick[§]

July 19, 2007

Abstract Beckmann's interaction model has each resident touching base in face-to-face activity with every other resident, per unit time, at the other's residence. We re-work his resulting "interaction city" with each resident "operating with" a Cobb-Douglas utility function. We then turn to a more satisfactory "technology" of residents interacting and solve for an interaction city with an explicit payoff to resident i for engaging in interaction.

Keywords Spatial interactions of city residents · productive face-to-face activity
JEL Classification Numbers R14 · D11

[*]Institute of Mathematics and Computer Science, Far Eastern National University, Vladivostok, 690600, Russia or Department of Economics, Queen's University, Kingston, ON K7L 3N6, Canada

[†]Corresponding author: Tel.: 1-(613)-533-2270. Fax: 1-(613)-533-6668. E-mail address: bazhanov@econ.queensu.ca (A.V. Bazhanov), hartwick@econ.queensu.ca (J.M. Hartwick).

[‡]Department of Economics, Queen's University, Kingston, ON, Canada

[§]Thanks from J.M. Hartwick to the SSHRC for financial support.

1 INTRODUCTION

Beckmann (1976, Chapter 8) set out an interesting model of an urban area, a model based on the interaction of each resident at a residential location with each other, on a regular basis.[1] Instead of every resident commuting to the center each day and interacting in a productive way as in a monocentric city, a resident in Beckmann's city commutes each day to the residence of every other person (resident) in the city. The cost per resident of interacting becomes complicated since person i needs to know where every other resident is located when she allocates her income toward commuting (interacting) costs, "housing" and other consumption goods. We re-solve Beckmann's model for the case of each resident having a Cobb-Douglas utility function defined over "housing" and other goods. Beckmann worked with an idiosyncratic separable utility function. Our re-solving in a sense brings Beckmann's analysis closer to that of textbook models of cities.

We then extend Beckmann's analysis by endogenizing the benefits of resident i interacting with each other resident over a period. We do this in a model with a simpler interaction "technology". Resident i visits or is visited by every other resident once per period. (Beckmann had resident i visiting every other resident at her residence once per period.) A productive interaction in our model is an uncertain "hit" of resident i with one other resident which yields to i a saleable patent. It is as if pairs of residents have coffee each period and one pairing results stochastically in a useful invention, and in income to one of the residents. Hence a resident's income is stochastic since she never knows which coffee she has will result in a saleable patent. And a resident's

[1] The model is interpreted slightly differently (total cost of interaction rather than average cost) in Fujita and Thisse (2002, pp. 174-179) and re-presented clearly. We follow their "formulation".

average income is a function of the number of other residents she is interacting with. City size becomes a function of the value of output, here the average number of useful patents "produced" per period. Production is a consequence of two residents pairing off for coffee on a regular basis. Every resident engages in interaction in every period but resident i never knows with which other resident or in which period "her" saleable patent will be realized.[2]

The idea that the core of a city should be treated as a group of interacting (cross-visiting) firms followed directly from Beckmann, notably in Borukov and Hochman (1977), Imai (1982), Fujita and Ogawa (1982)[3], Tauchen and Witte (1983) and (1984), Kanemoto (1990), and more recently in Berliant, Reed, and Wang (2000) and Helsely and Strange (2005). A central focus is on the inherent market failure associated with interactive cities. To a first approximation agent i locates to minimize her interaction costs without reckoning the costs she is imposing on the $N-1$ other firms by her choice of location. Each other firm faces a particular cost of interacting with her. One can envisage different degrees of interaction corresponding to different departures of equilibria from first best outcomes, a topic we hope to pursue in the future.[4]

2 THE MODEL

The city is located on a line with a resident at distance x from the center consuming $h(x)$ of land ("housing"). Land rent at x will be $r(x)$. Hence a household's budget

[2]Helsley and Strange (2005), for example, have the utility of an agent higher with more interactions but at the cost of more travel.

[3]Lucas and Rossi-Hansberg (2002) rework the approach of Fujita and Ogawa (1982) in a very general model which ends up analyzable only by means of numerical simulations. Of interest is their discovery of "the extreme sensitivity of the nature of equilibria to small changes in assumed travel costs" (p. 1447).

[4]Beckmann did not take up the issue of equilibria versus optima or a schedule of location taxes and subsidies that could implement a first best. Of interest would be the result that the first best Beckmann city was less technically complicated than the second best counterpart which we are reporting here.

constraint is

$$y - T(x) = cp + r(x)h(x)$$

with c other consumption goods (with price p set at unity), $T(x)$ interaction or total transportation costs per period per household, and y income per period. Consumption c will vary with distance x. The household has utility function $U = h(x)^{\alpha}c(x)^{1-\alpha}$. The utility level is fixed at $\overline{U}$ by free migration between cities (the open city assumption). Hence

$$h(x) = \overline{U}^{1/\alpha}/c(x)^{(1-\alpha)/\alpha}$$

Since $c(x) = (1-\alpha)[y - T(x)]$, we have $h(x) = \xi\left[1/\left[y - T(x)\right]^{(1-\alpha)/\alpha}\right]$ for $\xi = \left[\overline{U}^{1/(1-\alpha)}/(1-\alpha)\right]^{(1-\alpha)/\alpha}$ and population density function $n(x) = 1/h(x)$ in

$$n(x) = [y - T(x)]^{(1-\alpha)/\alpha}/\xi \tag{1}$$

And since $r(x)h(x) = \alpha[y - T(x)]$, we also have

$$r(x) = \alpha\left[y - T(x)\right]^{1/\alpha}/\xi$$

Consider exogenous edge rent $\overline{r}$ at edge b, positive and unspecified.[5] Cobb-Douglasness of utility gives us

$$\begin{aligned}
\alpha\left[y - T(b)\right] &= \overline{r}h(b) \\
\text{and } (1-\alpha)\left[y - T(b)\right] &= c\left(b\right) \\
\text{or } \alpha c(b) &= (1-\alpha)\overline{r}h(b)
\end{aligned}$$

In addition we have $\overline{U} = h(b)^{\alpha}c(b)^{1-\alpha}$. Hence we can solve for edge values $h(b)$ (=$1/n(b)$)

[5] With Cobb-Douglas utility one worries about a zero rent at the edge leading to an extremely large radius for a city. Hence $\overline{r}$ is treated as strictly positive.

and $c(b)$. These values then allow us to solve for $T(b)$

$$\begin{aligned} h(b) &= (\overline{r}/\alpha)^{\alpha-1} \cdot \xi^{\alpha} \\ c(b) &= (1-\alpha) \cdot (\overline{r}\xi/\alpha)^{\alpha} \end{aligned}$$

$$T(b) = y - (\overline{r}\xi/\alpha)^{\alpha} \tag{2}$$

Observe that

$$\begin{aligned} r(x) &= \alpha[y - T(x)]/h(x) \\ &= \alpha[y - T(x)] \cdot n(x) \\ &= \alpha[y - T(x)]^{1/\alpha}/\xi \\ &= \alpha\xi^{\alpha/(1-\alpha)}[n(x)]^{1/(1-\alpha)} \end{aligned} \tag{3}$$

Given $\overline{r}$ as rent at the edge, b, we can express $n(b)$ in terms of $\overline{r}$. That is

$$n(b) = \xi^{-\alpha} \left(\overline{r}/\alpha\right)^{1-\alpha} \tag{4}$$

3 MONOCENTRIC CITY AS BENCHMARK

We can fix ideas by appealing to the monocentric counterpart for comparison. Then $T(x) = t \cdot |x|$ and all interaction occurs at one point in the center. Given parameters α, y, ξ, t and edge rent $\overline{r}$, we can solve for edge, b in $\alpha \left[y - t \cdot b\right]^{1/\alpha}/\xi = \overline{r}$ and then city size N in

$$b = [y - (\overline{r}\xi/\alpha)^{\alpha}]/t$$

$$2\int_0^b \left[y - t \cdot x\right]^{(1-\alpha)/\alpha}/\xi dx = 2\alpha \left\{y^{1/\alpha} - (y - tb)^{1/\alpha}\right\}/\xi t = N.$$

We interpret this as parameters α, ξ, y, t and $\overline{r}$ yielding geographic size b and then b and $n(x)$ yielding population, N.[6] This sequence of links is somewhat different for a Beckmann city.[7]

In Beckmann's city, interaction occurs by one-on-one visiting of each person to all others, one trip per person visited per period. Hence each household incurs $N-1$ trips per period. Formally, then travel costs for interacting for a person at x miles from the center, at zero, are

$$T(x) = \int_{-b}^{x} t(x-z)n(z)dz + \int_{x}^{b} t(z-x)n(z)dz \tag{5}$$

The city ranges on the line from $-b$ to b. Observe that

$$d^2T(x)/dx^2 = 2tn(x)$$

Hence we can substitute for $n(x)$ and obtain the fundamental equation for a Beckmann city

$$d^2T(x)/dx^2 = 2t\left[y - T(x)\right]^{(1-\alpha)/\alpha}/\xi \tag{6}$$

We turn to solving the model.

4 SOLVING THE MODEL FOR $\alpha = .5$

Since the resident in the center at $x = 0$ will incur the least interaction costs, we have $T'(0) = 0$ and since $n(x)$ must be positive, we know that $T(x)$ is convex in x. For the

[6] Somewhat parenthetically we note that $dR = Ndy - \overline{r}db$, for $R = 2\int_0^b r(x)dx$. Roughly speaking, since the utility level is fixed, wage increments are fully capitalized in aggregate rent increments. This capitalization result turns on Leibnitz's Rule for differentiation of an integral.

[7] When we speak of a Beckmann city, we mean one generated with our Cobb-Douglas utility function, not one generated with Beckmann's utility function $u = \alpha \log h + c$. We have done no analysis with his utility function. In fact we started this analysis to see if we could re-work Beckmann's analysis with a Cobb-Douglas function.

case $\alpha = 1/2$, (6) is a linear nonhomogeneous equation of the second order

$$d^2T(x)/dx^2 + 2tT(x)/\xi = 2ty/\xi$$

which solves[8] to the closed form

$$T(x) = y - c_1 \cos(x\sqrt{2t/\xi}) \tag{7}$$

c_1 is a constant of integration.

Using the definition (1) of $n(x)$ we have $n(x) = c_1 \cos\left(x\sqrt{2t/\xi}\right)/\xi$. Substituting the last expression in definition (5) of $T(x)$ we obtain

$$\begin{aligned} T(x) &= c_1 t\left[\int_{-b}^{x} t(x-z)\cos\left(z\sqrt{2t/\xi}\right)dz + \int_{x}^{b} t(z-x)\cos\left(z\sqrt{2t/\xi}\right)dz\right]/\xi \\ &= c_1\left[\sqrt{2t/\xi}\cdot b\cdot\sin\left(b\sqrt{2t/\xi}\right) + \cos\left(b\sqrt{2t/\xi}\right) - \cos\left(x\sqrt{2t/\xi}\right)\right] \end{aligned} \tag{8}$$

We equate (7) and (8) at $x = 0$ to get c_1 as a function of the new "parameter", b, temporarily unspecified:

$$c_1 = y/\left[\sqrt{2t/\xi}\cdot b\cdot\sin\left(b\sqrt{2t/\xi}\right) + \cos\left(b\sqrt{2t/\xi}\right)\right] \tag{9}$$

Given boundary condition, $T(b) = y - \sqrt{2\overline{r}\xi}$, we have another nonlinear equation in c_1 and b.

$$c_1 = \left[y - \sqrt{2\overline{r}\xi}\right]/\left[\sqrt{2t/\xi}\cdot b\cdot\sin\left(b\sqrt{2t/\xi}\right)\right], \quad b \in \left(0, \frac{\pi}{2}\sqrt{\xi/2t}\right) \tag{10}$$

The above is a Beckmann city and an equilibrium is a positive pair (c_1^*, b^*) satisfying equations (9) and (10), given y, ξ, t and $\overline{r}$.

[8] We find a general solution $T_h(x)$ of the corresponding homogeneous equation $T'' + \frac{2t}{\xi}T = 0$ (see Murphy, 1960, p. 84) and the particular integral $T_p(x)$ of the nonhomogeneous one [Murphy, 1960, p. 146], and then $T(x) = T_h(x) + T_p(x)$, or by finding a solution of the equation with the missing $T'(x)$ [Murphy, 1960, p. 160]. We use $T'(x) = 0$ at 0.

To obtain b^*, we substitute for c_1 from (10) in (9) to get a nonlinear equation for b.

$$y - \sqrt{2\overline{r}\xi} \cdot \left[1 + \sqrt{2t/\xi} \cdot b \cdot \tan\left(b\sqrt{2t/\xi}\right)\right] = 0 \tag{11}$$

or

$$y - \sqrt{2\overline{r}\xi} = \sqrt{2\overline{r}\xi} \cdot \sqrt{2t/\xi} \cdot b \cdot \tan\left(b\sqrt{2t/\xi}\right)$$

which can be rewritten as

$$\frac{a}{\beta} = \tan(\beta)$$

where $\beta = b\sqrt{2t/\xi}$ and $a = \left[y - \sqrt{2\overline{r}\xi}\right]/\sqrt{2\overline{r}\xi}$. Since tan has intersections with the hyperbola only if $a > 0$, then we have a natural affordability condition[9] for the existence of positive root on b of equation (11), namely:

$$y - \sqrt{2\overline{r}\xi} \equiv T(b) > 0$$

We denote the solution of (11) as b^*. Then $c_1^* = \left[y - \sqrt{2\overline{r}\xi}\right] / \left[\sqrt{2t/\xi} \cdot b^* \cdot \sin\left(b^*\sqrt{2t/\xi}\right)\right]$ from (10). We use this expression for c_1 in $T(x)$[10] to get $T(x; b^*)$ and then get $n(x; b^*) = \mu \cos(x\sqrt{2t/\xi})$ for $\mu = \frac{1}{\xi}\left[y - \sqrt{2\overline{r}\xi}\right] / \left[\sqrt{2t/\xi} \cdot b^* \cdot \sin\left(b^*\sqrt{2t/\xi}\right)\right]$. The integral for total population, N is then $2\mu \int_0^{b^*} \cos(x\sqrt{2t/\xi})dx$ which works out to be

$$\begin{aligned} (y - \sqrt{2\overline{r}\xi})/tb^* &\equiv T(b^*)/tb^* \\ &= N \end{aligned} \tag{12}$$

We plot the function on the left of (11) in Figure 1 for parameter values $y = 10$, $\xi = 1 = t = \overline{r}$.

---

[9] $y - T(b)$ is income available for housing and $c(b)$ at b and $\sqrt{2\overline{R}\xi}$ is the cost of achieving utility level $\overline{U}$ at b.

[10] $T(x; b^*) = y - \left\{\left[y - \sqrt{2\overline{R}\xi}\right] / \left[\sqrt{2t/\xi} \cdot b^* \cdot \sin\left(b^*\sqrt{2t/\xi}\right)\right]\right\} \cos\left(x\sqrt{2t/\xi}\right)$

It follows directly that $db/dy > 0$, $db/d\bar{r} < 0$, $db/dt < 0$, and $db/d\xi > 0$. The edge increases with income, decreases with edge rent, transportation cost and increases with the open city utility level.

Given $T(x; b^*)$ immediately above we have $r(x; b^*)$ defined in terms of x and b^*. Thus, we have functions $T(x; b^*)$, $n(x; b^*)$, $r(x; b^*)$ over $(-b^*, +b^*)$ and population N in (12), given b^* an equilibrium value for b.

Here is an example with $y = 10$, $\xi = 1 = t = \bar{r}$, which yields $b = 0.9558$ (see Fig. 1), and then $N = 8.98$. The functions $T(x), n(x)$, and $r(x)$ come out as:

$$\begin{aligned} T(x) &= 10 - 6.5 \cdot cos(x\sqrt{2}) \\ n(x) &= 6.5 \cdot cos(x\sqrt{2}) \\ r(x) &= 21.2 \cdot cos(x\sqrt{2})^2 \end{aligned}$$

$T(x)$ plots as a strictly convex function i.e. U shaped over $(-b^*, b^*)$, $n(x)$ as strictly concave (inverted U shaped) and $r(x)$ is generally bell-shaped with two points of inflection (see Fig. 2). The total transportation cost for this example is 43.002.

5 $\alpha \neq .5$

For α not 1/2 we can solve for $T(x)$ for an approximate solution by series method which is a Taylor Series expansion in the neighborhood of $x = 0$. In this case, we obtain

$$T(x) = T(0) + x^2 t(y-1)^{(1-\alpha)/\alpha}/\xi + x^4(\alpha - 1)\left(t(y-1)^{(1-\alpha)/\alpha}\right)^2/[6\xi^2\alpha(y-1)] + O(x^6) \tag{13}$$

For the case of $\alpha = .5$ this solution tracks a plot of our exact solution above well in the neighborhood of $x = 0$ (see Fig. 3, with $T(x)$ analytical (circles) and $T(x)$ in form (13)(crosses)).

Comparison of $T(x)$ for different α is on Fig. 4 ($\alpha = .5$ - solid line, $\alpha = .45$ - crosses, and $\alpha = .55$ - circles). We do not pursue further analysis with $\alpha \neq .5$ since we cannot obtain closed form solutions.

6 BEYOND THE BECKMANN MODEL

Beckmann's idea of an interactive city is wanting in at least two dimensions. First the pattern of interaction involves much duplication in the costs of visiting. Each resident travels at a cost to the "home" of every other resident once per period. One could imagine scale economies in interaction costs. Once person i was at a site for a visit, she could visit all households nearby the one she is currently at. We have explored this alternate interactive "technology" in detail elsewhere. Another alternative is having resident i visited by or do visiting with every other resident once every period. This leads to a much simpler cost of visiting than Beckmann dealt with and we draw on it directly. The second large deficiency of Beckmann's model is an explicit and meaningful motivation for resident i to be doing interacting. We counter this criticism by constructing a simple model of an interaction being productive for resident i. Hence we exploit Beckmann's idea of resident i touching base once with every other resident, but in a simpler model of interacting costs, and we extend the analysis by endogenizing the payoff to resident i from doing interacting.

7 PRODUCTIVE INTERACTIONS

We hypothesize that if resident i visits each of M other randomly chosen residents in a period, she will come up with a saleable patent worth $\widetilde{w}$ to her as income, every z periods. Though the patent pops up stochastically after two residents have coffee, we assume that only one of the two residents receives the $\widetilde{w}$. We assume that $z(M)$ is decreasing in contact number M. In a city of N residents, we will assume that resident i makes $N-1$ visits per period and she will make a successful visit or contact once every $z(N-1)$ periods. Hence her average income per period will be $w = \widetilde{w}/[z(N-1)]$. For $\widetilde{w}$ exogenous, we will have average income larger in larger cities. Larger cities will also be producing a saleable patent more often than will smaller cities.

We are assuming then that over each period, each resident of our town either visits or is visited by every other resident "for a cup of coffee". Every once in a while the coffee chat yields a saleable patent to one of the two engaged in chat. Residents are assumed to be indistinguishable from one another in their ability to produce a saleable patent. Each resident however has a distinct location, x of residence along the line (our city). The city center is at $x = 0$ and is symmetric about this point. It is the face-to-face coffee meeting that is necessary for a saleable patent to emerge. Chats over a telephone or by email are assumed not to be substitutes for a face-to-face meeting. Hence resident i incurs visiting costs over each period, her bill for total visiting costs per period being a function of her location on the line.

The resident's home consumes space and each resident desires more "home", $h(x)$ or more space on the line. There is land rent $r(x)$ associated with a unit of land or space at location x. In addition each resident desires other consumption goods, represented by scalar $c(x)$, costing \$1 per unit. Utility is then $U(c(x), h(x))$ for bundle $(c(x), h(x))$ at location x. We set out a worked example and employ a Cobb-Douglas specification of the utility function, $c(x)^{\alpha} h(x)^{1-\alpha}$. The per period budget constraint[11] for the resident

[11] For resident i, we use her expected income in a period as the income she "works with" over that period. In other words we treat resident i's income as non-stochastic each period, a clear simplification. In a fully stochastic model, the income of resident i in any period would involve an expectation over a sequence of periods.

at x is $w - V(x) = c(x) + r(x)h(x)$, where $V(x)$ is the visiting cost per period.

Consider our visiting "technology". A resident at edge b departs from her home in a period and visits every other resident, incurring cost $2tb$ where t is roundtrip cost per unit distance travelled. Recall that the city center is at $x = 0$. We postulate that cost per visit at site is zero. The resident at the other edge departs from her home in a period and visits every other resident except the resident that has just visited her from the "opposite" end. This "second resident" incurs cost, $\cong 2tb - tdb$, for db a small distance related to the person she does not need to visit. And so on for other residents as we track each resident's visiting cost closer and closer to the center at 0. The person at 0 has been visited by every other resident and thus has visiting costs, zero. A smooth approximation of the visiting cost for a resident at x is simply $2t|x|$ $\{= V(x)\}$. Visiting is then done at distinct sites, not in the center as would be the case for a CBD type city, in which the CBD was the location for face-to-face coffee pairings. More on this below.

Thus utility maximization for the resident at x has $U_h/U_c = r(x)$ and $w - 2t|x| = c(x) + r(x)h(x)$. Given our Cobb-Douglas specification, this yields $w - 2t|x| = c(x)/\alpha$. We invoke the OPEN CITY assumption: each resident faces option $\overline{U}$ if she relocates to a nearby city at zero re-location cost. Hence

$$c(x) = \left[\overline{U}/h(x)^{1-\alpha}\right]^{1/\alpha}$$

When combined with the previous equation, one has

$$1/h(x) = \mu[w - 2t|x|]^{(\alpha/(1-\alpha))}$$

where $\mu = \alpha^{(\alpha/(1-\alpha))}/\overline{U}^{(1/(1-\alpha))}$. We can now solve for total population $N(b)$ in $2\int_0^b 1/h(x)dx =$

$N(b)$. When integrated, this becomes

$$N(b) = \mu(1-\alpha)/t\{w^{1/(1-\alpha)} - [w - 2tb]^{1/(1-\alpha)}\}$$

for $w = \widetilde{w}/z(N(b)-1)$ where $\widetilde{w}$ is an exogenous parameter.

Edge b is solved for in $r(b) = \overline{r}$, for $\overline{r}$ the exogenous land rent at the city edge. The equation for $r(x)$ is derived in much the same way we obtained the equation for $1/h(x)$. This equation is

$$r(x) = (1-\alpha)\mu[w - 2t\,|x|]^{(1/(1-\alpha))}$$

which gives us

$$b = (w - \{\overline{r}/[(1-\alpha)\mu]\}^{(1-\alpha)})/2t$$

Substituting this expression into the equation for $N(b)$ we have

$$N(b) = [\mu(1-\alpha)w^{(1/(1-\alpha))} - \overline{r}]/t$$

Hence, given a form for function $z(.)$, we have a non-linear equation in $N(b)$ to solve in order to determine the equilibrium size of our city with productive and costly face-to-face visiting. Since $z(.)$ is assumed to be declining in $N(b)-1$, we can work with $z(.)$ specified as $1/(N(b)-1)^{\zeta}$ or $w = \widetilde{w}(N(b)-1)^{\zeta}$. Then the equation for $N(b)$ is

$$N(b) = \{\mu(1-\alpha)\left[\widetilde{w}(N-1)^{\zeta}\right]^{(1/(1-\alpha))} - \overline{r}\}/t$$

which gives us an explicit formula for N for some selection of values for ζ and α.

8 NUMERICAL EXAMPLES

We will consider our numerical examples for $\alpha = \zeta = 0.5$. Then we have

$$N = (\overline{r}+t)/[\mu(1-\alpha)\widetilde{w}^{(1/(1-\alpha))} - t] + 1$$

We have summarized the results for given $t = 0.01, \overline{U} = 1, \overline{r} = 1$ and $\overline{r} = 2$ in the Table 1 and Table 2.

Plots of rent $r(x)$ and density functions for $\overline{r} = 1$ are depicted on Figures 5 and 6 in circles for $\widetilde{w} = 0.25$, in dashed lines for $\widetilde{w} = 0.5$, and in solid lines for $\widetilde{w} = 1$.

For $\overline{r} = 2$ we have qualitatively the same pictures for rent and density functions.

The reader will have become aware that our city immediately above is formally a variant of the textbook monocentric city. The economics or our interactive city are very different however. Output here is a useful invention and inventions pop up stochastically from the "process" of two residents engaged in FACE-TO-FACE chat. Explicit in our formulation of productive visiting is that it happens at a residence and not at a central meeting place. Visiting costs could be lowered if, per period, every resident travelled to a central place and there, each resident sat down and had coffee once per period with every other resident. It seems entirely appropriate to view the CBD of a modern city as a central meeting place for residents and that each resident travels there to have a coffee visit, yielding a productive output, with every other resident. The productive output could be a saleable patent and it could pop out of a visit stochastically, in the way we have treated productive visiting above. A CBD city could be viewed as an interactive city with a special low cost "visiting technology", a "technology" involving every resident meeting in a central place for face-to-face interacting, period after period. We have innovated by (a) exploring alternative costs for a resident to interact and (b) by introducing a simple model of the explicit productiveness of face-to-face interactions.

9 CONCLUDING REMARK

We have brought the Beckmann model into mainstream urban economics by making use of a Cobb-Douglas utility function. We observed interesting density and rent functions for the case of a Cobb-Douglas utility function. The Beckmann model strikes us as highly inefficient since each visit by resident i to j requires a separate costly trip. We introduced a "technology" in which resident i does visiting OR is visited by every other resident, once per period. We employed this simpler visiting "technology" in a model with an explicit payoff to resident i for making a visit or being visited. We solved our new interactive city and reported some numerical examples illustrating its properties. Our interactive city is built around the notion of a productive interaction between two residents in a face-to-face setting. Inventive activity here is based on individuals discussing things with each other in a face-to-face setting. Every once in a while a chat or interaction between a pair of residents results in a saleable invention from our city.

References

[1] Beckmann, M. 1976. "Spatial Equilibrium in the Dispersed City," in a G.J. Papageorgiou (ed.), *Mathematical Land Use Theory.* Lexington: Lexington Books.

[2] Berliant, M., J. Reed, and P. Wang. 2000. "Knowledge Exchange, Matching and Agglomeration." Vanderbilt Working Papers No. 33.

[3] Borukov, E. and O. Hochman. 1977. "Optimum and Market in a Model of a City without a Predetermined Center," *Environment and Planning A,* 9, 849-856.

[4] Fujita, M. and H. Ogawa. 1982. "Multiple Equilibria and Structural Transition of Non-monocentric Urban Configurations," *Regional Science and Urban Economics,* 12, 161-169.

[5] Fujita, M. and J.F. Thisse. 2002. *Economics of Agglomeration.* New York: Cambridge University Press.

[6] Helsley, R.W. and W.C. Strange. 2005. "Social Interactions and Urban Spatial Structure." Working Paper No. 2005-03, Centre for Urban Economics and Real Estate.

[7] Imai, H. 1982. "CBD Hypothesis and Economies of Agglomeration," *Journal of Economic Theory,* 28, 275-299.

[8] Kanemoto, Y. 1990. "Optimal Cities with Indivisibility in Production and Interaction between Firms," *Journal of Urban Economics,* 27, 46-59.

[9] Lucas, R. and E. Rossi-Hansberg. 2002. "On the Internal Structure of Cities," *Econometrica,* 70, 1445-1476.

[10] Murphy, G. 1960. *Ordinary Differential Equations and Their Solutions.* Princeton: Van Nostrad Company.

[11] Tauchen, H. and A. D. Witte. 1983. “An Equilibrium Model of Office Location and Contact Patterns”, *Environment and Planning A,* 15, 1311-1326.

[12] ____1984. “Socially Optimal and Equilibrium Distributions of Office Activity: Models with Exogenous and Endogenous Contacts”, *Journal of Urban Economics,* 15, 66-68.

TABLE 1: Productive Interaction City parameters for $\overline{r} = 1$

$\widetilde{w}$	N	$z(N)$	$w(N)$	b
0.25	180.6	0.07	3.35	67.5
0.5	20.2	0.228	2.19	9.6
1	5.2	0.487	2.05	2.57

TABLE 2: Productive Interaction City parameters for $\overline{r} = 2$

$\widetilde{w}$	N	$z(N)$	$w(N)$	b
0.25	358.3	0.05	4.73	94.9
0.5	39.3	0.162	3.09	13.3
1	9.4	0.346	2.89	3.28

10 FIGURES CAPTIONS

FIGURE 1: Beckmann b equation.

FIGURE 2: Beckmann density (+) and rent (o) functions.

FIGURE 3: Taylor series (+) and analytical (o) solutions.

FIGURE 4: $T(x)$, Taylor series: .45 (+), .5 (-), .55 (o).

FIGURE 5: Productive Interaction rent functions ($\bar{r} = 1$): for $\widetilde{w} = 0.25$ - circled line; for $\widetilde{w} = 0.5$ - dashed line; for $\widetilde{w} = 1$ - solid line.

FIGURE 6: Productive Interaction density functions ($\bar{r} = 1$): for $\widetilde{w} = 0.25$ - circled line; for $\widetilde{w} = 0.5$ - dashed line; for $\widetilde{w} = 1$ - solid line.

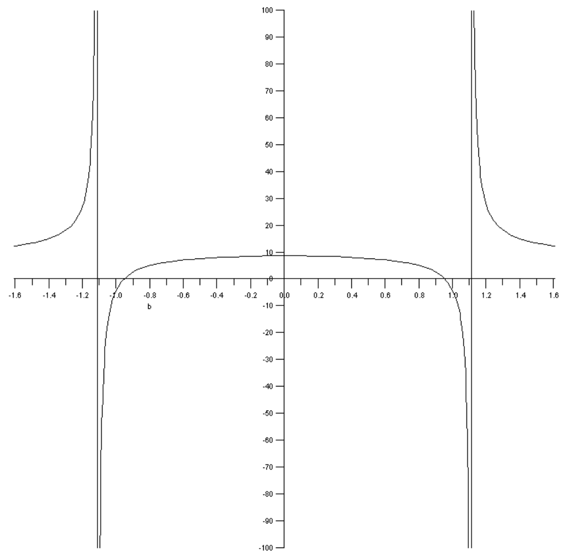

Fig. 1. Beckmann b equation.

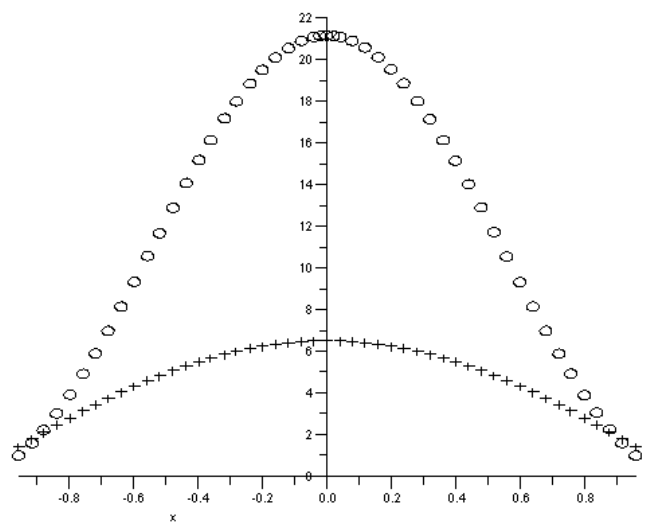

Fig. 2. Beckmann density (+) and rent (o) functions.

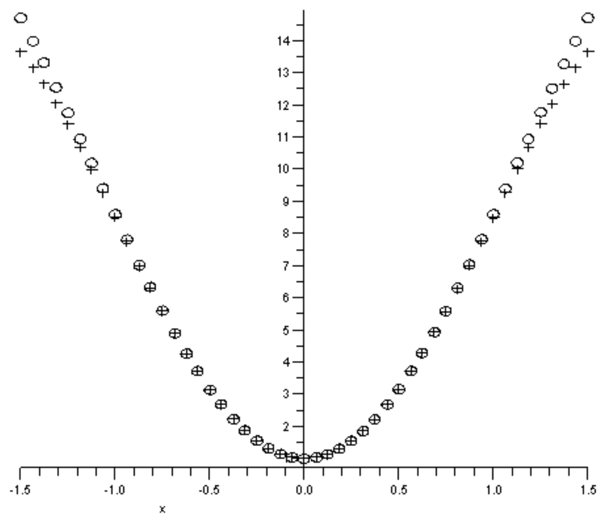

Fig. 3. T(x): Taylor series (+) and analytical (o) solutions.

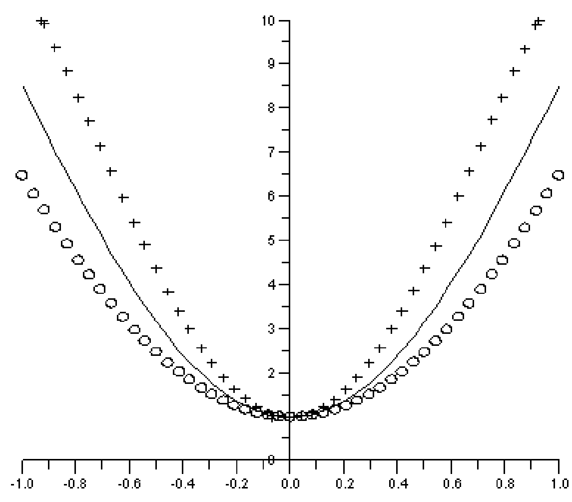

Fig. 4. T(x), Taylor series: 0.45 (+), 0.5(-), 0.55(o).

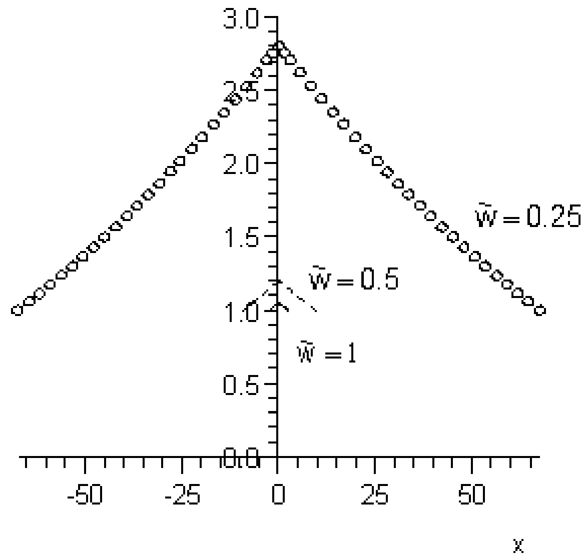

Fig. 5. Rent functions.

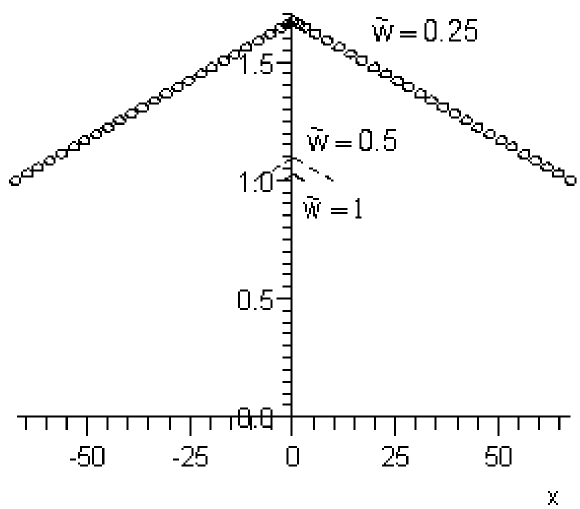

Fig. 6. Density functions.