
ECON 351* -- NOTE 8**Hypothesis Testing in the Classical Normal Linear Regression Model****1. Components of an Hypothesis Test**

1. A **testable hypothesis**, which consists of *two parts*:

Part 1: a **null hypothesis**, H_0

Part 2: an **alternative hypothesis**, H_1

2. A **feasible test statistic**.

Definition: A test statistic is a ***random variable*** whose value for given sample data determines whether the null hypothesis H_0 is rejected or not rejected.

Definition: A test statistic is ***feasible*** if it satisfies two conditions:

(1) Its **probability distribution**, or **sampling distribution**, ***must be known completely when the null hypothesis H_0 is true***, and it must have some other distribution when the null hypothesis is false.

(2) Its value can be calculated from the given ***sample data***.

3. A **decision rule** or **rejection rule**.

Definition: A decision rule ***specifies (1) a rejection region and (2) a non-rejection region for the test statistic***.

(1) ***Definition***: The **rejection region** is the ***set, or range, of values of the test statistic for which the null hypothesis H_0 is rejected*** -- i.e., that have a low probability of occurring when the null hypothesis is true.

(2) ***Definition***: The **nonrejection region** is the ***set, or range, of values of the test statistic for which the null hypothesis H_0 is not rejected, or retained***.

2. Procedure for Testing Hypotheses

Five Basic Steps

The procedure for testing hypotheses consists of *five basic steps*.

Step 1: *Formulate the null hypothesis H_0 and the alternative hypothesis H_1 .*

Step 2: *Specify the test statistic and its null distribution -- specifically its distribution when the null hypothesis H_0 is true.*

The distribution of the test statistic when the null hypothesis H_0 is true is known as the *null distribution of the test statistic*.

Step 3: *Calculate the sample value of the test statistic under the null hypothesis H_0 for the given sample data.*

Step 4: *Select the significance level α , and determine the corresponding rejection region and non-rejection region for the test statistic.*

Step 5: *Apply the decision rule and state the inference, or conclusion, implied by the sample value of the test statistic.*

We illustrate these five steps for an important class of hypothesis tests in applied econometrics -- namely **tests of equality restrictions on individual regression coefficients**.

Tests of Equality Restrictions on Individual Regression Coefficients

- These tests assess the probable empirical validity of statements or hypotheses of the following form:

$$\beta_j = \mathbf{b}_j \quad \text{where } \mathbf{b}_j \text{ is a } \textit{specified constant}. \quad (j = 1, 2)$$

- Such statements are conjectures about the population values of the regression coefficients β_j ($j = 1, 2$).

Examples

$$\beta_2 = 0 \quad \Rightarrow \quad \partial E(Y_i | X_i) / \partial X_i = 0, \text{ i.e., } X_i \text{ is unrelated to } E(Y_i | X_i)$$

$$\beta_2 = 1.0 \quad \Rightarrow \quad \partial E(Y_i | X_i) / \partial X_i = 1$$

$$\beta_2 = 0.8 \quad \Rightarrow \quad \partial E(Y_i | X_i) / \partial X_i = 0.8$$

$$\beta_2 = -1.0 \quad \Rightarrow \quad \partial E(Y_i | X_i) / \partial X_i = -1.0$$

Later we will consider more general hypotheses that take the form of **linear equality restrictions on two or more regression coefficients** β_j ($j = 1, 2$).

STEP 1: Formulation of the Null and Alternative Hypotheses

Step 1: Formulate the *null hypothesis* H_0 and the *alternative hypothesis* H_1 .

Components of a Statistical Test

A **statistical hypothesis test** consists of *two opposing statements or propositions or conjectures* about the model parameters:

1. The **null hypothesis**, denoted by H_0 .
 - H_0 is the *proposition being tested*.
 - It specifies our conjecture about the true value(s) of the regression coefficient(s).

2. The **alternative hypothesis**, denoted by H_1 .
 - H_1 is the *counter-proposition to the null hypothesis H_0* .
 - It specifies the set of alternative possibilities which is *presumed* to contain the truth if the null hypothesis is false.

Purpose of a Statistical Test

- A statistical test is designed and constructed so as **to provide sample evidence** respecting the *probable empirical validity, or truth, of the null hypothesis H_0* .
- The test addresses the question: **Are the sample estimates of the model parameters -- consistent or inconsistent (compatible or incompatible) with the truth of the null hypothesis?**

Consistency or compatibility means *sufficiently close to the value(s) specified by H_0 that we retain (do not reject) the null hypothesis*.

- A statistical test does not test the empirical validity, or truth, of the alternative hypothesis H_1 . Only the null hypothesis H_0 is being subjected to test.

Formulation of H_0 and H_1 -- Equality Restrictions on β_2

The Null Hypothesis H_0

$H_0: \beta_2 = b_2$ where b_2 is a *specified constant* (such as 0 or 0.9 or -1).

The Alternative Hypothesis H_1

For a null hypothesis of this general form, there are *three possible alternative hypotheses*.

(1) $H_1: \beta_2 \neq b_2$ a *two-sided* alternative hypothesis.

Rejection of the null hypothesis $H_0: \beta_2 = b_2$ implies that β_2 takes some other value, and that this other value is *either greater than or less than b_2* .

That is, $H_1: \beta_2 \neq b_2 \Rightarrow$ *either $\beta_2 > b_2$ or $\beta_2 < b_2$* .

(2) $H_1: \beta_2 > b_2$ a *one-sided (right-sided)* alternative hypothesis.

Rejection of the null hypothesis $H_0: \beta_2 = b_2$ in this case implies that β_2 takes some value that is *greater than b_2* .

This alternative hypothesis completely discounts the possibility that $\beta_2 < b_2$. It implies that values of β_2 *less than b_2* are considered to be logically unacceptable alternatives to the null hypothesis, an implication that presumably is based on economic theory.

(3) $H_1: \beta_2 < b_2$ a *one-sided (left-sided)* alternative hypothesis.

Rejection of the null hypothesis $H_0: \beta_2 = b_2$ in this case implies that β_2 takes some value that is *less than b_2* .

This alternative hypothesis completely excludes the possibility that $\beta_2 > b_2$. It implies that the value of β_2 could not be *greater than b_2* if in fact the null hypothesis H_0 is false.

STEP 2: Specify the Test Statistic and its Null Distribution

Step 2: Specify the *test statistic* and its *null distribution* when the null hypothesis H_0 is true.

Theoretical Prerequisite

Assumptions A1-A9 of the CNLRM -- especially the *normality assumption A9*.

Important Results

1. Under **assumptions A1-A9** of the CNLRM, the following **t-statistics** are *feasible test statistics* for the OLS coefficient estimators $\hat{\beta}_1$ and $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{\text{Var}}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{\text{se}}(\hat{\beta}_2)} = \frac{\hat{\beta}_2 - \beta_2}{\hat{\sigma}/(\sum_i x_i^2)^{1/2}} \sim t[N - 2]$$

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\text{Var}}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\text{se}}(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}(\sum_i X_i^2)^{1/2} / N^{1/2}(\sum_i x_i^2)^{1/2}} \sim t[N - 2].$$

2. For the *true (but unknown) values* of the population regression coefficients β_1 and β_2 , each of the test statistics $t(\hat{\beta}_1)$ and $t(\hat{\beta}_2)$ has the **t-distribution** with **$N - 2$ degrees of freedom**, denoted as **$t[N - 2]$** .

STEP 3: Evaluate the Test Statistic Under H_0

Step 3: Calculate the *sample value* of the test statistic under the *null hypothesis H_0* for the given sample data.

The **null hypothesis** is

$$H_0: \beta_2 = b_2 \quad \text{where } b_2 \text{ is a } \textit{specified constant} \text{ such as 0 or 0.9 or -1.}$$

From Step 2, the **feasible test statistic** is the t-statistic for the OLS estimator $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\text{s}\hat{\text{e}}(\hat{\beta}_2)} \sim t[N - 2].$$

To calculate the *sample value* of $t(\hat{\beta}_2)$ under the null hypothesis $H_0: \beta_2 = b_2$, simply **substitute** in the above formula for $t(\hat{\beta}_2)$

- **the value b_2 for β_2** , since b_2 is the value of β_2 specified by H_0 ;
- **the sample value of $\hat{\beta}_2$** , the point estimate of β_2 for the given sample data;
- **the sample value of $\text{s}\hat{\text{e}}(\hat{\beta}_2) = \sqrt{\text{Var}(\hat{\beta}_2)}$** , the estimated standard error of $\hat{\beta}_2$.

□ The *sample value* of $t(\hat{\beta}_2)$ evaluated under the null hypothesis H_0 is therefore

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - b_2}{\text{s}\hat{\text{e}}(\hat{\beta}_2)}.$$

Note: The subscript “0” on $t_0(\hat{\beta}_2)$ indicates the value of $t(\hat{\beta}_2)$ under H_0 .

□ The *null distribution* of $t_0(\hat{\beta}_2)$, the **calculated sample value** of $t(\hat{\beta}_2)$, is **$t[N-2]$, the *t-distribution* with $N-2$ degrees of freedom:**

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - b_2}{\text{s}\hat{\text{e}}(\hat{\beta}_2)} \sim t[N - 2] \quad \text{under } H_0: \beta_2 = b_2.$$

STEP 4: Determine the Rejection and Non-Rejection Regions

Step 4: Select the *significance level* α , and determine the corresponding *rejection region* and *non-rejection region* for the calculated test statistic.

Background: Type I and Type II Errors

In performing any hypothesis test -- i.e., in deciding to reject or retain a null hypothesis -- there is always some chance of making mistakes. Such mistakes arise whenever the decision to retain or reject H_0 does not reflect the true but unknown state of the world.

| <i>Decision</i> | <i>State of the World</i> | |
|---|---------------------------------------|--------------------------------------|
| | H_0 is true | H_0 is false |
| Retain (do not reject) H_0 | Correct Decision Pr = $1 - \alpha$ | Type II Error Pr = β |
| Reject H_0 | Type I Error Pr = α | Correct Decision Pr = $1 - \beta$ |

1. A *correct decision* is made if:

- the null hypothesis **H_0 is *false*** and the decision is to **reject** it.
- the null hypothesis **H_0 is *true*** and the decision is to **retain (not to reject)** it.

2. An *incorrect decision* is made if:

- the null hypothesis **H_0 is *true*** and the decision is to **reject** it (a Type I error).
- the null hypothesis **H_0 is *false*** and the decision is to **retain (not to reject)** it (a Type II error).

□ **Definitions:**

Type I error: *rejecting H_0 when H_0 is true.*

Type II error: *not rejecting H_0 when H_0 is false.*

□ **Probabilities of Type I and Type II Errors**

$\alpha \equiv \text{Pr}(\text{Type I Error}) = \text{Pr}(H_0 \text{ is rejected} \mid H_0 \text{ is true})$

$1 - \alpha = \text{Pr}(H_0 \text{ is not rejected} \mid H_0 \text{ is true}) =$ the **confidence level** of the test
= the *probability* of making a **correct decision** when the null hypothesis H_0 is *true*.

$\beta \equiv \text{Pr}(\text{Type II Error}) = \text{Pr}(H_0 \text{ is not rejected} \mid H_0 \text{ is false})$

$1 - \beta = \text{Pr}(H_0 \text{ is rejected} \mid H_0 \text{ is false}) =$ the **power** of the test
= the *probability* of making a **correct decision** when the null hypothesis H_0 is *false*.

The Significance Level of the Test

Definition: The *significance level* of the test is chosen to equal α , the **probability of making a Type I error**.

- **Significance level of the test = $\alpha \equiv \text{Pr}(\text{Type I Error})$
= $\text{Pr}(H_0 \text{ is rejected} \mid H_0 \text{ is true})$.**
- **Confidence level of the test = $1 - \alpha = \text{Pr}(H_0 \text{ is not rejected} \mid H_0 \text{ is true})$.**
- **Power of the Test**

Definition: The *power of the test* is defined to equal $1 - \beta$, the probability of making a correct decision when the null hypothesis H_0 is *false*.

$$\text{Power of the test} \equiv 1 - \beta = \text{Pr}(H_0 \text{ is rejected} \mid H_0 \text{ is false}) \\ = 1 - \text{Pr}(\text{Type II Error})$$

- **Relationship Between Type I and Type II Errors**

For any given sample size N ,

$$\alpha = \text{Pr}(\text{Type I Error}) \text{ is } \textit{inversely related} \text{ to } \text{Pr}(\text{Type II Error}) \\ \Rightarrow \alpha = \text{Pr}(\text{Type I Error}) \text{ is } \textit{directly related} \text{ to the } \text{power} \text{ of the test.}$$

Result: For any given sample size N , there exists a *trade-off* between

- (1) $\alpha = \text{Pr}(\text{Type I Error})$ = the **significance level** of the test
- and
- (2) $\beta = \text{Pr}(\text{Type II Error})$

Implications of Trade-Off Between α and β

- By choosing a *lower* significance level α -- and thereby *reducing* the **Pr(Type I Error)** -- we necessarily
 - (1) *increase* β = the **Pr(Type II Error)**, and
 - (2) *decrease* $1 - \beta$ = the *power* of the test.

- Conversely, by choosing a *higher* significance level α -- and thereby *increasing* the **Pr(Type I Error)** -- we necessarily
 - (1) *decrease* β = the **Pr(Type II Error)**, and
 - (2) *increase* $1 - \beta$ = the *power* of the test.

□ Comments on Choosing α , the Significance Level of the Test

- The value of the significance level α is usually chosen to be small. In practice, the values most frequently chosen are:
 - $\alpha = 0.01$ (a 1% significance level);
 - $\alpha = 0.05$ (a 5% significance level);
 - $\alpha = 0.10$ (a 10% significance level).

- But the choice of value for α is more or less arbitrary. It presumably reflects the investigator's relative willingness to accept, or relative aversion to, Type I and Type II errors.

Rejection and Non-Rejection Regions for a Test Statistic

Definition of the Rejection and Non-Rejection Regions

- The **rejection region** is the *set, or range, of values of the test statistic* for which the null hypothesis H_0 is *rejected*.

Values of the test statistic in the rejection region have a low probability of occurring when the null hypothesis H_0 is true.

- The **nonrejection region** is the *set, or range, of values of the test statistic* for which the null hypothesis H_0 is *not rejected, or retained*.

Distinguishing Between the Rejection and Nonrejection Regions

Question: How are the rejection and nonrejection regions delineated or separated?

Answer: By the **critical values of the test statistic** -- or more correctly, by the **critical values of the null distribution of the test statistic**.

Critical Values of a Test Statistic

Definition: The *critical values* of a test statistic are defined as those values

that separate the *rejection region* from the *non-rejection region*,

that partition the *sample values* of a test statistic into a *rejection region* and a *non-rejection region*.

Determinants

The *critical values* of a test statistic are determined by the following factors:

1. the *null distribution of the test statistic* -- the probability, or sampling, distribution of the test statistic when the null hypothesis H_0 is true;
2. the chosen *significance level for the test*, α ;
3. the *nature of the hypothesis test*, specifically whether the test is
 - (1) a *two-tail, or two-sided, test*
or
 - (2) a *one-tail, or one-sided, test*, of which there are two types,
 - (2.1) a *left-tail test*
 - (2.2) a *right-tail test*.

Two-Tail and One-Tail Tests: Which is it?

Important Point: Whether a two-tail or one-tail test is appropriate depends on the nature of the *alternative hypothesis* H_1 .

Definitions

- A **two-tail test** is one for which the alternative hypothesis H_1 is a ***two-sided hypothesis*** that incorporates the “not equal” condition “ \neq ”.

Example: $H_0: \beta_2 = b_2$ where b_2 is a specified constant
 $H_1: \beta_2 \neq b_2$ is a ***two-sided alternative hypothesis***
 \Rightarrow a ***two-tail test***
 \Rightarrow a ***two-tail rejection region***.

- A **one-tail test** is one for which the alternative hypothesis H_1 is a ***one-sided hypothesis*** that incorporates either the “less than” condition “ $<$ ” or the “greater than” condition “ $>$ ”.

- ***Case 1:*** A **left-tail test** is one in which H_1 incorporates the “less than” condition “ $<$ ”.

Example: $H_0: \beta_2 = b_2$ where b_2 is a specified constant
 $H_1: \beta_2 < b_2$ is a ***left-sided alternative hypothesis***
 \Rightarrow a ***one-tail test***, specifically a ***left-tail test***.
 \Rightarrow a ***left-tail rejection region***.

- ***Case 2:*** A **right-tail test** is one in which H_1 incorporates the “greater than” condition “ $>$ ”.

Example: $H_0: \beta_2 = b_2$ where b_2 is a specified constant
 $H_1: \beta_2 > b_2$ is a ***right-sided alternative hypothesis***
 \Rightarrow a ***one-tail test***, specifically a ***right-tail test***.
 \Rightarrow a ***right-tail rejection region***.

Note: The form of the alternative hypothesis H_1 -- specifically the *direction of the inequality* in H_1 -- determines whether a one-tail test is a right-tail test or a left-tail test.

Determining Critical Values for Two-Tail Tests

Problem: Determine the critical values for the following **two-tail t-test**:

$H_0: \beta_2 = b_2$ or $\beta_2 - b_2 = 0$ where b_2 is a specified constant

$H_1: \beta_2 \neq b_2$ or $\beta_2 - b_2 \neq 0 \iff$ a **two-sided alternative hypothesis**.

1. The **appropriate test statistic** is the **t-statistic for $\hat{\beta}_2$** , the OLS estimate of the slope coefficient β_2 :

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\text{s}\hat{e}(\hat{\beta}_2)}.$$

2. The **sample value of the test statistic $t(\hat{\beta}_2)$** is calculated by evaluating $t(\hat{\beta}_2)$ under the null hypothesis H_0 . That is, in the expression for $t(\hat{\beta}_2)$, set β_2 equal to b_2 , which is the value of β_2 specified by H_0 . The resulting sample value of the test statistic $t_0(\hat{\beta}_2)$ under H_0 is

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\text{s}\hat{e}(\hat{\beta}_2)}.$$

3. The **null distribution of $t_0(\hat{\beta}_2)$** is $t[N - 2]$, the t-distribution with $N - 2$ degrees of freedom. In other words, if the null hypothesis $H_0: \beta_2 = b_2$ is true, then

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\text{s}\hat{e}(\hat{\beta}_2)} \sim t[N - 2] \text{ under } H_0.$$

Question 1: What values of $\hat{\beta}_2$ and $t_0(\hat{\beta}_2)$ would lead us to reject H_0 against H_1 ?

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\widehat{se}(\hat{\beta}_2)}.$$

Remember:

$\hat{\beta}_2$ = the *estimated value of* β_2

b_2 = the *hypothesized value of* β_2

$\widehat{se}(\hat{\beta}_2) > 0$ ($\widehat{se}(\hat{\beta}_2)$ is *always a positive number*)

- We would reject H_0 against H_1 if $\hat{\beta}_2$, the estimated value of β_2 , is very different from b_2 , the hypothesized value of β_2 .
- More specifically, we would **reject H_0 against H_1** in either of the following two cases:

$$1. \hat{\beta}_2 \gg b_2 \quad \Rightarrow \quad \hat{\beta}_2 - b_2 \gg 0 \quad \Rightarrow \quad t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\widehat{se}(\hat{\beta}_2)} \gg 0$$

values of $\hat{\beta}_2$ much greater than b_2 imply **large positive values** of $t_0(\hat{\beta}_2)$;

or

$$2. \hat{\beta}_2 \ll b_2 \quad \Rightarrow \quad \hat{\beta}_2 - b_2 \ll 0 \quad \Rightarrow \quad t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\widehat{se}(\hat{\beta}_2)} \ll 0$$

values of $\hat{\beta}_2$ much less than b_2 imply **large negative values** of $t_0(\hat{\beta}_2)$.

Question 2: How much larger or smaller than zero does the value of $t_0(\hat{\beta}_2)$ have to be for us to reject H_0 in favour of H_1 ?

Answer: It depends on **the significance level we choose** for the test and **the null distribution of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
 = the probability of making a Type I error.

Allocate α equally between large positive values of t_0 and large negative values of t_0 . We therefore have both an upper critical value and a lower critical value of the null distribution of our test statistic, which is the **t[N – 2]-distribution**.

$t_{\alpha/2}[N - 2]$ = the **upper $\alpha/2$ critical value** of the **t[N – 2]-distribution**;
 $-t_{\alpha/2}[N - 2]$ = the **lower $\alpha/2$ critical value** of the **t[N – 2]-distribution**.

Implications: If $H_0: \beta_2 = b_2$ is **true**, then the following two probability statements hold.

$$(1) \quad \Pr\left(-t_{\alpha/2}[N - 2] \leq t_0(\hat{\beta}_2) \leq t_{\alpha/2}[N - 2]\right) = 1 - \alpha \quad (1)$$

$$(2) \quad \Pr\left(t_0(\hat{\beta}_2) < -t_{\alpha/2}[N - 2] \quad \text{or} \quad t_0(\hat{\beta}_2) > t_{\alpha/2}[N - 2]\right) \\ = \Pr\left(\left|t_0(\hat{\beta}_2)\right| > t_{\alpha/2}[N - 2]\right) = \alpha \quad (2)$$

where

$t_0(\hat{\beta}_2)$ = the **calculated sample value of the t-statistic** under H_0 ;
 $\left|t_0(\hat{\beta}_2)\right|$ = the **absolute value of $t_0(\hat{\beta}_2)$** ;
 $t_{\alpha/2}[N - 2]$ = the **upper $\alpha/2$ critical value** of the **t[N – 2]-distribution**;
 $-t_{\alpha/2}[N - 2]$ = the **lower $\alpha/2$ critical value** of the **t[N – 2]-distribution**;
 α = the **significance level** for the test;
 $1 - \alpha$ = the **confidence level** for the test.

Determining the Rejection and Nonrejection Regions -- Two-Tail Test

- The **non-rejection region** for $t_0(\hat{\beta}_2)$ is defined by the double inequality in probability statement (1) above. It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$-t_{\alpha/2}[N-2] \leq t_0(\hat{\beta}_2) \leq t_{\alpha/2}[N-2] \Leftrightarrow \text{non-rejection region for } H_0: \beta_2 = b_2.$$

- The **rejection region** for $t_0(\hat{\beta}_2)$ is the *two-sided region* or *two-tail region* defined in probability statement (2) above. It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$t_0(\hat{\beta}_2) < -t_{\alpha/2}[N-2] \quad \text{or} \quad t_0(\hat{\beta}_2) > t_{\alpha/2}[N-2]$$

or

$$|t_0(\hat{\beta}_2)| > t_{\alpha/2}[N-2] \quad \Leftrightarrow \text{rejection region for } H_0: \beta_2 = b_2$$

is a *two-tail rejection region*.

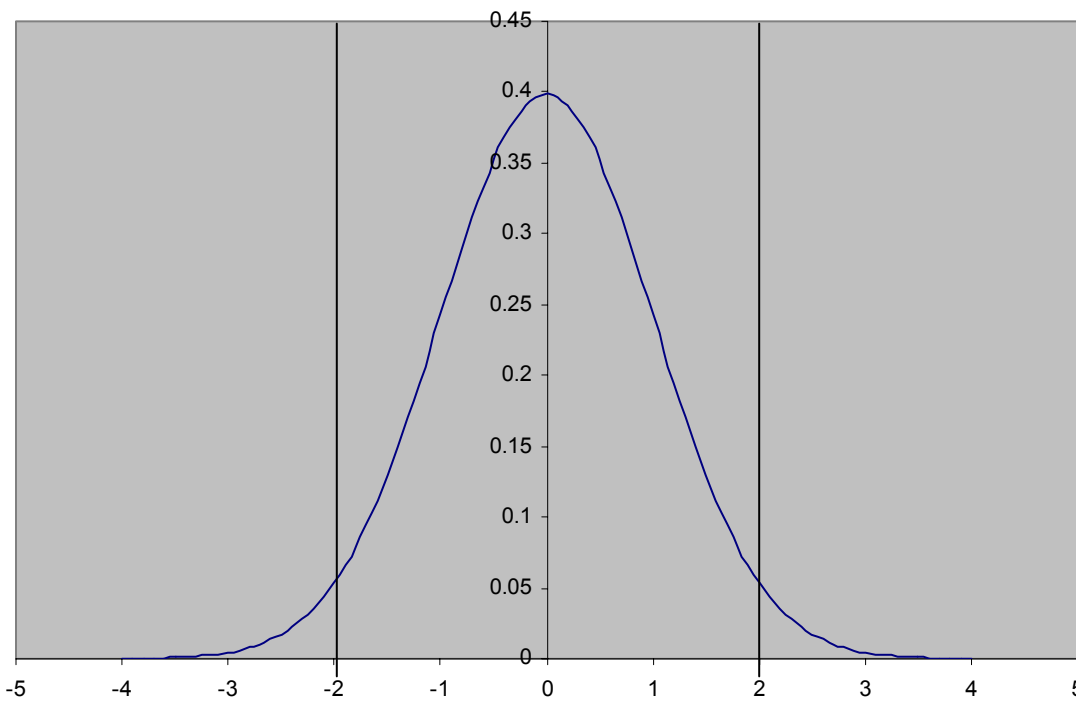
NOTE: For a *two-tail test*, the **rejection region is a two-tail rejection region** consisting of **two parts**.

- 1) The ***lower or left-tail rejection region*** $t_0(\hat{\beta}_2) < -t_{\alpha/2}[N-2]$, which contains **unexpectedly small values of $t_0(\hat{\beta}_2)$** under H_0 -- i.e., values that we would only expect to obtain with “small” probability $\alpha/2$ if the null hypothesis $H_0: \beta_2 = b_2$ is true.
- 2) The ***upper or right-tail rejection region*** $t_0(\hat{\beta}_2) > t_{\alpha/2}[N-2]$, which contains **unexpectedly large values of $t_0(\hat{\beta}_2)$** under H_0 -- i.e., values that we would only expect to obtain with “small” probability $\alpha/2$ if the null hypothesis $H_0: \beta_2 = b_2$ is true.

- 3) The **rejection region** for a *two-tail test* thus consists of **both the lower left-hand tail and the upper right-hand tail** of the $t[N - 2]$ -distribution.
- The area in each tail under the $t[N - 2]$ -distribution is $\alpha/2$.
 - The sum of these two tail area probabilities equals the significance level α .
 - Thus, for a two-tail test, the significance level α is **allocated equally between the lower $\alpha/2$ rejection region and the upper $\alpha/2$ rejection region**.

Rejection and Nonrejection Regions for a Two-Tail Test

| | | | | |
|-------------------------------|---|---------------------|---|--------------------------------|
| left-tail rejection region | ↓ | nonrejection region | ↓ | right-tail rejection region |
| area = $\alpha/2$ | ↓ | area = $1 - \alpha$ | ↓ | area = $\alpha/2$ |



| | | | | |
|--|---|---|---|--------------------------------------|
| $-\ t_{\alpha/2}$ | ↑ | $t_{\alpha/2}$ | ↑ | |
| $\Pr(t_0 < - t_{\alpha/2}) = \alpha/2$ | ↑ | $\Pr(- t_{\alpha/2} \leq t_0 \leq t_{\alpha/2}) = 1 - \alpha$ | ↑ | $\Pr(t_0 > t_{\alpha/2}) = \alpha/2$ |

STEP 5: Apply the Decision Rule and State Inference

Step 5: Apply the *decision rule* of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Two-Tail Test

Formulation 1: Determine if the *sample value* t_0 of the test statistic lies in the *rejection or nonrejection region* at the chosen significance level α .

Decision Rule for a Two-Tail Test -- Formulation 1

1. If the *sample value* t_0 of the test statistic lies in the *rejection region* at the chosen significance level, then **reject the null hypothesis H_0** .

For a **two-tail** test: $|t_0| > t_{\alpha/2}[N-2] \Rightarrow$ **reject H_0 at significance level α** .

Reject H_0 in favour of H_1 at significance level α if

(1) $t_0 > t_{\alpha/2}[N-2]$ meaning t_0 lies in the *upper tail* rejection area;

or

(2) $t_0 < -t_{\alpha/2}[N-2]$ meaning t_0 lies in the *lower tail* rejection area.

Inference: Reject H_0 in favour of H_1 at significance level α .

2. If the *sample value* t_0 of the test statistic lies in the *nonrejection region* at the chosen significance level, then **retain (do not reject) the null hypothesis H_0** .

For a **two-tail** test: $|t_0| \leq t_{\alpha/2}[N-2] \Rightarrow$ **retain H_0 at significance level α** .

Retain H_0 against H_1 at significance level α if

$-t_{\alpha/2}[N-2] \leq t_0 \leq t_{\alpha/2}[N-2]$ meaning t_0 lies in the *nonrejection area*.

Inference: Retain H_0 against H_1 at significance level α .

Examples of Two-Tail Hypothesis Tests

The Model:

DATA: `auto1.dta` (a Stata-format data file)

MODEL: $\text{price}_i = \beta_1 + \beta_2 \text{weight}_i + u_i \quad (i = 1, \dots, N)$

```
. regress price weight
```

| Source | SS | df | MS | | | |
|----------|-----------|----|------------|-----------------|--------|--|
| Model | 184233937 | 1 | 184233937 | Number of obs = | 74 | |
| Residual | 450831459 | 72 | 6261548.04 | F(1, 72) = | 29.42 | |
| Total | 635065396 | 73 | 8699525.97 | Prob > F = | 0.0000 | |
| | | | | R-squared = | 0.2901 | |
| | | | | Adj R-squared = | 0.2802 | |
| | | | | Root MSE = | 2502.3 | |

| price | Coef. | Std. Err. | t | P> t | [95% Conf. Interval] | |
|--------|-----------------|-----------------|--------------|-------|----------------------|----------|
| weight | 2.044063 | .3768341 | 5.424 | 0.000 | 1.292858 | 2.795268 |
| _cons | -6.707353 | 1174.43 | -0.006 | 0.995 | -2347.89 | 2334.475 |

$$N = 74 \quad N - 2 = 74 - 2 = \mathbf{72}$$

$$\hat{\beta}_2 = \mathbf{2.0441} \quad s\hat{e}(\hat{\beta}_2) = \mathbf{0.376834}$$

$$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$$

$$t_{\alpha/2}[N - 2] = t_{0.025}[72] = \mathbf{1.9935}$$

Test 1: Test the proposition that weight_i is unrelated to price_i at the 5 percent significance level ($\alpha = 0.05$).

- **Null and Alternative Hypotheses**

$$H_0: \beta_2 = 0$$

$$H_1: \beta_2 \neq 0 \quad \text{a } \underline{\text{two-sided}} \text{ alternative hypothesis.}$$

- The *feasible test statistic* is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} \sim t[N - 2].$$

- **Compute the sample value of $t(\hat{\beta}_2)$** under the null hypothesis H_0

Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\hat{s}e(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The *two-tail critical value* of the $t[N - 2]$ distribution at the 5 percent significance level (at $\alpha = 0.05$) is $t_{\alpha/2}[N - 2] = t_{0.025}[72] = \mathbf{1.9935}$.

- **Decision Rule:**

$$\text{If } |t_0| > t_{\alpha/2}[N - 2] \quad \text{reject } H_0 \text{ at significance level } \alpha;$$

$$\text{If } |t_0| \leq t_{\alpha/2}[N - 2] \quad \text{retain } H_0 \text{ at significance level } \alpha.$$

- **Inference:**

$$|t_0| = 5.424 > 1.9935 = t_{0.025}[72] \Rightarrow \text{reject } H_0 \text{ at significance level } \alpha = 0.05$$

Reject $H_0: \beta_2 = 0$ in favour of **$H_1: \beta_2 \neq 0$** at the **5 percent** significance level.

Test 2: Test the proposition that a **1-pound increase in weight_i** is associated with an **increase in average price_i of 1 dollar**. Use the 5 percent significance level ($\alpha = .05$).

- ***Null and Alternative Hypotheses***

$$H_0: \beta_2 = 1$$

$$H_1: \beta_2 \neq 1 \quad \text{a } \underline{\text{two-sided}} \text{ alternative hypothesis.}$$

- The ***feasible test statistic*** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\widehat{\text{Var}}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} \sim t[N - 2].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H_0

Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 1$ and $\widehat{\text{se}}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} = \frac{2.0441 - 1}{0.376834} = \frac{1.0441}{0.376834} = \mathbf{2.771}$$

- The ***two-tail critical value* of the $t[N - 2]$ distribution at the 5 percent significance level (at $\alpha = 0.05$)** is $t_{\alpha/2}[N - 2] = t_{0.025}[72] = \mathbf{1.9935}$.

- ***Decision Rule:***

If $|t_0| > t_{\alpha/2}[N - 2]$ **reject H_0 at significance level α** ;

If $|t_0| \leq t_{\alpha/2}[N - 2]$ **retain H_0 at significance level α** .

- ***Inference:***

$|t_0| = 2.771 > 1.9935 = t_{0.025}[72] \Rightarrow$ **reject H_0 at significance level $\alpha = 0.05$**

Reject $H_0: \beta_2 = 1$ in favour of $H_1: \beta_2 \neq 1$ at the 5 percent significance level.

Test 3: Test the proposition that a **1-pound increase in weight_i** is associated with an **increase in average price_i of 2 dollars**. Perform the test at the 5 percent significance level ($\alpha = .05$).

- ***Null and Alternative Hypotheses***

$$H_0: \beta_2 = 2$$

$$H_1: \beta_2 \neq 2 \quad \text{a } \underline{\text{two-sided}} \text{ alternative hypothesis.}$$

- The ***feasible test statistic*** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} \sim t[N - 2].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H_0

Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 2$ and $\hat{s}e(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} = \frac{2.0441 - 2}{0.376834} = \frac{0.0441}{0.376834} = \mathbf{0.1170}$$

- The ***two-tail critical value* of the $t[N - 2]$ distribution at the 5 percent significance level** (at $\alpha = 0.05$) is $t_{\alpha/2}[N - 2] = t_{0.025}[72] = \mathbf{1.9935}$.

- ***Decision Rule:***

If $|t_0| > t_{\alpha/2}[N - 2]$ **reject H_0 at significance level α ;**

If $|t_0| \leq t_{\alpha/2}[N - 2]$ **retain H_0 at significance level α .**

- ***Inference:***

$|t_0| = 0.1170 < 1.9935 = t_{0.025}[72] \Rightarrow$ **retain H_0 at significance level $\alpha = 0.05$**

Retain $H_0: \beta_2 = 2$ against $H_1: \beta_2 \neq 2$ at the 5 percent significance level.

How to perform all three of these two-tail hypothesis tests at once

Test 1: $H_0: \beta_2 = 0$ versus $H_1: \beta_2 \neq 0$

Test 2: $H_0: \beta_2 = 1$ versus $H_1: \beta_2 \neq 1$

Test 3: $H_0: \beta_2 = 2$ versus $H_1: \beta_2 \neq 2$

Compute the *two-sided 95 percent confidence interval* for β_2 .

$$\hat{\beta}_{2U} = \hat{\beta}_2 + t_{\alpha/2}[N-2]s\hat{e}(\hat{\beta}_2) = 2.0441 + 0.75121 = 2.79531 = \underline{\underline{2.795}}$$

$$\hat{\beta}_{2L} = \hat{\beta}_2 - t_{\alpha/2}[N-2]s\hat{e}(\hat{\beta}_2) = 2.0441 - 0.75121 = 1.29289 = \underline{\underline{1.293}}$$

Result: The two-sided 95% confidence interval for β_2 is [1.293, 2.795].

Decision Rule:

- If the *hypothesized value* of β_2 lies *outside* the two-sided 95 percent confidence interval for β_2 , *reject* the null hypothesis H_0 at the 5 percent significance level.
- If the *hypothesized value* of β_2 lies *inside* the two-sided 95 percent confidence interval for β_2 , *retain* the null hypothesis H_0 at the 5 percent significance level.

Test 1: Since the value 0 lies *outside* the two-sided 95 percent confidence interval for β_2 , *reject* $H_0: \beta_2 = 0$ in favour of $H_1: \beta_2 \neq 0$ at the 5 percent significance level.

Test 2: Since the value 1 lies *outside* the two-sided 95 percent confidence interval for β_2 , *reject* $H_0: \beta_2 = 1$ in favour of $H_1: \beta_2 \neq 1$ at the 5 percent significance level.

Test 3: Since the value 2 lies *inside* the two-sided 95 percent confidence interval for β_2 , *retain* $H_0: \beta_2 = 2$ against $H_1: \beta_2 \neq 2$ at the 5 percent significance level.

Determining Critical Values for One-Tail Tests

CASE 1 -- A Left-Tail Test

Problem: Determine the critical values for the following **one-tail t-test**:

$$\begin{aligned}
 H_0: \beta_2 = b_2 \text{ or } \beta_2 - b_2 = 0 & \quad \text{where } b_2 \text{ is a specified constant} \\
 H_1: \beta_2 < b_2 \text{ or } \beta_2 - b_2 < 0 & \quad \Leftarrow \text{ a } \textit{left-sided alternative hypothesis}. \\
 & \quad \Rightarrow \text{ a } \textit{one-tail test}, \text{ specifically a } \textit{left-tail test}.
 \end{aligned}$$

1. The **appropriate test statistic** is the **t-statistic for $\hat{\beta}_2$** , the OLS estimate of the slope coefficient β_2 :

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{s\hat{e}(\hat{\beta}_2)}.$$

2. The **sample value of the test statistic $t(\hat{\beta}_2)$** is calculated by evaluating $t(\hat{\beta}_2)$ **under the null hypothesis H_0** . This involves evaluating $t(\hat{\beta}_2)$ using the **equality form of the null hypothesis H_0** , which is $\beta_2 = b_2$. Setting $\beta_2 = b_2$ in the expression for $t(\hat{\beta}_2)$ yields the sample value of the test statistic $t(\hat{\beta}_2)$ under H_0 :

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{s\hat{e}(\hat{\beta}_2)}.$$

Note: The **null hypothesis for a left-tail test** is sometimes written as $H_0: \beta_2 \geq b_2$ rather than $H_0: \beta_2 = b_2$. But the computational procedure for testing

$$H_0: \beta_2 \geq b_2 \text{ against } H_1: \beta_2 < b_2$$

is **exactly the same** as the procedure for testing

$$H_0: \beta_2 = b_2 \text{ against } H_1: \beta_2 < b_2.$$

Question: Why do we use the equality form of the null hypothesis, $\beta_2 = b_2$, to calculate the sample value of the test statistic?

Answer: A two-part answer:

- A test that takes the null hypothesis as $H_0: \beta_2 = b_2$ is the most favorable to the null hypothesis, and hence is the least favorable to the alternative hypothesis $H_1: \beta_2 < b_2$. This means that, at any chosen significance level α , if we reject $H_0: \beta_2 = b_2$ in favor of the alternative hypothesis $H_1: \beta_2 < b_2$, then we would also reject $H_0: \beta_2 = b_2 + c$ in favor of $H_1: \beta_2 < b_2 + c$, where $c > 0$ is any *positive* constant.
 - Calculating the value of $t(\hat{\beta}_2)$ for *all* values of $\beta_2 > b_2$ would be extremely tedious!! (How would you know when you're done?) Moreover, its unnecessary.
3. The **null distribution of $t_0(\hat{\beta}_2)$ is $t[N - 2]$** , the t-distribution with $N - 2$ degrees of freedom. In other words, if the null hypothesis H_0 is true -- i.e., if $\beta_2 = b_2$, then

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{s\hat{e}(\hat{\beta}_2)} \sim t[N - 2] \text{ under } H_0.$$

Question 1: What values of $\hat{\beta}_2$ and $t_0(\hat{\beta}_2)$ would lead us to reject H_0 against H_1 ?

$$H_0: \beta_2 = b_2 \text{ or } \beta_2 - b_2 = 0$$

$$H_1: \beta_2 < b_2 \text{ or } \beta_2 - b_2 < 0 \quad \Leftarrow \text{ a *left-sided alternative hypothesis*.}$$

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\text{se}(\hat{\beta}_2)}.$$

Remember:

$\hat{\beta}_2$ = the *estimated value* of β_2

b_2 = the *hypothesized value* of β_2

$\text{se}(\hat{\beta}_2) > 0$ (se($\hat{\beta}_2$) is *always a positive number*)

- We would reject H_0 against H_1 if $\hat{\beta}_2$ is much less than b_2 , if **the *estimated value* of β_2 is much less than b_2 , the *hypothesized value* of β_2 .**
- More specifically, we would **reject H_0 against H_1** in the following case:

$$\hat{\beta}_2 \ll b_2 \quad \Rightarrow \quad \hat{\beta}_2 - b_2 \ll 0 \quad \Rightarrow \quad t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\text{se}(\hat{\beta}_2)} \ll 0$$

Values of $\hat{\beta}_2$ *much less than* b_2 imply **large *negative* values** of $t_0(\hat{\beta}_2)$.

Question 2: How much less than zero does the value of $t_0(\hat{\beta}_2)$ have to be for us to reject H_0 in favour of H_1 ?

Answer: It depends on the **significance level** we choose for the test and the **null distribution of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
 = the probability of making a Type I error.

Because only large negative values of t_0 favour the alternative hypothesis, there is only one critical value – a lower critical value of the $t[N - 2]$ -distribution that delineates a lower or left-tail rejection area equal to α .

$-t_{\alpha}[N - 2]$ = the **lower α critical value** of the $t[N - 2]$ -distribution.

Implications: If $H_0: \beta_2 = b_2$ is **true**, then the following two probability statements hold.

$$(1) \quad \Pr\left(t_0(\hat{\beta}_2) \geq -t_{\alpha}[N - 2]\right) = 1 - \alpha \quad (1)$$

$$(2) \quad \Pr\left(t_0(\hat{\beta}_2) < -t_{\alpha}[N - 2]\right) = \alpha \quad (2)$$

where

$t_0(\hat{\beta}_2)$ = the **calculated sample value of the t-statistic** under H_0 ;
 $-t_{\alpha}[N - 2]$ = the **lower α -level critical value** of the $t[N - 2]$ -distribution;
 α = the **significance level** for the test;
 $1 - \alpha$ = the **confidence level** for the test.

Determine the Rejection and Nonrejection Regions -- Left-Tail Test

- The ***non-rejection region*** for $t_0(\hat{\beta}_2)$ is the set of values defined by the inequality in probability statement (1) above:

$$(1) \quad \Pr\left(t_0(\hat{\beta}_2) \geq -t_\alpha[N-2]\right) = 1 - \alpha \quad (1)$$

It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$t_0(\hat{\beta}_2) \geq -t_\alpha[N-2] \quad \Leftarrow \text{non-rejection region under } H_0: \beta_2 = b_2.$$

- The ***rejection region*** for $t_0(\hat{\beta}_2)$ is the set of values defined by the inequality in probability statement (2) above:

$$(2) \quad \Pr\left(t_0(\hat{\beta}_2) < -t_\alpha[N-2]\right) = \alpha \quad (2)$$

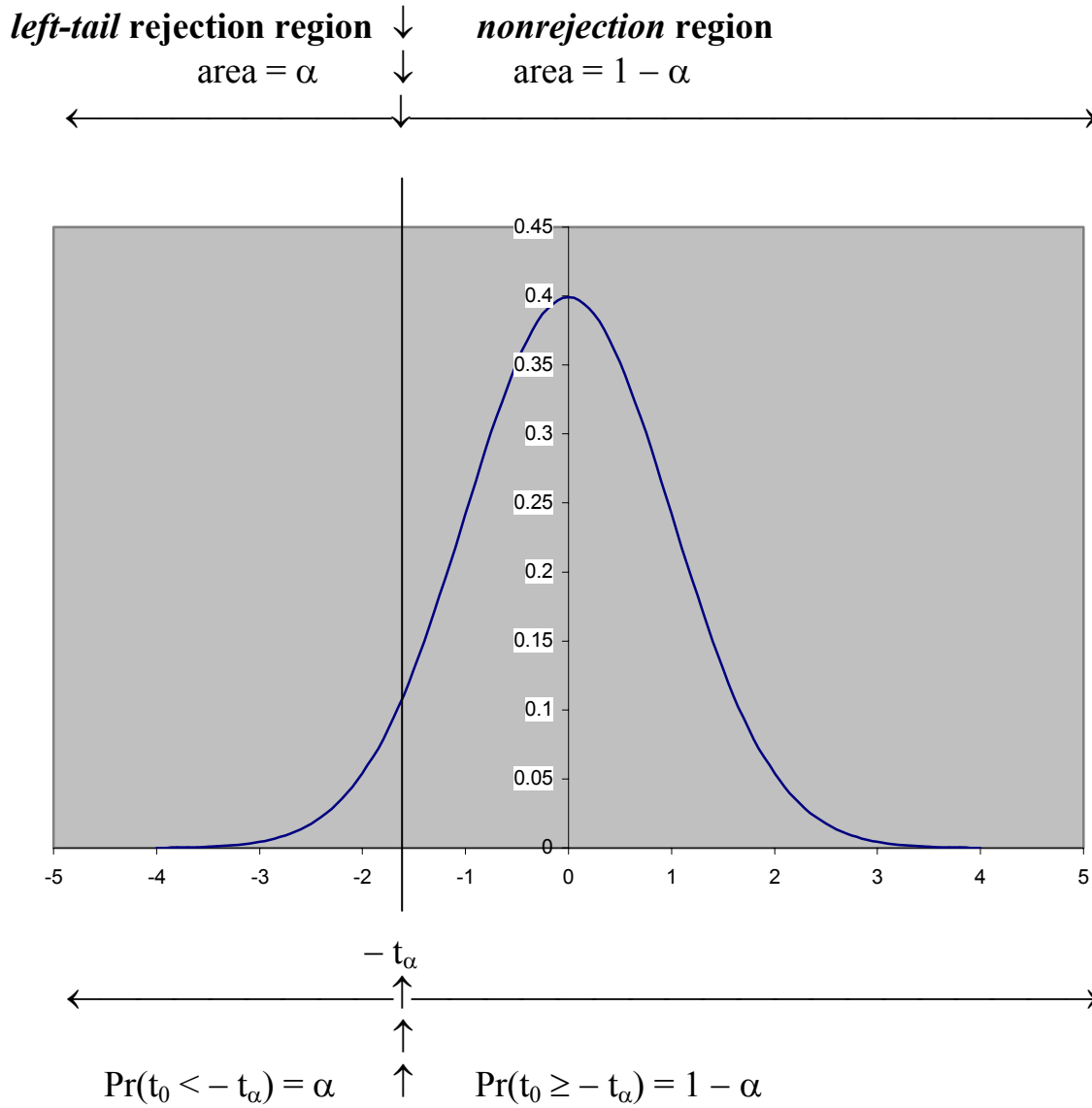
It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$t_0(\hat{\beta}_2) < -t_\alpha[N-2] \quad \Leftarrow \text{rejection region under } H_0: \beta_2 = b_2 \\ \text{is a one-tail } \underline{\text{left-tail}} \text{ rejection region.}$$

NOTE:

- 1) For a ***left-tail test***, the rejection region $t_0(\hat{\beta}_2) < -t_\alpha[N-2]$ consists only of the ***lower left-hand tail*** of the t-distribution with $N - 2$ degrees of freedom.
- 2) This ***left-tail rejection region*** contains ***unexpectedly small values*** of $t_0(\hat{\beta}_2)$ under H_0 -- i.e., values that we would only expect to obtain with “small” probability α if the null hypothesis $H_0: \beta_2 = b_2$ or $\beta_2 = b_2$ is true.
- 3) The ***left tail area under the t[N - 2]-distribution*** in this lower tail ***equals the significance level α*** . This area is called the ***lower α -level (or lower 100 α percent) tail area*** of the t[N - 2]-distribution.

Rejection and Nonrejection Regions for a Left-Tail Test



STEP 5: Apply the Decision Rule and State Inference

Step 5: Apply the *decision rule* of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Left-Tail Test

Formulation 1: Determine if the *sample value* t_0 of the test statistic lies in the *rejection or nonrejection region* at the chosen significance level α .

Decision Rule for a Left-Tail Test -- Formulation 1

1. If the *sample value* t_0 of the test statistic lies in the *left-tail rejection region* at the chosen significance level, then **reject the null hypothesis H_0** .

For a **left-tail** test: $t_0 < -t_{\alpha}[N-2] \Rightarrow$ **reject H_0 at significance level α** .

Reject H_0 in favour of H_1 at significance level α if

$t_0 < -t_{\alpha}[N-2]$ meaning t_0 lies in the *lower left-tail rejection area*.

Inference: Reject H_0 in favour of H_1 at significance level α .

2. If the *sample value* t_0 of the test statistic lies in the *nonrejection region* at the chosen significance level, then **retain (do not reject) the null hypothesis H_0** .

For a **left-tail** test: $t_0 \geq -t_{\alpha}[N-2] \Rightarrow$ **retain H_0 at significance level α** .

Retain H_0 against H_1 at significance level α if

$t_0 \geq -t_{\alpha}[N-2]$ meaning t_0 lies in the *nonrejection area*.

Inference: Retain H_0 against H_1 at significance level α .

CASE 2 -- A Right-Tail Test

Problem: Determine the critical values for the following **one-tail t-test**:

$H_0: \beta_2 = b_2$ or $\beta_2 - b_2 = 0$ where b_2 is a specified constant

$H_1: \beta_2 > b_2$ or $\beta_2 - b_2 > 0 \iff$ a **right-sided alternative hypothesis**.
 \implies a **one-tail test**, specifically a **right-tail test**.

NOTE: This one-tail test is a **right-tail test** because, as we will see, the **rejection region** for the test consists of the **upper right-hand tail** of the appropriate t-distribution.

1. Again, the **appropriate test statistic** is the **t-statistic for $\hat{\beta}_2$** , the OLS estimate of the slope coefficient β_2 :

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{\text{Var}}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{\text{se}}(\hat{\beta}_2)}.$$

2. The **sample value of the test statistic $t(\hat{\beta}_2)$** is calculated by evaluating $t(\hat{\beta}_2)$ **under the null hypothesis H_0** . This involves evaluating $t(\hat{\beta}_2)$ using the **equality form of the null hypothesis H_0** , which is $\beta_2 = b_2$. Setting $\beta_2 = b_2$ in the expression for $t(\hat{\beta}_2)$ yields the sample value of the test statistic $t(\hat{\beta}_2)$ under H_0 :

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\hat{\text{se}}(\hat{\beta}_2)}.$$

Note: The **null hypothesis for a right-tail test** can be written as $H_0: \beta_2 \leq b_2$ rather than $H_0: \beta_2 = b_2$. But the computational procedure for testing

$$H_0: \beta_2 \leq b_2 \text{ against } H_1: \beta_2 > b_2$$

is **exactly the same** as the procedure for testing

$$H_0: \beta_2 = b_2 \text{ against } H_1: \beta_2 > b_2.$$

Question: Why do we use the equality form of the null hypothesis, $\beta_2 = b_2$, to calculate the sample value of the test statistic?

Answer: A test that takes the null hypothesis as $H_0: \beta_2 = b_2$ is the most favorable to the null hypothesis, and hence is the least favorable to the alternative hypothesis $H_1: \beta_2 > b_2$. This means that, at any chosen significance level α , if we reject $H_0: \beta_2 = b_2$ in favor of the alternative hypothesis $H_1: \beta_2 > b_2$, then we would also reject $H_0: \beta_2 = b_2 - c$ in favor of $H_1: \beta_2 > b_2 - c$, where $c > 0$ is any *positive* constant.

3. The **null distribution of $t_0(\hat{\beta}_2)$ is $t[N - 2]$** , the t-distribution with $N - 2$ degrees of freedom. In other words, if the null hypothesis H_0 is true -- i.e., if $\beta_2 = b_2$, then

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\hat{s}e(\hat{\beta}_2)} \sim t[N - 2] \quad \text{under } H_0.$$

Question 1: What values of $\hat{\beta}_2$ and $t_0(\hat{\beta}_2)$ would lead us to reject H_0 against H_1 ?

$$H_0: \beta_2 = b_2 \text{ or } \beta_2 - b_2 = 0$$

$$H_1: \beta_2 > b_2 \text{ or } \beta_2 - b_2 > 0 \quad \Leftarrow \text{ a *right-sided alternative hypothesis*.}$$

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\widehat{se}(\hat{\beta}_2)}.$$

Remember:

$\hat{\beta}_2$ = the *estimated value of* β_2

b_2 = the *hypothesized value of* β_2

$se(\hat{\beta}_2) > 0$ ($se(\hat{\beta}_2)$ is *always a positive number*)

- We would reject H_0 against H_1 if $\hat{\beta}_2$ is much greater than b_2 , if **the *estimated value of* β_2 is *much greater than* b_2 the *hypothesized value of* β_2 .**
- More specifically, we would **reject H_0 against H_1** in the following case:

$$\hat{\beta}_2 \gg b_2 \quad \Rightarrow \quad \hat{\beta}_2 - b_2 \gg 0 \quad \Rightarrow \quad t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - b_2}{\widehat{se}(\hat{\beta}_2)} \gg 0$$

Values of $\hat{\beta}_2$ *much greater than* b_2 imply **large positive values of** $t_0(\hat{\beta}_2)$.

Question 2: How much *greater* than zero does the value of $t_0(\hat{\beta}_2)$ have to be for us *to reject* H_0 in favour of H_1 ?

Answer: It depends on **the *significance level* we choose** for the test and **the *null distribution* of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
= the probability of making a Type I error.

Because only large positive values of t_0 favour the alternative hypothesis, there is only one critical value – an upper critical value of the $t[N - 2]$ -distribution that delineates an upper or right-tail rejection area equal to α .

$t_{\alpha}[N - 2]$ = the ***upper α critical value*** of the ***$t[N - 2]$ -distribution***.

Implications: If $H_0: \beta_2 = b_2$ is *true*, then the following two probability statements hold.

$$(1) \quad \Pr\left(t_0(\hat{\beta}_2) \leq t_{\alpha}[N - 2]\right) = 1 - \alpha \quad (1)$$

$$(2) \quad \Pr\left(t_0(\hat{\beta}_2) > t_{\alpha}[N - 2]\right) = \alpha \quad (2)$$

where

$t_0(\hat{\beta}_2)$ = the **calculated *sample* value of the t-statistic** under H_0 ;
 $t_{\alpha}[N - 2]$ = the **upper α -level critical value** of the $t[N - 2]$ -distribution;
 α = the ***significance level*** for the test;
 $1 - \alpha$ = the ***confidence level*** for the test.

Determine the Rejection and Nonrejection Regions -- Right-Tail Test

- The **non-rejection region** for $t_0(\hat{\beta}_2)$ is the set of values defined by the inequality in probability statement (1) above:

$$(1) \quad \Pr\left(t_0(\hat{\beta}_2) \leq t_{\alpha}[N-2]\right) = 1 - \alpha \quad (1)$$

It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$t_0(\hat{\beta}_2) \leq t_{\alpha}[N-2] \quad \Leftarrow \text{non-rejection region under } H_0: \beta_2 = b_2.$$

- The **rejection region** for $t_0(\hat{\beta}_2)$ is the set of values defined by the inequality in probability statement (2) above.

$$(2) \quad \Pr\left(t_0(\hat{\beta}_2) > t_{\alpha}[N-2]\right) = \alpha \quad (2)$$

It consists of all values of $t_0(\hat{\beta}_2)$ such that

$$t_0(\hat{\beta}_2) > t_{\alpha}[N-2] \quad \Leftarrow \text{rejection region under } H_0: \beta_2 = b_2 \\ \text{is a } \textit{one-tail right-tail} \text{ rejection region.}$$

NOTE:

- 1) For a **right-tail test**, the rejection region $t_0(\hat{\beta}_2) > t_{\alpha}[N-2]$ consists only of the **upper right-hand tail** of the t-distribution with $N - 2$ degrees of freedom.
 - 2) This **right-tail rejection region** contains **unexpectedly large values** of $t_0(\hat{\beta}_2)$ under H_0 -- i.e., values that we would only expect to obtain with “small” probability α if the null hypothesis $H_0: \beta_2 = b_2$ is true.
 - 3) The **right tail area under the t[N - 2]-distribution** in this upper tail **equals the significance level α** . This area is called the **upper α -level (or upper 100 α percent) tail area** of the t[N - 2]-distribution.
-

STEP 5: Apply the Decision Rule and State Inference

Step 5: Apply the *decision rule* of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Right-Tail Test

Formulation 1: Determine if the *sample value* t_0 of the test statistic lies in the *rejection or nonrejection region* at the chosen significance level α .

Decision Rule for a Right-Tail Test -- Formulation 1

1. If the *sample value* t_0 of the test statistic lies in the *right-tail rejection region* at the chosen significance level, then **reject the null hypothesis H_0** .

For a **right-tail** test: $t_0 > t_{\alpha}[N-2] \Rightarrow$ **reject H_0 at significance level α** .

Reject H_0 in favour of H_1 at significance level α if

$t_0 > t_{\alpha}[N-2]$ meaning t_0 lies in the *upper right-tail rejection area*.

Inference: Reject H_0 in favour of H_1 at significance level α .

2. If the *sample value* t_0 of the test statistic lies in the *nonrejection region* at the chosen significance level, then **retain (do not reject) the null hypothesis H_0** .

For a **right-tail** test: $t_0 \leq t_{\alpha}[N-2] \Rightarrow$ **retain H_0 at significance level α** .

Retain H_0 against H_1 at significance level α if

$t_0 \leq t_{\alpha}[N-2]$ meaning t_0 lies in the *nonrejection area*.

Inference: Retain H_0 against H_1 at significance level α .

Examples of One-Tail Hypothesis Tests

The Model:

DATA: `auto1.dta` (a Stata-format data file)

MODEL: $\text{price}_i = \beta_1 + \beta_2 \text{weight}_i + u_i \quad (i = 1, \dots, N)$

```
. regress price weight
```

| Source | SS | df | MS | | | |
|----------|-----------|----|------------|-----------------|--------|--|
| Model | 184233937 | 1 | 184233937 | Number of obs = | 74 | |
| Residual | 450831459 | 72 | 6261548.04 | F(1, 72) = | 29.42 | |
| Total | 635065396 | 73 | 8699525.97 | Prob > F = | 0.0000 | |
| | | | | R-squared = | 0.2901 | |
| | | | | Adj R-squared = | 0.2802 | |
| | | | | Root MSE = | 2502.3 | |

| price | Coef. | Std. Err. | t | P> t | [95% Conf. Interval] | |
|--------|-----------------|-----------------|--------------|-------|----------------------|----------|
| weight | 2.044063 | .3768341 | 5.424 | 0.000 | 1.292858 | 2.795268 |
| _cons | -6.707353 | 1174.43 | -0.006 | 0.995 | -2347.89 | 2334.475 |

$$N = 74 \quad N - 2 = 74 - 2 = \mathbf{72}$$

$$\hat{\beta}_2 = \mathbf{2.0441} \quad s\hat{e}(\hat{\beta}_2) = \mathbf{0.376834}$$

$$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$$

$$t_{\alpha/2}[N-2] = t_{0.025}[72] = \mathbf{1.9935}$$

$$t_{\alpha}[N-2] = t_{0.05}[72] = \mathbf{1.6663}$$

Test 1 – A Left-Tail Test: Test the proposition that **weight_i has a *negative* effect on price_i**. Perform the test at the 5 percent significance level ($\alpha = 0.05$).

- ***Null and Alternative Hypotheses***

$$H_0: \beta_2 = 0$$

$$H_1: \beta_2 < 0 \quad \text{a *one-sided left-sided* alternative hypothesis.}$$

- The ***feasible test statistic*** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\widehat{\text{Var}}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} \sim t[N - 2].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H_0 .

Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\widehat{\text{se}}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The ***one-tail critical value*** of the $t[N - 2]$ distribution at the **5 percent significance level** (at $\alpha = 0.05$) is $t_{\alpha}[N - 2] = t_{0.05}[72] = \mathbf{1.6663}$.

- ***Decision Rule -- Left-Tail Test:***

$$\text{If } t_0 < -t_{\alpha}[N - 2] \quad \text{reject } H_0 \text{ at significance level } \alpha;$$

$$\text{If } t_0 \geq -t_{\alpha}[N - 2] \quad \text{retain } H_0 \text{ at significance level } \alpha.$$

- ***Inference:***

$$t_0 = 5.424 > -1.6663 = -t_{0.05}[72] \Rightarrow \text{retain } H_0 \text{ at significance level } \alpha = 0.05$$

Retain $H_0: \beta_2 = 0$ against $H_1: \beta_2 < 0$ at the 5 percent significance level.

Test 2 – A Right-Tail Test: Test the proposition that **weight_i** has a *positive* effect on **price_i**. Perform the test at the 5 percent significance level ($\alpha = 0.05$).

- **Null and Alternative Hypotheses**

$$H_0: \beta_2 = 0$$

$$H_1: \beta_2 > 0 \quad \text{a one-sided right-sided alternative hypothesis.}$$

- The *feasible* test statistic is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\text{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} \sim t[N - 2].$$

- Compute the *sample value* of $t(\hat{\beta}_2)$ under the null hypothesis H_0 .

Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\hat{s}e(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{s}e(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The *one-tail critical value* of the $t[N - 2]$ distribution at the 5 percent significance level (at $\alpha = 0.05$) is $t_{\alpha}[N - 2] = t_{0.05}[72] = \mathbf{1.6663}$.

- **Decision Rule -- Right-Tail Test:**

If $t_0 > t_{\alpha}[N - 2]$ **reject H_0 at significance level α ;**

If $t_0 \leq t_{\alpha}[N - 2]$ **retain H_0 at significance level α .**

- **Inference:**

$$t_0 = 5.424 > \mathbf{1.6663} = t_{0.05}[72] \Rightarrow \text{reject } H_0 \text{ at significance level } \alpha = 0.05$$

Reject $H_0: \beta_2 = 0$ against $H_1: \beta_2 > 0$ at the 5 percent significance level.

The **sample evidence favours** the alternative hypothesis **$H_1: \beta_2 > 0$** .

Interpretation of the Decision Rules

- An hypothesis test can lead to *only two* possible decisions:

either

- (1) a decision *to reject the null hypothesis H_0* against the alternative hypothesis H_1 , in which case the **sample evidence favours H_1 over H_0** ;

or

- (2) a decision *to retain (not to reject) the null hypothesis H_0* against the alternative hypothesis H_1 , in which case the **sample evidence favours H_0 over H_1** .

- **Points to remember** in interpreting these alternative decisions.

- 1) An hypothesis test can never be interpreted as **either *proving or disproving the truth of the null hypothesis***.

Reason: The decision to reject or retain H_0 on the basis of sample evidence is always subject -- explicitly or implicitly -- to some uncertainty, or margin of statistical error. That is, there is always some non-zero probability of committing a Type I or Type II error.

- 2) A decision *to retain (not to reject) the null hypothesis H_0* should not be interpreted to mean that we “accept” H_0 , or that H_0 is true.

Reason: Saying “we accept H_0 ” implies that we are concluding that the null hypothesis is true, but such a conclusion is incorrect. Nonrejection (or retention) of H_0 means only that the sample data provide insufficient evidence to reject H_0 ; it does not mean that H_0 is true beyond any doubt.

Explanation: see example in Gujarati (1995, p. 129).

We obtain an estimate of the slope coefficient of $\hat{\beta}_2 = 0.5091$ and a corresponding estimated standard error of $s\hat{e}(\hat{\beta}_2) = 0.0357$.

- First, we perform a **two-tail test of $H_0: \beta_2 = 0.50$** against $H_1: \beta_2 \neq 0.50$.

The sample value of the t-statistic under H_0 is calculated as:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{s\hat{e}(\hat{\beta}_2)} = \frac{0.5091 - 0.50}{0.0357} = 0.25.$$

But the sample value 0.25 is clearly insignificant at, say, the 5% significance level ($\alpha = 0.05$). Suppose we “accept” H_0 and conclude that the true value of β_2 is 0.50.

- Next, we perform a **two-tail test of $H_0: \beta_2 = 0.48$** against $H_1: \beta_2 \neq 0.48$.

The sample value of the t-statistic under this H_0 is calculated as:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{s\hat{e}(\hat{\beta}_2)} = \frac{0.5091 - 0.48}{0.0357} = 0.82.$$

But the sample value 0.82 is also clearly insignificant at the 5% significance level ($\alpha = 0.05$). Do we now “accept” this H_0 and conclude that the true value of β_2 is 0.48?

- **Question:** Do we “accept” either or both of these hypothesized values of β_2 as the true value?

The correct answer is NO. We do not know the true value of β_2 .

- ◆ **Conclusion:** All we can legitimately conclude from these two hypothesis tests is that the sample evidence is consistent or compatible with both the null hypotheses we have tested. But the tests provide no reason to conclude that either hypothesized value, 0.50 or 0.48, is the true value of β_2 .

- 3) A decision *to reject* H_0 on the basis of given sample data **does not imply that we *must accept* the alternative hypothesis H_1** , or that H_1 *must be true*, or that H_1 does in fact *contain the truth*.

Reason: An hypothesis test is designed only to assess the probable empirical validity of the null hypothesis H_0 ; it is not designed to test the alternative hypothesis H_1 .

- Regardless of whether a test outcome for some particular sample data indicates rejection or nonrejection of H_0 , the set of alternative possibilities specified by the alternative hypothesis H_1 may or may not contain the truth.
- It is quite possible for a test of some null hypothesis H_0 against some alternative hypothesis H_1 to indicate rejection of H_0 *when H_1 is false* -- that is, when the alternative possibilities specified by H_1 do not contain the truth.