QUEEN'S UNIVERSITY AT KINGSTON
Department of Economics

ECONOMICS 351* - Section A

Introductory Econometrics

Fall Term 2001 **MID-TERM EXAM ANSWERS** M.G. Abbott

DATE: **Monday October 22, 2001.**

TIME: **80 minutes; 2:30 p.m. - 3:50 p.m.**

INSTRUCTIONS: The exam consists of **FIVE (5)** questions. Students are required to answer **ALL FIVE (5)** questions.

Answer all questions in the exam booklets provided. Be sure your ***name*** and ***student number*** are printed clearly on the front of all exam booklets used.

Do not write answers to questions on the front page of the first exam booklet.

Please label clearly each of your answers in the exam booklets with the appropriate number and letter.

Please write legibly.

MARKING: The marks for each question are indicated in parentheses immediately above each question. **Total marks** for the exam **equal 100**.

QUESTIONS: **Answer ALL FIVE questions.**

All questions pertain to the simple (two-variable) linear regression model for which the population regression equation can be written in conventional notation as:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \qquad \textbf{(1)}$$

where Y_i and X_i are observable variables, β_1 and β_2 are unknown (constant) regression coefficients, and u_i is an unobservable random error term. The Ordinary Least Squares (OLS) sample regression equation corresponding to regression equation (1) is

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \qquad (i = 1, ..., N) \qquad \textbf{(2)}$$

where $\hat{\beta}_1$ is the OLS estimator of the intercept coefficient β_1, $\hat{\beta}_2$ is the OLS estimator of the slope coefficient β_2, $\hat{u}_i$ is the OLS residual for the i-th sample observation, and N is sample size (the number of observations in the sample).

(15 marks)

1. State the Ordinary Least Squares (OLS) estimation criterion. State the OLS normal equations. Derive the OLS normal equations from the OLS estimation criterion.

<u>**ANSWER**</u>**:**

(3 marks)

- State the Ordinary Least Squares (OLS) estimation criterion. **(3 marks)**

The OLS coefficient estimators are **those formulas or expressions for $\hat{\beta}_1$ and $\hat{\beta}_2$ that <u>minimize</u> the sum of <u>squared</u> residuals RSS** for any given sample of size N.

The **OLS estimation criterion** is therefore:

$$\underset{\{\hat{\beta}_j\}}{\text{Minimize}}\ \text{RSS}\left(\hat{\beta}_1, \hat{\beta}_2\right) = \sum_{i=1}^{N} \hat{u}_i^2 = \sum_{i=1}^{N} \left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right)^2$$

(4 marks)

- State the OLS normal equations. **(4 marks)**

The ***first*** **OLS normal equation** can be written in *any one* of the following forms:

$$\begin{aligned} \Sigma_i\, Y_i - N\hat{\beta}_1 - \hat{\beta}_2\, \Sigma_i\, X_i &= 0 \\ -N\hat{\beta}_1 - \hat{\beta}_2\, \Sigma_i\, X_i &= -\Sigma_i\, Y_i \qquad \textbf{(N1)} \\ N\hat{\beta}_1 + \hat{\beta}_2\, \Sigma_i\, X_i &= \Sigma_i\, Y_i \end{aligned}$$

The ***second*** **OLS normal equation** can be written in *any one* of the following forms:

$$\begin{aligned} \Sigma_i\, X_i Y_i - \hat{\beta}_1\, \Sigma_i\, X_i - \hat{\beta}_2\, \Sigma_i\, X_i^2 &= 0 \\ -\hat{\beta}_1\, \Sigma_i\, X_i - \hat{\beta}_2\, \Sigma_i\, X_i^2 &= -\Sigma_i\, X_i Y_i \qquad \textbf{(N2)} \\ \hat{\beta}_1\, \Sigma_i\, X_i + \hat{\beta}_2\, \Sigma_i\, X_i^2 &= \Sigma_i\, X_i Y_i \end{aligned}$$

Question 1 (continued)

(8 marks)

- Show how the OLS normal equations are derived from the OLS estimation criterion.

(4 marks)

<u>**Step 1**</u>**: Partially differentiate** the RSS $\left(\hat{\beta}_1, \hat{\beta}_2\right)$ function with respect to $\hat{\beta}_1$ and $\hat{\beta}_2$, using

$$\hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i \qquad \Rightarrow \qquad \frac{\partial \hat{u}_i}{\partial \hat{\beta}_1} = -1 \qquad \text{and} \qquad \frac{\partial \hat{u}_i}{\partial \hat{\beta}_2} = -X_i .$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = \sum_{i=1}^{N} 2\hat{u}_i \left(\frac{\partial \hat{u}_i}{\partial \hat{\beta}_1} \right) = \sum_{i=1}^{N} 2\hat{u}_i(-1) = -2\sum_{i=1}^{N} \hat{u}_i = -2\sum_{i=1}^{N}\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) \tag{1}$$

$$\frac{\partial RSS}{\partial \hat{\beta}_2} = \sum_{i=1}^{N} 2\hat{u}_i \left(\frac{\partial \hat{u}_i}{\partial \hat{\beta}_2} \right) = \sum_{i=1}^{N} 2\hat{u}_i(-X_i) = -2\sum_{i=1}^{N} X_i \hat{u}_i$$

$$= -2\sum_{i=1}^{N} X_i\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) \qquad \text{since } \hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i \tag{2}$$

$$= -2\sum_{i=1}^{N}\left(X_i Y_i - \hat{\beta}_1 X_i - \hat{\beta}_2 X_i^2\right)$$

(4 marks)

<u>**Step 2**</u>**: Obtain the first-order conditions (FOCs)** for a minimum of the RSS function by **setting the partial derivatives (1) and (2) *equal to zero*** and then dividing each equation by –2 and re-arranging:

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = 0 \quad \Rightarrow \quad -2\sum_i \hat{u}_i = 0 \quad \Rightarrow \quad -2\sum_i\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) = 0$$

$$\Rightarrow \quad \sum_i\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) = 0$$

$$\Rightarrow \quad \sum_i Y_i - N\hat{\beta}_1 - \hat{\beta}_2 \sum_i X_i = 0$$

$$\Rightarrow \quad \sum_i Y_i = N\hat{\beta}_1 + \hat{\beta}_2 \sum_i X_i \qquad \textbf{(N1)}$$

$$\frac{\partial RSS}{\partial \hat{\beta}_2} = 0 \quad \Rightarrow \quad -2\sum_i X_i \hat{u}_i = 0 \quad \Rightarrow \quad -2\sum_i X_i\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) = 0$$

$$\Rightarrow \quad \sum_i X_i\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) = 0$$

$$\Rightarrow \quad \sum_i\left(X_i Y_i - \hat{\beta}_1 X_i - \hat{\beta}_2 X_i^2\right) = 0$$

$$\Rightarrow \quad \sum_i X_i Y_i - \hat{\beta}_1 \sum_i X_i - \hat{\beta}_2 \sum_i X_i^2 = 0$$

$$\Rightarrow \quad \sum_i X_i Y_i = \hat{\beta}_1 \sum_i X_i + \hat{\beta}_2 \sum_i X_i^2 \qquad \textbf{(N2)}$$

(15 marks)

2. Give a general definition of the t-distribution. Starting from this definition, derive the t-statistic for the OLS slope coefficient estimator $\hat{\beta}_2$. State all assumptions required for the derivation.

<u>ANSWER</u>:

(2 marks)

- **General Definition of the t-Distribution**

  A random variable has the **t-distribution with *m* degrees of freedom** if it can be constructed by dividing

  (1) a **standard normal random variable Z ~ N(0, 1)**

  by

  (2) the **square root** of an ***independent*** **chi-square random variable V** that has been divided by its degrees of freedom *m* .

  <u>Formally</u>: Consider the two random variables Z and V.

  If (1) $Z \sim N(0,1)$
  (2) $V \sim \chi^2[m]$
  (3) Z and V are ***independent***,

  then the random variable

$$t = \frac{Z}{\sqrt{V/m}} \sim t[m],$$ the **t-distribution** with ***m*** **degrees of freedom**.

(2 marks)

- **<u>Error Normality Assumption (A9)</u>:** The random error terms u_i are **independently and identically distributed (iid)** as the **normal distribution** with **zero mean** and **constant variance σ^2**.

  the u_i are iid as $N(0, \sigma^2)$ for all i.

Question 2 (continued)

(2 marks)

- ***Two implications of error normality assumption (A9)***: (follow from ***linearity property* of the normal distribution** whereby any *linear* function of a normally distributed random variable is itself normally distributed).

  (1 mark)

  (1) The **Y_i values are normally distributed**: Y_i are NID$\left(\beta_1 + \beta_2 X_i, \sigma^2\right)$
  Why? Because the PRE states that **the Y_i values are *linear* functions of the u_i**: $Y_i = \beta_1 + \beta_2 X_i + u_i$.

  (1 mark)

  (2) The OLS slope coefficient estimator $\hat{\beta}_2$ **is normally distributed:** $\hat{\beta}_2 \sim N\left(\beta_2, Var(\hat{\beta}_2)\right)$
  Why? Because $\hat{\beta}_2$ **can be written as a *linear* function of the Y_i values**: $\hat{\beta}_2 = \sum_i k_i Y_i$.

(3 marks)

- ***Numerator* of the t-statistic for $\hat{\beta}_2$**: the $Z(\hat{\beta}_2)$ statistic.

  The normality of the sampling distribution of $\hat{\beta}_2$ implies that $\hat{\beta}_2$ can be written in the form of a **standard normal variable** with mean zero and variance one, denoted as N(0,1).

$$\hat{\beta}_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum_i x_i^2}\right) \quad \Rightarrow \quad Z(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{Var(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)} \sim N(0,1)$$

  where the **Z-statistic for $\hat{\beta}_2$** can be written as

$$Z(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{Var(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)} = \frac{\hat{\beta}_2 - \beta_2}{\sigma \big/ \sqrt{\sum_i x_i^2}} = \frac{\left(\hat{\beta}_2 - \beta_2\right)\sqrt{\sum_i x_i^2}}{\sigma}. \qquad (1)$$

(2 marks)

- ***Denominator* of the t-statistic for $\hat{\beta}_2$**:

  The error normality assumption implies that the statistic $\hat{\sigma}^2/\sigma^2$ has a degrees-of-freedom-adjusted chi-square distribution with (N – 2) degrees of freedom; that is

$$\frac{(N-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2[N-2] \quad \Rightarrow \quad \frac{\hat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2[N-2]}{(N-2)} \quad \Rightarrow \quad \frac{\hat{\sigma}}{\sigma} \sim \sqrt{\frac{\chi^2[N-2]}{(N-2)}}. \qquad (2)$$

  The last term $\hat{\sigma}/\sigma$ in (2) is the denominator of the t-statistic for $\hat{\beta}_2$: it is distributed as the square root of a degrees-of-freedom-adjusted chi-square variable with (N – 2) degrees of freedom:

Question 2 (continued)

(4 marks)

- **The t-statistic for $\hat{\beta}_2$.**

The t-statistic for $\hat{\beta}_2$ is the ratio of (1) to (2): i.e.,

$$t(\hat{\beta}_2) = \frac{Z(\hat{\beta}_2)}{\hat{\sigma}/\sigma} = \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum_i x_i^2}\,/\sigma}{\hat{\sigma}/\sigma} = \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum_i x_i^2}}{\hat{\sigma}}. \quad (3)$$

- Dividing the numerator and denominator of (3) by $\sqrt{\sum_i x_i^2}$ yields

$$t(\hat{\beta}_2) = \frac{(\hat{\beta}_2 - \beta_2)}{\hat{\sigma}/\sqrt{\sum_i x_i^2}}. \quad (4)$$

- But the denominator of (4) is simply the ***estimated*** **standard error of $\hat{\beta}_2$**; i.e.,

$$\frac{\hat{\sigma}}{\sqrt{\sum_i x_i^2}} = \sqrt{\hat{Var}(\hat{\beta}_2)} = \hat{se}(\hat{\beta}_2).$$

- **Result:** The **t-statistic for $\hat{\beta}_2$** thus takes the form

$$t(\hat{\beta}_2) = \frac{(\hat{\beta}_2 - \beta_2)}{\hat{\sigma}/\sqrt{\sum_i x_i^2}} = \frac{(\hat{\beta}_2 - \beta_2)}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{(\hat{\beta}_2 - \beta_2)}{\hat{se}(\hat{\beta}_2)} \sim t[N-2]. \quad \mathbf{(5)}$$

(10 marks)

3. Explain what is meant by each of the following statements about the estimator $\hat{\theta}$ of the population parameter θ.

(a) $\hat{\theta}$ is an unbiased estimator of θ.
(b) $\hat{\theta}$ is a consistent estimator of θ.

For $\hat{\theta}$ to be a consistent estimator of θ, must it be an unbiased estimator of θ, yes or no?

ANSWER:

(4 marks)

- (a) $\hat{\theta}$ is an unbiased estimator of θ. **(4 marks)**

  The mean, or expectation, of the estimator $\hat{\theta}$ equals the true parameter value θ for any finite sample size. Unbiasedness is a *small sample* property that holds for a sample of any finite size $n < \infty$.

  $$E(\hat{\theta}) = \theta \qquad \Rightarrow \qquad \text{Bias}(\hat{\theta}) \equiv E(\hat{\theta}) - \theta = 0.$$

(5 marks)

- (b) $\hat{\theta}$ is a consistent estimator of θ.

  The estimator $\hat{\theta}$ is a consistent estimator if its sampling distribution converges to, or collapses on, the parameter θ as sample size becomes indefinitely large (as $n \to \infty$); if $\hat{\theta}$ gets closer and closer to θ as sample size gets larger and larger. Consistency is a *large sample* property.

  More formally, $\hat{\theta}$ is a *consistent* estimator of θ if (1) its probability limit equals θ, i.e., if $\text{plim}(\hat{\theta}) = \theta$; or (2) if $\hat{\theta}$ converges in probability to θ as $n \to \infty$. The probability that $\hat{\theta}$ is arbitrarily close to θ approaches 1 as same size increases without limit.

(1 mark)

- For $\hat{\theta}$ to be a consistent estimator of θ, must it be an unbiased estimator of θ, yes or no?

  No.

(10 marks)
4. State the Gauss-Markov theorem. Explain fully what it means.

<u>ANSWER</u>:

(5 marks)

- **<u>Statement of Gauss-Markov theorem</u>:**

Under assumptions A1-A8 of the Classical Linear Regression Model (CLRM), the OLS coefficient estimators $\hat{\beta}_j$ (j = 1, 2) are the ***minimum variance estimators*** **of the regression coefficients $\boldsymbol{\beta_j}$** (j = 1, 2) in the class of ***all linear unbiased estimators*** of β_j.

That is, under assumptions A1-A8, the **OLS coefficient estimators** $\hat{\beta}_j$ are **the <u>B</u>est <u>L</u>inear <u>U</u>nbiased <u>E</u>stimators -- or BLUE -- of $\boldsymbol{\beta_j}$** (j = 1, 2), where

1) **BLUE** $\equiv$ **B**est **L**inear **U**nbiased **E**stimator
2) "Best" means "minimum variance" or "smallest variance" (in the class of all linear unbiased estimators) .

(5 marks)

- **<u>Explanation of Gauss-Markov theorem</u>:**

1. Let $\tilde{\beta}_j$ be ***any other linear unbiased estimator*** of β_j.
Let $\hat{\beta}_j$ be the ***OLS estimator*** of β_j; it too is linear and unbiased.

2. Both estimators $\tilde{\beta}_j$ and $\hat{\beta}_j$ are **unbiased estimators** of β_j:

$$E(\hat{\beta}_j) = \beta_j \qquad \text{and} \qquad E(\tilde{\beta}_j) = \beta_j .$$

3. But the OLS estimator $\hat{\beta}_j$ has a ***smaller*** **variance** than $\tilde{\beta}_j$:

$$\text{Var}(\hat{\beta}_j) \le \text{Var}(\tilde{\beta}_j) \qquad \Rightarrow \qquad \hat{\beta}_j \text{ is \textbf{efficient} relative to } \tilde{\beta}_j .$$

This means that the OLS estimator $\hat{\beta}_j$ is **statistically more precise** than $\tilde{\beta}_j$, ***any other linear unbiased*** **estimator of $\boldsymbol{\beta_j}$**.

Alternatively, the Gauss-Markov theorem says that the OLS coefficient estimators $\hat{\beta}_j$ are the ***best*** **of all** ***linear unbiased*** **estimators** of β_j, where **"best"** means **"minimum variance"**.

(50 marks)

5. A researcher is using data for a sample of 88 houses sold in an urban area during a recent year to investigate the relationship between house prices Y_i (measured in *thousands* of dollars) and house size X_i (measured in square meters). Preliminary analysis of the sample data produces the following sample information:

$$N = 88 \qquad \sum_{i=1}^{N} Y_i = 25{,}832.05 \qquad \sum_{i=1}^{N} X_i = 16{,}462.34 \qquad \sum_{i=1}^{N} Y_i^2 = 8{,}500{,}750.6$$

$$\sum_{i=1}^{N} X_i^2 = 3{,}329{,}789.6 \qquad \sum_{i=1}^{N} X_i Y_i = 5{,}209{,}990.7 \qquad \sum_{i=1}^{N} x_i y_i = 377{,}534.76$$

$$\sum_{i=1}^{N} y_i^2 = 917{,}854.51 \qquad \sum_{i=1}^{N} x_i^2 = 250{,}144.32 \qquad \sum_{i=1}^{N} \hat{u}_i^2 = 348{,}053.43$$

where $x_i \equiv X_i - \overline{X}$ and $y_i \equiv Y_i - \overline{Y}$ for i = 1, ..., N. Use the above sample information to answer all the following questions. **Show explicitly all formulas and calculations**.

(10 marks)

(a) Use the above information to compute OLS estimates of the intercept coefficient β_1 and the slope coefficient β_2.

- $\hat{\beta}_2 = \dfrac{\sum_i x_i y_i}{\sum_i x_i^2} = \dfrac{377{,}534.76}{250{,}144.32} = 1.509268 = \underline{\mathbf{1.5093}}$ **(5 marks)**

- $\hat{\beta}_1 = \overline{Y} - \hat{\beta}_2 \overline{X}$

$$\overline{Y} = \frac{\sum_{i=1}^{N} Y_i}{N} = \frac{25{,}832.05}{88} = 293.546 \quad \text{and} \quad \overline{X} = \frac{\sum_{i=1}^{N} X_i}{N} = \frac{16{,}462.34}{88} = 187.072$$

Therefore

$$\hat{\beta}_1 = \overline{Y} - \beta_2 \overline{X} = 293.546 - (1.509268)(187.072) = 293.546 - 282.342 = \underline{\mathbf{11.204}}$$

(5 marks)

(5 marks)

(b) Interpret the slope coefficient estimate you calculated in part (a) -- i.e., explain what the numeric value you calculated for $\hat{\beta}_2$ means.

<u>Note</u>: $\hat{\beta}_2$ = **1.50927**. Y_i is measured in ***<u>thousands</u> of dollars***, and X_i is measured in ***square meters***.

The estimate **1.50927** of β_2 means that an ***increase*** **(decrease) in house size** X_i of ***1 square meter*** is associated on average with **an increase (decrease) in house price of *1.50927 <u>thousands</u> of dollars***, or ***1,509.27 <u>dollars</u>***.

(5 marks)

(c) Calculate an estimate of σ^2, the error variance.

$$\hat{\sigma}^2 = \frac{RSS}{N-2} = \frac{\sum_{i=1}^{N}\hat{u}_i^2}{N-2} = \frac{348{,}053.43}{88-2} = \frac{348{,}053.43}{86} = \underline{\mathbf{4{,}047.133}}$$

(6 marks)

(d) Compute the value of R^2, the coefficient of determination for the estimated OLS sample regression equation. Briefly explain what the calculated value of R^2 means.

(1) $$ESS = TSS - RSS = \sum_{i=1}^{N} y_i^2 - \sum_{i=1}^{N}\hat{u}_i^2 = 917{,}854.51 - 348{,}053.43 = 569{,}801.08$$

$$R^2 = \frac{ESS}{TSS} = \frac{\sum_{i=1}^{N}\hat{y}_i^2}{\sum_{i=1}^{N} y_i^2} = \frac{569{,}801.08}{917{,}854.51} = \underline{\mathbf{0.6208}}$$

or **(4 marks)**

(2) $$R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{i=1}^{N}\hat{u}_i^2}{\sum_{i=1}^{N} y_i^2} = 1 - \frac{348{,}053.43}{917{,}854.51} = 1 - 0.3792 = \underline{\mathbf{0.6208}}$$

<u>Interpretation of R^2</u> = 0.6208: The value of 0.6208 indicates that **62.08 percent of the total sample (or observed) variation in Y_i** (house prices) **is *attributable to*, or *explained by*, the regressor X_i** (house size, measured in square meters). **(2 marks)**

(6 marks)

(e) What are the values of $\sum_{i=1}^{N}\hat{u}_i$ and $\sum_{i=1}^{N}X_i\hat{u}_i$ for the sample regression equation you have estimated? Explain briefly how you obtained your answer.

$\sum_{i=1}^{N}\hat{u}_i = \mathbf{0}$ from normal equation N1 **(3 marks)**

$\sum_{i=1}^{N}X_i\hat{u}_i = \mathbf{0}$ from normal equation N2 **(3 marks)**

These computational properties of the OLS sample regression equation follow from the ***first-order conditions* for the OLS coefficient estimators**, which are called the ***OLS normal equations***.

(12 marks)

(f) Perform a test of the null hypothesis H_0: $\beta_2 = 0$ against the alternative hypothesis H_1: $\beta_2 \neq 0$ at the 5% significance level (i.e., for significance level $\alpha = 0.05$). Show how you calculated the test statistic. State the decision rule you use, and the inference you would draw from the test. Briefly explain what the test outcome means.

H_0: $\beta_2 = 0$
H_1: $\beta_2 \neq 0$ a ***two-sided* alternative hypothesis** $\Rightarrow$ a ***two-tailed* test**

- Test statistic is $t(\hat{\beta}_2) = \dfrac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2]$. (1)
- Calculate the estimated standard error of $\hat{\beta}_2$:

$$\hat{Var}(\hat{\beta}_2) = \frac{\hat{\sigma}^2}{\sum_i x_i^2} = \frac{4047.133}{250{,}144.32} = 0.0161792$$

$$\hat{se}(\hat{\beta}_2) = \sqrt{\hat{Var}(\hat{\beta}_2)} = \sqrt{0.0161792} = \mathbf{0.1271975} = \underline{\mathbf{0.12720}}.$$ **(1 mark)**

- Calculate the ***sample value* of the t-statistic** (1) under H_0: set $\beta_2 = 0$ in (1).

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{1.509268 - 0.0}{0.12720} = \frac{1.509268}{0.12720} = 11.865 = \underline{\mathbf{11.87}}$$ **(3 marks)**

- **Null distribution** of $t_0(\hat{\beta}_2)$ is $\mathbf{t[N-2] = t[86]}$. **(1 mark)**

<u>Decision Rule</u>: At significance level α, **(2 marks)**

- ***reject* H_0** if $|t_0(\hat{\beta}_2)| > t_{\alpha/2}[86]$,
  i.e., if either (1) $t_0(\hat{\beta}_2) > t_{\alpha/2}[86]$ or (2) $t_0(\hat{\beta}_2) < -t_{\alpha/2}[86]$;
- ***retain* H_0** if $|t_0(\hat{\beta}_2)| \leq t_{\alpha/2}[86]$, i.e., if $-t_{\alpha/2}[86] \leq t_0(\hat{\beta}_2) \leq t_{\alpha/2}[86]$.

Critical value of t[86]-distribution: from t-table, use **df = 86,** 60 < 86 < 120.

- ***two-tailed* 5 percent critical value** = $t_{\alpha/2}[86] = t_{0.025}[86] = \underline{\mathbf{1.988}}$ **(1 mark)**
  (any value between **2.00** and **1.98** is acceptable)

Question 5(f) -- continued

<u>Inference</u>:

- At **5 percent significance level**, i.e., for $\alpha = 0.05$, **(3 marks)**

  $\left|t_0(\hat{\beta}_2)\right| = 11.87 > 1.988 = t_{0.025}[86] \Rightarrow$ ***reject*** $\mathbf{H_0}$ vs. H_1 **at 5 percent level**.

- **<u>Inference</u>: At the 5% significance level**, the null hypothesis $\beta_2 = 0$ is rejected in favour of the alternative hypothesis $\beta_2 \neq 0$.

Meaning of test outcome: Rejection of the null hypothesis $\beta_2 = 0$ against the alternative hypothesis $\beta_2 \neq 0$ means that **the sample evidence favours the existence of a relationship between *house prices* and *house size***. **(1 mark)**

Question 5(g)

(6 marks)

(g) Compute the two-sided 95% confidence interval for the slope coefficient β_2.

The **two-sided (1 – α)-level,** or **100(1 – α) percent, confidence interval for β_2** is computed as

$$\hat{\beta}_2 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2) \le \beta_2 \le \hat{\beta}_2 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2)$$ **(2 marks)**

where

- $\hat{\beta}_{2L} = \hat{\beta}_2 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2)$ = the ***lower*** **100(1 – α)% confidence limit for β_2**
- $\hat{\beta}_{2U} = \hat{\beta}_2 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2)$ = the ***upper*** **100(1 – α)% confidence limit for β_2**
- $t_{\alpha/2}[N-2]$ = the **α/2 critical value of the t-distribution with N–2 degrees of freedom**.

- Required results and intermediate calculations:

$\hat{\beta}_2$ = **1.50927** $\hat{se}(\hat{\beta}_2) = \sqrt{\hat{Var}(\hat{\beta}_2)} = \sqrt{0.0161792}$ = **0.12720**

$1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow$ **α/2 = 0.025**: $t_{\alpha/2}[N-2] = t_{0.025}[86]$ = **1.988**

$t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2) = t_{0.025}[86]\,\hat{se}(\hat{\beta}_2)$ = 1.988(0.12720) = **0.252874**

- **Lower 95% confidence limit for β_2** is: **(4 marks)**

$$\hat{\beta}_{2L} = \hat{\beta}_2 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2) = \hat{\beta}_2 - t_{0.025}[86]\,\hat{se}(\hat{\beta}_2)$$
= 1.50927 – 1.988(0.12720) = 1.50927 – 0.252874 = 1.2564 **= <u>1.256</u>**

- **Upper 95% confidence limit for β_2** is: **(4 marks)**

$$\hat{\beta}_{2U} = \hat{\beta}_2 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_2) = \hat{\beta}_2 + t_{0.025}[86]\,\hat{se}(\hat{\beta}_2)$$
= 1.50927 + 1.988(0.12720) = 1.50927 + 0.252874 = 1.7621 = **<u>1.762</u>**

- **<u>Result</u>:** The **two-sided 95% confidence interval for β_2** is:

[1.256, 1.762] ***or*** **[1.26, 1.76]**

Percentage Points of the t-Distribution

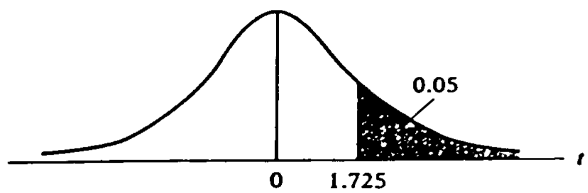
TABLE D.2
Percentage points of the t distribution

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.