Examples of Multiple Linear Regression Models

Data: *Stata* tutorial data set in text file **auto1.raw** or **auto1.txt**.

Sample data: A cross-sectional sample of 74 cars sold in North America in 1978.

Variable definitions:

$price_i$ = the price of the i-th car (in US dollars);
wgt_i = the weight of the i-th car (in pounds);
mpg_i = the fuel efficiency of the i-th car (in miles per gallon);
$foreign_i$ = 1 if the i-th car is manufactured outside North America, = 0 otherwise.

Multiple Linear Regression Model (1)

The PRE is:

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + u_i . \qquad \textbf{(1)}$$

k = 3; k − 1 = 2

- The regressor wgt_i^2 is called an ***interaction*** **variable**. It is the product of wgt_i with itself; it is a ***second-order*** **polynomial term** in the variable wgt_i.

Marginal or partial effect of wgt$_i$

The marginal effect of wgt_i on price_i is obtained by partially differentiating regression equation (2) with respect to wgt_i.

$$\frac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i)}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i .$$

- **Marginal effect of *wgt$_i$* on *price$_i$* is a *linear* function of *wgt$_i$*.** It is not a constant.

Hypotheses of interest

1. The ***marginal*** **effect of *wgt$_i$* on *price$_i$* is *zero***: i.e., ***wgt$_i$*** **has no effect on *price$_i$***; or **car *price$_i$* is unrelated to car *wgt$_i$*.**

- H_0: $\beta_2 = 0$ *and* $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i = 0$.

  Restricted **model** corresponding to H_0: set $\beta_2 = 0$ and $\beta_3 = 0$ in PRE (1).

  $$\text{price}_i = \beta_1 + u_i .$$

  $k_0 = 1$; $k_0 - 1 = 0$

- H_1: $\beta_2 \neq 0$ *and/or* $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i$.

  Unrestricted **model** corresponding to H_1: is PRE (1).

2. The ***marginal* effect of *wgt*$_i$ on *price*$_i$ is *constant***: i.e., it does not depend on ***wgt*$_i$**.

- H_0: $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i = \beta_2$.

***Restricted* model** corresponding to H_0: set $\beta_3 = 0$ in PRE (1).

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + u_i .$$

$k_0 = 2$; $k_0 - 1 = 1$

- H_1: $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i$.

***Unrestricted* model** corresponding to H_1: is PRE (1).

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + u_i . \qquad \textbf{(1)}$$

$k = 3$; $k - 1 = 2$

The OLS SRE for Model (1)

```
. regress price wgt wgtsq

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  2,    71) =   23.09
   Model |   250285462     2   125142731               Prob > F      =  0.0000
Residual |   384779934    71  5419435.69               R-squared     =  0.3941
---------+------------------------------               Adj R-squared =  0.3770
   Total |   635065396    73  8699525.97               Root MSE      =  2328.0

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |  -7.273097   2.691747     -2.702   0.009      -12.64029   -1.905906
   wgtsq |   .0015142   .0004337      3.491   0.001       .0006494     .002379
   _cons |    13418.8   3997.822      3.357   0.001       5447.372    21390.23
------------------------------------------------------------------------------
```

```
. test wgt wgtsq
```

F-test of hypothesis 1

```
 ( 1)  wgt = 0.0
 ( 2)  wgtsq = 0.0

       F(  2,    71) =   23.09
            Prob > F =    0.0000
```

3. The ***marginal* effect of *wgt_i* on *price_i* is *decreasing in wgt_i***: i.e., the marginal effect of wgt_i on $price_i$ exhibits ***decreasing marginal returns* in *wgt_i***.

- H_0: $\beta_3 = 0$ *or* $\beta_3 \geq 0$ $\Rightarrow$ $\dfrac{\partial^2 \text{price}_i}{\partial \text{wgt}_i^2} = 2\beta_3 \geq 0$.

- H_1: $\beta_3 < 0$ $\Rightarrow$ $\dfrac{\partial \text{price}_i}{\partial \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i$ *and* $\dfrac{\partial^2 \text{price}_i}{\partial \text{wgt}_i^2} = 2\beta_3 < 0$.

  $\Rightarrow$ a ***one-sided* alternative hypothesis**

  $\Rightarrow$ a ***left-tail* test**

Perform a ***left-tail* t-test** using the OLS coefficient estimate $\hat{\beta}_3$ of β_3 for the ***unrestricted* model** corresponding to H_1, which is PRE (1):

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + u_i \,. \qquad \textbf{(1)}$$

k = 3; k − 1 = 2; N − k = N − 3.

4. The ***marginal* effect of *wgt_i* on *price_i* is *increasing in wgt_i***: i.e., the marginal effect of wgt_i on $price_i$ exhibits ***increasing marginal returns* in *wgt_i***.

- H_0: $\beta_3 = 0$ *or* $\beta_3 \leq 0$ $\Rightarrow$ $\dfrac{\partial^2 \text{price}_i}{\partial \text{wgt}_i^2} = 2\beta_3 \leq 0$.

- H_1: $\beta_3 > 0$ $\Rightarrow$ $\dfrac{\partial \text{price}_i}{\partial \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i$ *and* $\dfrac{\partial^2 \text{price}_i}{\partial \text{wgt}_i^2} = 2\beta_3 > 0$.

  $\Rightarrow$ a ***one-sided* alternative hypothesis**

  $\Rightarrow$ a ***right-tail* test**

Perform a ***right-tail* t-test** using the OLS coefficient estimate $\hat{\beta}_3$ of β_3 for the ***unrestricted* model** corresponding to H_1, which is PRE (1):

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + u_i \,. \qquad \textbf{(1)}$$

k = 3; k − 1 = 2; N − k = N − 3.

Multiple Linear Regression Model (2)

The PRE is:

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + u_i \,. \qquad \textbf{(2)}$$

$k = 4; \quad k - 1 = 3.$

Marginal or partial effect of wgt_i

The marginal effect of wgt_i on price_i is obtained by partially differentiating regression equation (2) with respect to wgt_i.

$$\frac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i \,.$$

- **Marginal effect of** wgt_i **on** $price_i$ **is a *linear* function of** wgt_i; it is **not a *constant***.

Marginal or partial effect of mpg_i

The marginal or partial effect of mpg_i mpg_i on price_i is obtained by partially differentiating regression equation (2) with respect to mpg_i.

$$\frac{\partial\, \text{price}_i}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{mpg}_i} = \beta_4 \,.$$

- **Marginal effect of** mpg_i **on** $price_i$ **is *constant***: it does not vary with any observable variable.

Hypotheses of interest

1. The ***marginal*** **effect of** wgt_i **on** $price_i$ **is *zero***: i.e., wgt_i **has no effect on** $price_i$; or **car** $price_i$ **is unrelated to car** wgt_i.

2. The ***marginal*** **effect of** wgt_i **on** $price_i$ **is *constant***: i.e., it does not depend on wgt_i.

3. The ***marginal*** **effect of** mpg_i **on** $price_i$ **is *zero***: i.e., mpg_i **has no effect on** $price_i$; or **car** $price_i$ **is unrelated to fuel efficiency** as measured by mpg_i.

1. The ***marginal*** **effect of** ***wgt_i*** **on** ***price_i*** **is** ***zero***: i.e., ***wgt_i*** **has no effect on** ***price_i***; or **car** ***price_i*** **is unrelated to car** ***wgt_i***.

- H_0: $\beta_2 = 0$ *and* $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{wgt}_i} = \beta_2 + 2\beta_3\text{wgt}_i = 0$.

  Restricted **model** corresponding to H_0: set $\beta_2 = 0$ and $\beta_3 = 0$ in PRE (2).

  $$\text{price}_i = \beta_1 + \beta_4\text{mpg}_i + u_i.$$

  $k_0 = 2$; $k_0 - 1 = 1$.

- H_1: $\beta_2 \neq 0$ *and/or* $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{wgt}_i} = \beta_2 + 2\beta_3\text{wgt}_i$.

  Unrestricted **model** corresponding to H_1: is PRE (2).

  $$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + u_i. \qquad \textbf{(2)}$$

  $k = 4$; $k - 1 = 3$.

2. The ***marginal*** **effect of** ***wgt_i*** **on** ***price_i*** **is** ***constant***: i.e., it does not depend on ***wgt_i***.

- H_0: $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{wgt}_i} = \beta_2 + 2\beta_3\text{wgt}_i = \beta_2$.

  Restricted **model** corresponding to H_0: set $\beta_3 = 0$ in PRE (2).

  $$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_4\text{mpg}_i + u_i.$$

  $k_0 = 3$; $k_0 - 1 = 2$

- H_1: $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{wgt}_i} = \beta_2 + 2\beta_3\text{wgt}_i$.

  Unrestricted **model** corresponding to H_1: is PRE (2).

  $$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + u_i. \qquad \textbf{(2)}$$

  $k = 4$; $k - 1 = 3$.

3. The ***marginal* effect of *mpg_i* on *$price_i$* is *zero***: i.e., ***mpg_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to fuel efficiency** as measured by ***mpg_i***.

- H_0: $\beta_4 = 0 \quad \Rightarrow \quad \dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4 = 0$.

  ***Restricted* model** corresponding to H_0: set $\beta_4 = 0$ in PRE (2).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + u_i.$$

  $k_0 = 3$; $k_0 - 1 = 2$

- H_1: $\beta_4 \neq 0 \quad \Rightarrow \quad \dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4$.

  ***Unrestricted* model** corresponding to H_1: is PRE (2).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + u_i. \qquad \textbf{(2)}$$

  $k = 4$; $k - 1 = 3$.

The OLS SRE for Model (2)

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + u_i . \qquad \textbf{(2)}$$

```
. regress price wgt wgtsq mpg

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  3,    70) =   16.47
   Model |   262753599     3  87584533.2               Prob > F      =  0.0000
Residual |   372311797    70  5318739.95               R-squared     =  0.4137
---------+------------------------------               Adj R-squared =  0.3886
   Total |   635065396    73  8699525.97               Root MSE      =  2306.2

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |  -9.039563   2.905512     -3.111   0.003      -14.83442   -3.244703
   wgtsq |   .0016794    .000443      3.791   0.000       .0007958     .002563
     mpg |  -124.7675   81.49014     -1.531   0.130      -287.2945    37.75945
   _cons |   19804.79   5751.709      3.443   0.001       8333.365    31276.21
------------------------------------------------------------------------------
```

```
. test wgt wgtsq
```

F-test of hypothesis 1

```
 ( 1)  wgt = 0.0
 ( 2)  wgtsq = 0.0

       F(  2,    70) =   11.59
            Prob > F =    0.0000
```

```
. test wgtsq
```

F-test of hypothesis 2

```
 ( 1)  wgtsq = 0.0

       F(  1,    70) =   14.37
            Prob > F =    0.0003
```

```
. test mpg
```

F-test of hypothesis 3

```
 ( 1)  mpg = 0.0

       F(  1,    70) =    2.34
            Prob > F =    0.1303
```

```
. lincom mpg
```

t-test of hypothesis 3

```
 ( 1)  mpg = 0.0

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     (1) |  -124.7675   81.49014     -1.531   0.130      -287.2945    37.75945
------------------------------------------------------------------------------
```

Multiple Linear Regression Model (3)

The PRE is:

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + u_i . \qquad \textbf{(3)}$$

$k = 5$; $k - 1 = 4$.

Marginal or partial effect of wgt

$$\frac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i .$$

- **Marginal effect of *wgt_i* on *$price_i$* is a *linear* function of *wgt_i***; it is **not a *constant***.

Marginal or partial effect of mpg_i

$$\frac{\partial\, \text{price}_i}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{mpg}_i} = \beta_4 + 2\beta_5 \text{mpg}_i .$$

- **Marginal effect of *mpg_i* on *$price_i$* is a *linear* function of *mpg_i***; it is **not a *constant***.

Hypotheses of interest

1. The ***marginal* effect of *wgt_i* on *$price_i$* is *zero***: i.e., ***wgt_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to car *wgt_i***.

2. The ***marginal* effect of *wgt_i* on *$price_i$* is *constant***: i.e., it does not depend on ***wgt_i*** or ***mpg_i***.

3. The ***marginal* effect of *mpg_i* on *$price_i$* is *zero***: i.e., ***mpg_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to fuel efficiency** as measured by ***mpg_i***.

4. The ***marginal* effect of *mpg_i* on *$price_i$* is *constant***: i.e., it does not depend on ***mpg_i*** or ***wgt_i***.

1. The ***marginal*** **effect of** $\boldsymbol{wgt_i}$ **on** $\boldsymbol{price_i}$ **is** ***zero***: i.e., $\boldsymbol{wgt_i}$ **has no effect on** $\boldsymbol{price_i}$; or **car** $\boldsymbol{price_i}$ **is unrelated to car** $\boldsymbol{wgt_i}$.

- H_0: $\beta_2 = 0$ *and* $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, wgt_i} = \beta_2 + 2\beta_3 wgt_i = 0$.

  Restricted **model** corresponding to H_0: set $\beta_2 = 0$ and $\beta_3 = 0$ in PRE (3).

  $$price_i = \beta_1 + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i.$$

  $k_0 = 3$; $k_0 - 1 = 2$.

- H_1: $\beta_2 \neq 0$ *and/or* $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, wgt_i} = \beta_2 + 2\beta_3 wgt_i$

  Unrestricted **model** corresponding to H_1: is PRE (3).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i. \qquad \textbf{(3)}$$

  $k = 5$; $k - 1 = 4$.

2. The ***marginal*** **effect of** $\boldsymbol{wgt_i}$ **on** $\boldsymbol{price_i}$ **is** ***constant***: i.e., it does not depend on $\boldsymbol{wgt_i}$ or $\boldsymbol{mpg_i}$.

- H_0: $\beta_3 = 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, wgt_i} = \beta_2$

  Restricted **model** corresponding to H_0: set $\beta_3 = 0$ in PRE (3).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i.$$

  $k_0 = 4$; $k_0 - 1 = 3$.

- H_1: $\beta_3 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, wgt_i} = \beta_2 + 2\beta_3 wgt_i$

  Unrestricted **model** corresponding to H_1: is PRE (3).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i. \qquad \textbf{(3)}$$

  $k = 5$; $k - 1 = 4$.

3. The ***marginal*** **effect of** ***mpg_i*** **on** ***$price_i$*** **is** ***zero***: i.e., ***mpg_i*** **has no effect on** ***$price_i$***; or **car** ***$price_i$*** **is unrelated to fuel efficiency** as measured by ***mpg_i***.

- H_0: $\beta_4 = 0$ *and* $\beta_5 = 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{mpg}_i} = \beta_4 + 2\beta_5\text{mpg}_i = 0$.

Restricted **model** corresponding to H_0: set $\beta_4 = 0$ and $\beta_5 = 0$ in PRE (3).

$$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + u_i.$$

$k_0 = 3$; $k_0 - 1 = 2$.

- H_1: $\beta_4 \neq 0$ *and/or* $\beta_5 \neq 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{mpg}_i} = \beta_4 + 2\beta_5\text{mpg}_i$.

Unrestricted **model** corresponding to H_1: is PRE (3).

$$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + \beta_5\text{mpg}_i^2 + u_i. \qquad \textbf{(3)}$$

$k = 5$; $k - 1 = 4$.

4. The ***marginal*** **effect of** ***mpg_i*** **on** ***$price_i$*** **is** ***constant***: i.e., it does not depend on ***mpg_i*** or ***wgt_i***.

- H_0: $\beta_5 = 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{mpg}_i} = \beta_4 + 2\beta_5\text{mpg}_i = \beta_4$.

Restricted **model** corresponding to H_0: set $\beta_5 = 0$ in PRE (3).

$$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + u_i.$$

$k_0 = 4$; $k_0 - 1 = 3$.

- H_1: $\beta_5 \neq 0$ $\Rightarrow$ $\dfrac{\partial\,\text{price}_i}{\partial\,\text{mpg}_i} = \beta_4 + 2\beta_5\text{mpg}_i$

Unrestricted **model** corresponding to H_1: is PRE (3).

$$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + \beta_5\text{mpg}_i^2 + u_i. \qquad \textbf{(3)}$$

$k = 5$; $k - 1 = 4$.

Model (3)

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + u_i . \qquad \textbf{(3)}$$

The OLS SRE for Model (3)

```
. regress price wgt wgtsq mpg mpgsq

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  4,    69) =   12.93
   Model |   272062621     4  68015655.2               Prob > F      =  0.0000
Residual |   363002775    69  5260909.79               R-squared     =  0.4284
---------+------------------------------               Adj R-squared =  0.3953
   Total |   635065396    73  8699525.97               Root MSE      =  2293.7

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |   -7.99723   2.994029     -2.671   0.009      -13.97015   -2.024306
   wgtsq |   .0014407   .0004757      3.028   0.003       .0004916    .0023898
     mpg |  -615.2419   377.5204     -1.630   0.108      -1368.375    137.8907
   mpgsq |   9.323582   7.009083      1.330   0.188      -4.659156    23.30632
   _cons |   24884.95   6878.057      3.618   0.001       11163.61     38606.3
------------------------------------------------------------------------------
```

```
. test wgt wgtsq                        F-test of hypothesis 1

 ( 1)  wgt = 0.0
 ( 2)  wgtsq = 0.0

       F(  2,    69) =    5.34
            Prob > F =    0.0070

. test wgtsq                            F-test of hypothesis 2

 ( 1)  wgtsq = 0.0

       F(  1,    69) =    9.17
            Prob > F =    0.0035

. test mpg mpgsq                        F-test of hypothesis 3

 ( 1)  mpg = 0.0
 ( 2)  mpgsq = 0.0

       F(  2,    69) =    2.07
            Prob > F =    0.1340

. test mpgsq                            F-test of hypothesis 4

 ( 1)  mpgsq = 0.0

       F(  1,    69) =    1.77
            Prob > F =    0.1878
```

Multiple Linear Regression Model (4)

The PRE is:

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + u_i . \qquad \textbf{(4)}$$

k = 6; k − 1 = 5.

Marginal or partial effect of wgt_i

$$\frac{\partial\, \text{price}_i}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{wgt}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{wgt}_i} = \beta_2 + 2\beta_3 \text{wgt}_i + \beta_6 \text{mpg}_i .$$

- **Marginal effect of *wgt_i* on *$price_i$* is a *linear* function of *wgt_i* and *mpg_i***; it is **not a *constant***.

Marginal or partial effect of mpg_i

$$\frac{\partial\, \text{price}_i}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i)}{\partial\, \text{mpg}_i} = \frac{\partial\, E(\text{price}_i \mid \bullet)}{\partial\, \text{mpg}_i} = \beta_4 + 2\beta_5 \text{mpg}_i + \beta_6 \text{wgt}_i .$$

- **Marginal effect of *mpg_i* on *$price_i$* is a *linear* function of *mpg_i* and *wgt_i***; it is **not a *constant***.

Hypotheses of interest

1. The ***marginal*** **effect of *wgt_i* on *$price_i$* is *zero***: i.e., ***wgt_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to car *wgt_i***.

2. The ***marginal*** **effect of *wgt_i* on *$price_i$* is *constant***: i.e., it does not depend on ***wgt_i*** and/or ***mpg_i***.

3. The ***marginal*** **effect of *mpg_i* on *$price_i$* is *zero***: i.e., ***mpg_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to fuel efficiency** as measured by ***mpg_i***.

4. The ***marginal*** **effect of *mpg_i* on *$price_i$* is *constant***: i.e., it does not depend on ***mpg_i*** and/or ***wgt_i***.

1. The ***marginal*** **effect of** $\boldsymbol{wgt_i}$ **on** $\boldsymbol{price_i}$ **is *zero***: i.e., $\boldsymbol{wgt_i}$ **has no effect on** $\boldsymbol{price_i}$; or **car** $\boldsymbol{price_i}$ **is unrelated to car** $\boldsymbol{wgt_i}$.

- H_0: $\beta_2 = 0$ *and* $\beta_3 = 0$ *and* $\beta_6 = 0$ $\Rightarrow$ $\dfrac{\partial \, price_i}{\partial \, wgt_i} = \beta_2 + 2\beta_3 wgt_i + \beta_6 mpg_i = 0$

  Restricted **model** corresponding to H_0: set $\beta_2 = 0$ and $\beta_3 = 0$ and $\beta_6 = 0$ in PRE (4).

  $$price_i = \beta_1 + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i .$$

  $k_0 = 3$; $k_0 - 1 = 2$.

- H_1: $\beta_2 \neq 0$ *and/or* $\beta_3 \neq 0$ *and/or* $\beta_6 \neq 0$ $\Rightarrow$ $\dfrac{\partial \, price_i}{\partial \, wgt_i} = \beta_2 + 2\beta_3 wgt_i + \beta_6 mpg_i$

  Unrestricted **model** corresponding to H_1: is PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + \beta_6 wgt_i mpg_i + u_i . \quad \textbf{(4)}$$

  $k = 6$; $k - 1 = 5$.

2. The ***marginal*** **effect of** $\boldsymbol{wgt_i}$ **on** $\boldsymbol{price_i}$ **is *constant***: i.e., it does not depend on $\boldsymbol{wgt_i}$ and/or $\boldsymbol{mpg_i}$.

- H_0: $\beta_3 = 0$ *and* $\beta_6 = 0$ $\Rightarrow$ $\dfrac{\partial \, price_i}{\partial \, wgt_i} = \beta_2 + 2\beta_3 wgt_i + \beta_6 mpg_i = \beta_2$

  Restricted **model** corresponding to H_0: set $\beta_3 = 0$ and $\beta_6 = 0$ in PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_4 mpg_i + \beta_5 mpg_i^2 + u_i .$$

  $k_0 = 4$; $k_0 - 1 = 3$.

- H_1: $\beta_3 \neq 0$ *and/or* $\beta_6 \neq 0$ $\Rightarrow$ $\dfrac{\partial \, price_i}{\partial \, wgt_i} = \beta_2 + 2\beta_3 wgt_i + \beta_6 mpg_i$

  Unrestricted **model** corresponding to H_1: is PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + \beta_6 wgt_i mpg_i + u_i . \quad \textbf{(4)}$$

  $k = 6$; $k - 1 = 5$.

3. The ***marginal*** **effect of** $\boldsymbol{mpg_i}$ **on** $\boldsymbol{price_i}$ **is** ***zero***: i.e., $\boldsymbol{mpg_i}$ **has no effect on** $\boldsymbol{price_i}$; or **car** $\boldsymbol{price_i}$ **is unrelated to fuel efficiency** as measured by $\boldsymbol{mpg_i}$.

- H_0: $\beta_4 = 0$ *and* $\beta_5 = 0$ *and* $\beta_6 = 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4 + 2\beta_5 mpg_i + \beta_6 wgt_i = 0$

  Restricted **model** corresponding to H_0: set $\beta_4 = 0$ and $\beta_5 = 0$ and $\beta_6 = 0$ in PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + u_i.$$

  $k_0 = 3$; $k_0 - 1 = 2$.

- H_1: $\beta_4 \neq 0$ *and/or* $\beta_5 \neq 0$ *and/or* $\beta_6 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4 + 2\beta_5 mpg_i + \beta_6 wgt_i$

  Unrestricted **model** corresponding to H_1: is PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + \beta_6 wgt_i mpg_i + u_i. \quad \textbf{(4)}$$

  $k = 6$; $k - 1 = 5$.

4. The ***marginal*** **effect of** $\boldsymbol{mpg_i}$ **on** $\boldsymbol{price_i}$ **is** ***constant***: i.e., it does not depend on $\boldsymbol{mpg_i}$ and/or $\boldsymbol{wgt_i}$.

- H_0: $\beta_5 = 0$ *and* $\beta_6 = 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4 + 2\beta_5 mpg_i + \beta_6 wgt_i = \beta_4$

  Restricted **model** corresponding to H_0: set $\beta_5 = 0$ and $\beta_6 = 0$ in PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + u_i.$$

  $k_0 = 4$; $k_0 - 1 = 3$.

- H_1: $\beta_5 \neq 0$ *and/or* $\beta_6 \neq 0$ $\Rightarrow$ $\dfrac{\partial\, price_i}{\partial\, mpg_i} = \beta_4 + 2\beta_5 mpg_i + \beta_6 wgt_i$

  Unrestricted **model** corresponding to H_1: is PRE (4).

  $$price_i = \beta_1 + \beta_2 wgt_i + \beta_3 wgt_i^2 + \beta_4 mpg_i + \beta_5 mpg_i^2 + \beta_6 wgt_i mpg_i + u_i. \quad \textbf{(4)}$$

  $k = 6$; $k - 1 = 5$.

Model (4)

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + u_i. \qquad \textbf{(4)}$$

The OLS SRE for Model (4)

```
. regress price wgt wgtsq mpg mpgsq wgtmpg

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  5,    68) =   12.84
   Model |   308384833     5  61676966.6               Prob > F      =  0.0000
Residual |   326680563    68  4804125.93               R-squared     =  0.4856
---------+------------------------------               Adj R-squared =  0.4478
   Total |   635065396    73  8699525.97               Root MSE      =  2191.8

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |  -31.88985   9.148215     -3.486   0.001      -50.14483   -13.63487
   wgtsq |   .0034574   .0008629      4.007   0.000       .0017355    .0051792
     mpg |  -3549.495   1126.464     -3.151   0.002      -5797.318   -1301.672
   mpgsq |   38.74472   12.62339      3.069   0.003       13.55514    63.9343
  wgtmpg |   .5421927   .1971854      2.750   0.008       .1487154    .9356701
   _cons |   92690.55   25520.53      3.632   0.001       41765.12      143616
------------------------------------------------------------------------------
```

F-test of hypothesis 1

```
. test wgt wgtsq wgtmpg

 ( 1)  wgt = 0.0
 ( 2)  wgtsq = 0.0
 ( 3)  wgtmpg = 0.0

       F(  3,    68) =    6.42
            Prob > F =    0.0007
```

F-test of hypothesis 2

```
. test wgtsq wgtmpg

 ( 1)  wgtsq = 0.0
 ( 2)  wgtmpg = 0.0

       F(  2,    68) =    8.80
            Prob > F =    0.0004
```

F-test of hypothesis 3

```
. test mpg mpgsq wgtmpg

 ( 1)  mpg = 0.0
 ( 2)  mpgsq = 0.0
 ( 3)  wgtmpg = 0.0

       F(  3,    68) =    4.03
            Prob > F =    0.0106
```

F-test of hypothesis 4

```
. test mpgsq wgtmpg

 ( 1)  mpgsq = 0.0
 ( 2)  wgtmpg = 0.0

       F(  2,    68) =    4.75
            Prob > F =    0.0117
```

Multiple Linear Regression Model (5)

The PRE is:

$$\text{price}_i = \beta_1 + \beta_2\text{wgt}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i + \beta_5\text{mpg}_i^2 + \beta_6\text{wgt}_i\text{mpg}_i + \beta_7\text{foreign}_i + u_i \qquad \textbf{... (5)}$$

k = 7; k − 1 = 6.

Marginal or partial effect of wgt_i

$$\frac{\partial\,\text{price}_i}{\partial\,\text{wgt}_i} = \frac{\partial\,E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i)}{\partial\,\text{wgt}_i} = \frac{\partial\,E(\text{price}_i \mid \bullet)}{\partial\,\text{wgt}_i} = \beta_2 + 2\beta_3\text{wgt}_i + \beta_6\text{mpg}_i$$

- **Marginal effect of *wgt_i* on *$price_i$* is a *linear* function of *wgt_i* and *mpg_i***; it is **not a *constant***.

Marginal or partial effect of mpg_i

$$\frac{\partial\,\text{price}_i}{\partial\,\text{mpg}_i} = \frac{\partial\,E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i)}{\partial\,\text{mpg}_i} = \frac{\partial\,E(\text{price}_i \mid \bullet)}{\partial\,\text{mpg}_i} = \beta_4 + 2\beta_5\text{mpg}_i + \beta_6\text{wgt}_i$$

- **Marginal effect of *mpg_i* on *$price_i$* is a *linear* function of *mpg_i* and *wgt_i***; it is **not a *constant***.

Hypotheses of interest

1. The ***marginal* effect of *wgt_i* on *$price_i$* is *zero***: i.e., ***wgt_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to car *wgt_i***.

2. The ***marginal* effect of *wgt_i* on *$price_i$* is *constant***: i.e., it does not depend on ***wgt_i*** and/or ***mpg_i***.

3. The ***marginal* effect of *mpg_i* on *$price_i$* is *zero***: i.e., ***mpg_i* has no effect on *$price_i$***; or **car *$price_i$* is unrelated to fuel efficiency** as measured by ***mpg_i***.

4. The ***marginal* effect of *mpg_i* on *$price_i$* is *constant***: i.e., it does not depend on ***mpg_i*** and/or ***wgt_i***.

Interpretation of the slope coefficient β_7 on the $foreign_i$ dummy variable regressor

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + \beta_7 \text{foreign}_i + u_i \quad \ldots (5)$$

Compare the expressions implied by regression equation (5) for the conditional mean prices of foreign and domestic cars that have the same weight and fuel efficiency.

- The **conditional mean price of *foreign* cars** is obtained from equation (5) by **setting *foreign*** $foreign_i = 1$ and taking the conditional expectation of $price_i$:

$$\begin{aligned} &E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 1) \\ &\quad = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + \beta_7 \end{aligned}$$

- The **conditional mean price of *domestic* cars** is obtained from equation (5) by **setting** $foreign_i = 0$ and taking the conditional expectation of $price_i$:

$$\begin{aligned} &E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 0) \\ &\quad = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i \end{aligned}$$

- Take the difference between the conditional mean price of foreign cars and the conditional mean price of "similar" domestic cars, where "similar" means foreign and domestic cars with **the *same* values of** wgt_i **and** mpg_i.

$$\begin{aligned} &E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 1) - E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 0) \\ &\quad = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + \beta_7 \\ &\qquad - \beta_1 - \beta_2 \text{wgt}_i - \beta_3 \text{wgt}_i^2 - \beta_4 \text{mpg}_i - \beta_5 \text{mpg}_i^2 - \beta_6 \text{wgt}_i \text{mpg}_i \\ &\quad = \beta_7 \end{aligned}$$

Result: The coefficient β_7 on the foreign_i dummy variable in equation (5) is

$$\beta_7 = E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 1) - E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 0)$$

= the mean price of ***foreign* cars** with given values of ***wgt$_i$* and *mpg$_i$***
minus
the mean price of ***domestic* cars** with the same values of ***wgt$_i$* and *mpg$_i$***

= the ***adjusted* mean price difference** between ***foreign* and *domestic* cars of *the same weight* and *fuel efficiency***, that have **the *same* values of *wgt$_i$* and *mpg$_i$***

- Compare β_7 in equation (5) with β_2 in the following simple regression equation:

$$\text{price}_i = \beta_1 + \beta_2 \text{foreign}_i + u_i$$

$$\beta_2 = E(\text{price}_i \mid \text{foreign}_i = 1) - E(\text{price}_i \mid \text{foreign}_i = 0)$$

= the mean price of all ***foreign* cars**
minus
the mean price of all ***domestic* cars**

= the ***unadjusted* mean price difference** between ***all foreign* and *all domestic* cars** regardless of their *weight* and *fuel efficiency*

Hypothesis of interest

5. There is **no difference between the mean price of *foreign* and *domestic* cars** that have **the *same weight* and *fuel efficiency***.

- H_0: $\beta_7 = 0 \;\Rightarrow$

$$E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 1) - E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 0) = 0$$

- H_1: $\beta_7 \neq 0 \;\Rightarrow$

$$E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 1) - E(\text{price}_i \mid \text{wgt}_i, \text{mpg}_i, \text{foreign}_i = 0) \neq 0$$

Model (5)

$$\text{price}_i = \beta_1 + \beta_2 \text{wgt}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i + \beta_5 \text{mpg}_i^2 + \beta_6 \text{wgt}_i \text{mpg}_i + \beta_7 \text{foreign}_i + u_i.$$

The OLS SRE for Model (5)

```
. regress price wgt wgtsq mpg mpgsq wgtmpg foreign

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  6,    67) =   14.48
   Model |   358503838     6  59750639.7               Prob > F      =  0.0000
Residual |   276561558    67  4127784.45               R-squared     =  0.5645
---------+------------------------------               Adj R-squared =  0.5255
   Total |   635065396    73  8699525.97               Root MSE      =  2031.7

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |   -15.3063   9.724082     -1.574   0.120      -34.71565    4.103051
   wgtsq |   .0020985   .0008898      2.358   0.021       .0003224    .0038747
     mpg |  -1407.999   1211.602     -1.162   0.249      -3826.366    1010.368
   mpgsq |   14.23503   13.65253      1.043   0.301      -13.01554     41.4856
  wgtmpg |   .2373812   .2026331      1.171   0.246      -.1670761    .6418385
 foreign |   2749.963   789.1946      3.485   0.001       1174.724    4325.202
   _cons |   39826.01    28102.9      1.417   0.161       -16267.6    95919.63
------------------------------------------------------------------------------
```

```
. test wgt wgtsq wgtmpg
```

F-test of hypothesis 1

```
 ( 1)  wgt = 0.0
 ( 2)  wgtsq = 0.0
 ( 3)  wgtmpg = 0.0

       F(  3,    67) =    8.46
            Prob > F =    0.0001
```

```
. test wgtsq wgtmpg
```

F-test of hypothesis 2

```
 ( 1)  wgtsq = 0.0
 ( 2)  wgtmpg = 0.0

       F(  2,    67) =    4.55
            Prob > F =    0.0140
```

```
. test mpg mpgsq wgtmpg
```

F-test of hypothesis 3

```
 ( 1)  mpg = 0.0
 ( 2)  mpgsq = 0.0
 ( 3)  wgtmpg = 0.0

       F(  3,    67) =    0.56
            Prob > F =    0.6419
```

```
. test mpgsq wgtmpg
```

F-test of hypothesis 4

```
 ( 1)  mpgsq = 0.0
 ( 2)  wgtmpg = 0.0

       F(  2,    67) =    0.69
            Prob > F =    0.5068
```

The OLS SRE for Model (5)

```
. regress price wgt wgtsq mpg mpgsq wgtmpg foreign

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  6,    67) =   14.48
   Model |   358503838     6  59750639.7               Prob > F      =  0.0000
Residual |   276561558    67  4127784.45               R-squared     =  0.5645
---------+------------------------------               Adj R-squared =  0.5255
   Total |   635065396    73  8699525.97               Root MSE      =  2031.7

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |   -15.3063   9.724082     -1.574   0.120      -34.71565    4.103051
   wgtsq |   .0020985   .0008898      2.358   0.021       .0003224    .0038747
     mpg |  -1407.999   1211.602     -1.162   0.249      -3826.366    1010.368
   mpgsq |   14.23503   13.65253      1.043   0.301      -13.01554     41.4856
  wgtmpg |   .2373812   .2026331      1.171   0.246      -.1670761    .6418385
 foreign |   2749.963   789.1946      3.485   0.001       1174.724    4325.202
   _cons |   39826.01    28102.9      1.417   0.161       -16267.6    95919.63
------------------------------------------------------------------------------
```

```
. test foreign = 0
```

F-test of hypothesis 5

```
 ( 1)  foreign = 0.0

       F(  1,    67) =   12.14
            Prob > F =    0.0009
```

```
. test foreign
```

F-test of hypothesis 5 (again)

```
 ( 1)  foreign = 0.0

       F(  1,    67) =   12.14
            Prob > F =    0.0009
```

```
. lincom foreign
```

t-test of hypothesis 5

```
 ( 1)  foreign = 0.0

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     (1) |   2749.963   789.1946      3.485   0.001       1174.724    4325.202
------------------------------------------------------------------------------
```