ECON 351* -- Addendum to NOTE 8

The P-Value Decision Rule for Hypothesis Tests

Formulation 2 of the Decision Rule for t-Tests

Formulation 2: Determine if the **p-value for t_0,** the calculated sample value of the test statistic, **is *smaller* or *larger* than the chosen significance level** α.

- ***Definition:*** The **p-value** (or **probability value**) associated with the calculated sample value of the test statistic is defined as the ***lowest*** **significance level at which the null hypothesis H_0 *can be rejected***, given the calculated sample value of the test statistic.

- ***Interpretation***

- The **p-value** is the **probability of obtaining a *sample value* of the test statistic *as extreme as* the one we computed *if the null hypothesis H_0 is true***.

- **P-values** are ***inverse*** **measures** of **the strength of evidence *against* the *null* hypothesis H_0**.

  - ***Small*** **p-values** -- **p-values** ***close to zero*** -- constitute ***strong*** **evidence against the null hypothesis H_0**.

  - ***Large*** **p-values** -- **p-values** ***close to one*** -- provide only ***weak*** **evidence against the null hypothesis H_0**.

Decision Rule -- Formulation 2: the P-Value Decision Rule

1. If the **p-value** for the calculated sample value of the test statistic ***is less than*** the chosen **significance level** α, ***reject*** **the null hypothesis** at significance level α.

   p-value $<\ \alpha \ \Rightarrow$ ***reject*** $\mathbf{H_0}$ at **significance level** α.

2. If the **p-value** for the calculated sample value of the test statistic ***is greater than or equal to*** the chosen **significance level** α, ***retain (i.e., do not reject)*** **the null hypothesis** at significance level α.

   p-value $\geq\ \alpha \ \Rightarrow$ ***retain*** $\mathbf{H_0}$ at **significance level** α.

P-Values for Two-Tail and One-Tail t-Tests

Let t_0 be the calculated sample value of a t-statistic under some null hypothesis H_0.

- ***Two-tail*** **t-tests**

  H_0: $\beta_2 = b_2$
  H_1: $\beta_2 \neq b_2$ a ***two-sided*** **alternative hypothesis**

  two-tail **p-value for** $\mathbf{t_0} = \mathbf{Pr(|t| > |t_0|)}$

- ***Left-tail*** **t-tests**

  H_0: $\beta_2 = b_2$
  H_1: $\beta_2 < b_2$ a ***one-sided left-sided*** **alternative hypothesis**

  left-tail **p-value for** $\mathbf{t_0} = \mathbf{Pr(t < t_0)}$

- ***Right-tail*** **t-tests**

  H_0: $\beta_2 = b_2$
  H_1: $\beta_2 > b_2$ a ***one-sided right-sided*** **alternative hypothesis**

  right-tail **p-value for** $\mathbf{t_0} = \mathbf{Pr(t > t_0)}$

❑ P-values for *two-tail* t-tests

- ***Null*** **and** ***Alternative*** **Hypotheses**

  H_0: $\beta_2 = b_2$
  H_1: $\beta_2 \neq b_2$ a ***two-sided*** **alternative hypothesis**.

- **Definition of *two-tail* p-value for t_0**

  t_0 = the **calculated *sample value* of the t-statistic** for a given null hypothesis.

  The ***two-tail*** **p-value of t_0** is the probability that the null distribution of the test statistic takes an ***absolute*** **value *greater than* the *absolute* value of t_0**, where the absolute value of t_0 is denoted as $|t_0|$. That is,

  $$\textit{two-tail}\ \textbf{p-value for } t_0 = \Pr(|t| > |t_0|)$$
  $$= \Pr(t > t_0) + \Pr(t < -t_0) = 2\Pr(t > t_0) \quad \text{if } t_0 > 0$$
  $$= \Pr(t < t_0) + \Pr(t > -t_0) = 2\Pr(t < t_0) \quad \text{if } t_0 < 0$$

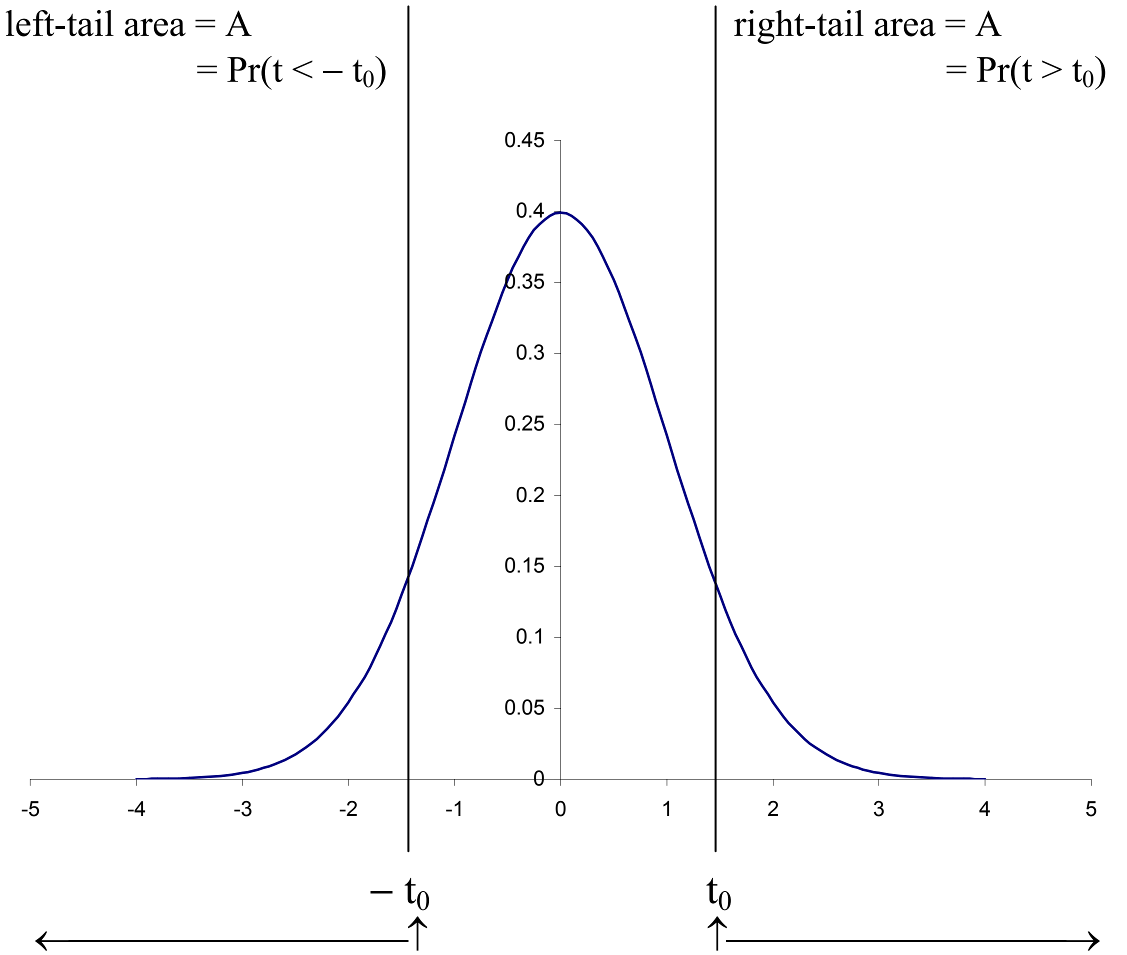
  Two-tail **p-value of t_0** is **the *probability* of obtaining a t value *greater in absolute size* than the sample value t_0** if the null hypothesis H_0: $\beta_2 = b_2$ is in fact true.

  Remember that a t-distribution is symmetric around its mean of zero.

Two-tail p-values for sample t-statistic t_0

Case 1: $t_0 > 0$

$$t_0 > 0 \quad \Rightarrow \quad -t_0 < 0$$



left-tail area = Pr($t < -t_0$) = Pr($t > t_0$)

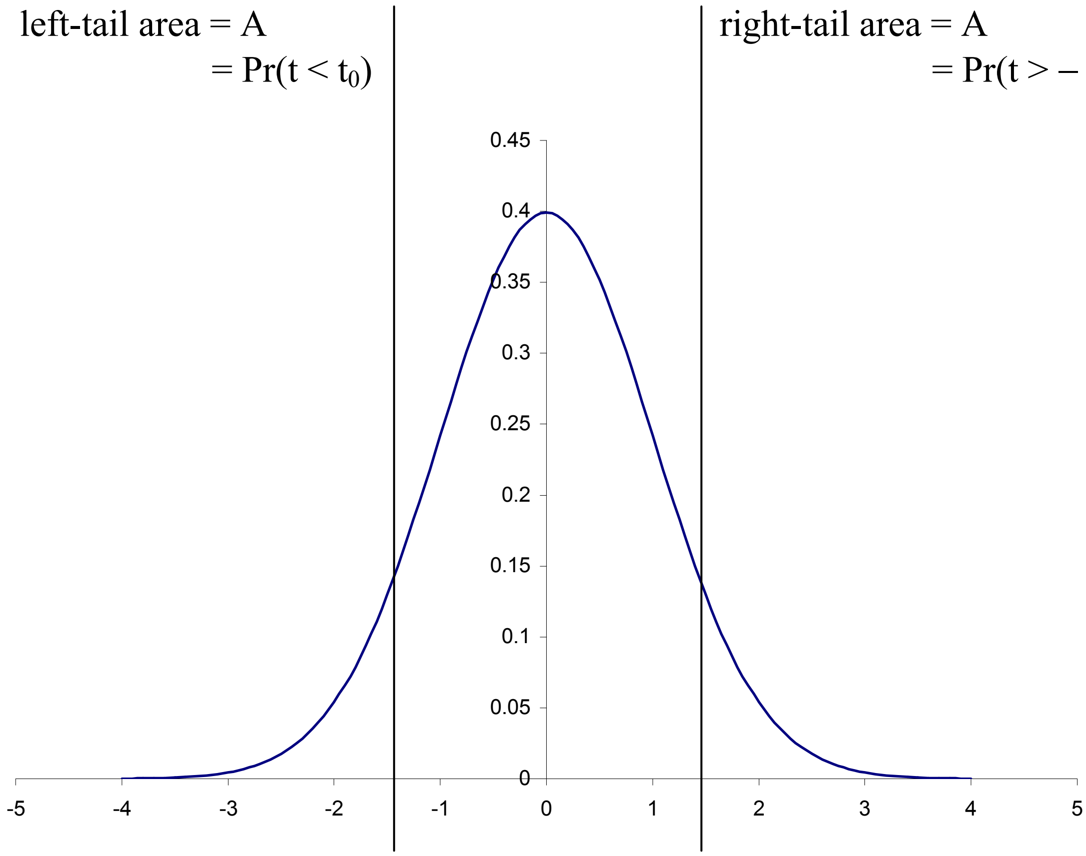
right-tail area = Pr($t > t_0$)

two-tail **p-value for t_0**
$$= \text{left-tail area A} + \text{right-tail area A}$$
$$= \Pr(t < -t_0) + \Pr(t > t_0)$$
$$= \Pr(t > t_0) + \Pr(t > t_0)$$
$$= 2\Pr(t > t_0)$$
$$= \Pr(|t| > |t_0|)$$

Two-tail p-values for sample t-statistic t_0

Case 2: $t_0 < 0$

$$t_0 < 0 \quad \Rightarrow \quad -t_0 > 0$$



t_0 $\qquad\qquad$ $-t_0$

left-tail area = $\Pr(t < t_0)$ $\qquad$ **right-tail area = $\Pr(t > -t_0)$**
$= \Pr(t < t_0)$

two-tail **p-value for t_0**

$$
\begin{aligned}
&= \text{left-tail area A} + \text{right-tail area A}\\
&= \Pr(t < t_0) + \Pr(t > -t_0)\\
&= \Pr(t < t_0) + \Pr(t < t_0)\\
&= 2\Pr(t < t_0)\\
&= 2\Pr(t > -t_0)\\
&= \Pr(|t| > |t_0|)
\end{aligned}
$$

❑ Computing *two-tail* critical values and p-values for t-statistics in *Stata 7*

Basic Syntax: The new *Stata 7* statistical functions for the t-distribution are **ttail(*df*, *t₀*)** and **invttail(*df*, *p*)**. The new *Stata 7* function **ttail(*df*, t_0)** replaces the **tprob(*df*, t_0)** function of previous releases, and in many ways is easier to use. Similarly, the new *Stata 7* function **invttail(*df*, t_0)** replaces the **invt(*df*, t_0)** function of previous releases.

- **ttail(*df*, t_0)** computes the ***right-tail* (upper-tail) p-value of a t-statistic** that has degrees of freedom ***df*** and calculated sample value t_0. It returns the probability that $t > t_0$, i.e., the value of $\Pr(t > t_0)$.

- **invttail(*df*, *p*)** computes the ***right-tail* critical value of a t-distribution** with degrees of freedom ***df*** and ***probability level p***. Let α denote the chosen significance level of the test. For two-tail t-tests, set $p = \alpha/2$. For one-tail t-tests, set $p = \alpha$.

- If **ttail(*df*, t_0) = *p***, then **invttail(*df*, *p*) = t_0**.

Usage: The statistical functions **ttail(*df*, t_0)** and **invttail(*df*, *p*)** must be used with *Stata 7* commands such as **display**, **generate**, **replace**, or **scalar**; they cannot be used by themselves. For example, simply typing **ttail(72, 2.0)** will produce an error message. Instead, to obtain the right-tail p-value for a calculated t-statistic that equals 2.0 and has the t-distribution with 60 degrees of freedom, enter the **display** command:

```
display ttail(72, 2.0)
.02463658
```

Examples: Suppose that sample size N = 74 and K = 2, so that the degrees of freedom for t-tests based on a linear regression with two regression coefficients equal N – K = N – 2 = 74 – 2 = 72.

Example 1: Two-tail t-tests

- The following are the ***two-tail*** **critical values** $\mathbf{t_{\alpha/2}[72]}$ of the t[72] distribution, where α is the chosen significance level for the two-tail t-test.

$$\alpha = 0.01 \Rightarrow \quad \alpha/2 = 0.005: \quad t_{\alpha/2}[72] = t_{0.005}[72] = 2.646;$$
$$\alpha = 0.02 \Rightarrow \quad \alpha/2 = 0.01: \quad t_{\alpha/2}[72] = t_{0.01}[72] = 2.379;$$
$$\alpha = 0.05 \Rightarrow \quad \alpha/2 = 0.025: \quad t_{\alpha/2}[72] = t_{0.025}[72] = 1.993;$$
$$\alpha = 0.10 \Rightarrow \quad \alpha/2 = 0.05: \quad t_{\alpha/2}[72] = t_{0.05}[72] = 1.666.$$

- The following commands use the **invttail(*df*, *p*)** statistical function to display these ***two-tail*** **critical values of the t[72] distribution** at the four chosen significance levels α, namely α = 0.01, 0.02, 0.05, and 0.10:

```
display invttail(72, 0.005)
display invttail(72, 0.01)
display invttail(72, 0.025)
display invttail(72, 0.05)
```

```
. display invttail(72, 0.005)
2.6458519

. display invttail(72, 0.01)
2.3792621

. display invttail(72, 0.025)
1.9934635

. display invttail(72, 0.05)
1.6662937
```

- Now use the **ttail(*df*, t_0)** statistical function to display the ***two-tail*** **p-values** of the four sample values t_0 = 2.660, 2.390, 2.000, and 1.671, which you already know equal the corresponding values of α (0.01, 0.02, 0.05, and 0.10):

```
display 2*ttail(60, 2.660)
display 2*ttail(60, 2.390)
display 2*ttail(60, 2.000)
display 2*ttail(60, 1.671)
```

Note that to compute the ***two-tail*** **p-values** of the calculated t-statistics, the values of the **ttail(*df*, t_0)** function must be multiplied by 2.

```
. display 2*ttail(72, 2.646)
.00999602

. display 2*ttail(72, 2.379)
.02001316

. display 2*ttail(72, 1.993)
.05005189

. display 2*ttail(72, 1.666)
.1000587
```

This example demonstrates the relationship between the two statistical functions **ttail(*df*, t_0)** and **invttail(*df*, *p*)** for the t-distribution.

- Suppose the sample t-values are *negative*, rather than *positive*. For example, consider the sample t-values $t_0 = -2.646$, and -1.993; you already know that their two-tail p-values are, respectively, 0.01, and 0.05. There are (at least) two alternative ways of using the **ttail(*df*, t_0)** statistical function to compute the correct ***two-tail* p-values** for negative values of t_0. To illustrate, enter the following **display** commands:

```
display 2*(1 - ttail(72, -2.646))
display 2*ttail(72, abs(-2.646))

display 2*(1 - ttail(72, -1.993))
display 2*ttail(72, abs(-1.993))
```

Note the use of the *Stata* absolute value operator **abs( )**.

```
. display 2*(1 - ttail(72, -2.646))
.00999602

. display 2*ttail(72, abs(-2.646))
.00999602

. display 2*(1 - ttail(72, -1.993))
.05005189

. display 2*ttail(72, abs(-1.993))
.05005189
```

- **Recommendation for computing *two-tail* p-values of sample t-statistics**

Let ***t0*** be any calculated sample value of a t-statistic that is distributed under the null hypothesis as a **t[*df*]** distribution, where ***t0*** may be either positive or negative.

The following command will always display the correct ***two-tail* p-value of *t0***:

```
display 2*ttail(df, abs(t0))
```

Examples of Two-Tail Hypothesis Tests

The Model:

DATA: **auto1.dta** (a Stata-format data file)

MODEL: **$price_i = \beta_1 + \beta_2\, weight_i + u_i \quad (i = 1, ..., N)$**

```
. regress price weight

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  1,    72) =   29.42
   Model |   184233937     1   184233937               Prob > F      =  0.0000
Residual |   450831459    72  6261548.04               R-squared     =  0.2901
---------+------------------------------               Adj R-squared =  0.2802
   Total |   635065396    73  8699525.97               Root MSE      =  2502.3

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
  weight |   2.044063   .3768341      5.424   0.000       1.292858    2.795268
   _cons |  -6.707353    1174.43     -0.006   0.995       -2347.89    2334.475
------------------------------------------------------------------------------
```

$N = 74 \qquad N - 2 = 74 - 2 = \mathbf{72}$

$\hat{\beta}_2 = \mathbf{2.0441} \qquad \hat{se}(\hat{\beta}_2) = \mathbf{0.376834}$

$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$

$t_{\alpha/2}[N-2] = t_{0.025}[72] = \mathbf{1.993}$

Test 1: Test the proposition that **weight$_i$ is *unrelated* to price$_i$**.

- ***Null*** **and** ***Alternative*** **Hypotheses**

  $\mathbf{H_0}$**:** $\boldsymbol{\beta_2 = 0}$
  $\mathbf{H_1}$**:** $\boldsymbol{\beta_2 \neq 0}$ a ***two-sided*** **alternative hypothesis**.

- The ***feasible*** **test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the *sample value* of $\mathbf{t(\hat{\beta}_2)}$** under the null hypothesis H_0
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\hat{se}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- ***P-Value Decision Rule:***

  If ***two-tail*** **p-value of $\mathbf{t_0}$ <** α, ***reject*** $\mathbf{H_0}$ at **significance level** α;
  If ***two-tail*** **p-value of $\mathbf{t_0}$** $\geq \alpha$, ***retain*** $\mathbf{H_0}$ at **significance level** α.

- Use *Stata 7* function **ttail(*df*, *t$_0$*)** to compute two-tail p-value of $t_0 = 5.424$, with ***df*** = 72. Enter either of the following *Stata* commands:

```
display 2*ttail(72, 5.424)
7.425e-07
display 2*ttail(72, abs(5.424))
7.425e-07
```

- ***Inference:***

  two-tail **p-value of $\mathbf{t_0}$ = <u>0.0000</u> << 0.01** $\Rightarrow$ ***reject*** $\mathbf{H_0}$ at $\boldsymbol{\alpha = 0.01}$

  Reject $\mathbf{H_0}$**:** $\boldsymbol{\beta_2 = 0}$ in favour of $\mathbf{H_1}$**:** $\boldsymbol{\beta_2 \neq 0}$ at ***less than*** **the** ***1 percent*** **significance level**.

<u>Test 2</u>: Test the proposition that a **1-pound increase in weight$_i$** is associated with an **increase in average price$_i$ of 1 dollar**.

- ***Null*** **and** ***Alternative*** **Hypotheses**

  H$_0$: $\beta_2 = 1$
  H$_1$: $\beta_2 \neq 1$ a ***<u>two-sided</u>*** **alternative hypothesis**.

- The ***feasible*** **test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the** ***sample value*** **of $t(\hat{\beta}_2)$** under the null hypothesis H$_0$
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 1$ and $\hat{se}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{2.0441 - 1}{0.376834} = \frac{1.0441}{0.376834} = \mathbf{2.771}$$

- ***P-Value Decision Rule:***

  If ***two-tail*** **p-value of $t_0 < \alpha$**, ***reject*** **H$_0$** at **significance level α**;
  If ***two-tail*** **p-value of $t_0 \geq \alpha$**, ***retain*** **H$_0$** at **significance level α**.

- Use *Stata 7* function **ttail(*df*, *t$_0$*)** to compute two-tail p-value of $t_0 = 2.771$, with ***df*** = 72. Enter either of the following *Stata* commands:

```
display 2*ttail(72, 2.771)
.00710627

display 2*ttail(72, abs(2.771))
.00710627
```

- ***Inference:***

  two-tail **p-value of t_0 = <u>0.0071</u> < 0.01** ⇒ ***reject*** **H$_0$** at **α = 0.01**

  Reject **H$_0$: $\beta_2 = 1$** in favour of **H$_1$: $\beta_2 \neq 1$** at the ***1 percent*** **significance level**.

Test 3: Test the proposition that a **1-pound increase in weight$_i$** is associated with an **increase in average price$_i$ of 2 dollars**.

- ***Null* and *Alternative* Hypotheses**

  H$_0$: $\beta_2 = 2$
  H$_1$: $\beta_2 \neq 2$ a ***two-sided*** **alternative hypothesis**.

- The ***feasible*** **test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\widehat{\text{Var}}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H$_0$
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 2$ and $\widehat{\text{se}}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\widehat{\text{se}}(\hat{\beta}_2)} = \frac{2.0441 - 2}{0.376834} = \frac{0.0441}{0.376834} = \mathbf{0.1170}$$

- ***P-Value Decision Rule:***

  If ***two-tail*** **p-value of $t_0 < \alpha$**, ***reject*** **H$_0$** at **significance level α**;
  If ***two-tail*** **p-value of $t_0 \geq \alpha$**, ***retain*** **H$_0$** at **significance level α**.

- Use *Stata 7* function **ttail(*df*, *t$_0$*)** to compute two-tail p-value of $t_0 = 0.1170$, with ***df*** = 72. Enter either of the following *Stata* commands:

```
display 2*ttail(72, 0.1170)
.90718581

display 2*ttail(72, abs(0.1170))
.90718581
```

- ***Inference:***

  two-tail **p-value of t_0 = <u>0.9072</u> >> 0.10** ⇒ ***retain*** **H$_0$** at **α = 0.10**

  Retain **H$_0$: $\beta_2 = 2$** against **H$_1$: $\beta_2 \neq 2$** at the ***10 percent*** **significance level**.

❑ P-values for *one-tail* t-tests

Right-tail **t-tests**

- ***Null*** **and** ***Alternative*** **Hypotheses**

  $\mathbf{H_0}$: $\boldsymbol{\beta_2 = b_2}$
  $\mathbf{H_1}$: $\boldsymbol{\beta_2 > b_2}$ a ***one-sided right-sided*** **alternative hypothesis**.

- **Definition of** ***right-tail*** **p-value for** $\mathbf{t_0}$

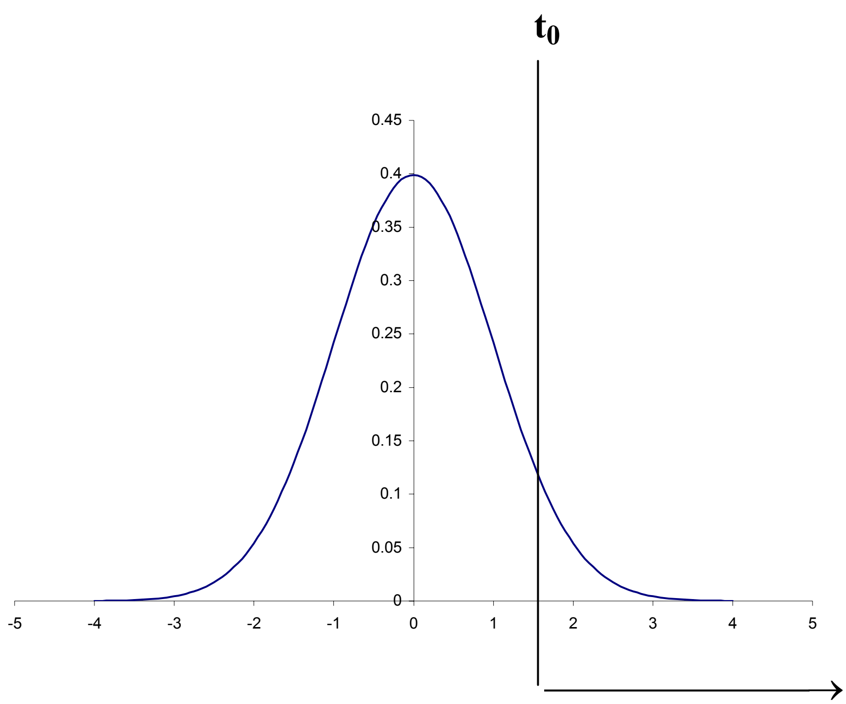
  For a ***right-tail*** **t-test**, the **p-value for** $\mathbf{t_0}$ is the probability that the null distribution of the test statistic takes a value ***greater than*** **the calculated** ***sample value*** $\mathbf{t_0}$ -- i.e.,

  $$\textit{right-tail}\ \textbf{p-value for}\ t_0 = \Pr(t > t_0).$$

  Right-tail **p-value of** $\mathbf{t_0}$ is **the** ***probability*** **of obtaining a t value** ***greater than*** **the** ***sample value*** $\mathbf{t_0}$ if the null hypothesis H_0: $\beta_2 = b_2$ is in fact true.

Right-tail p-values for sample t-statistic t_0

Case 1: $t_0 > 0$



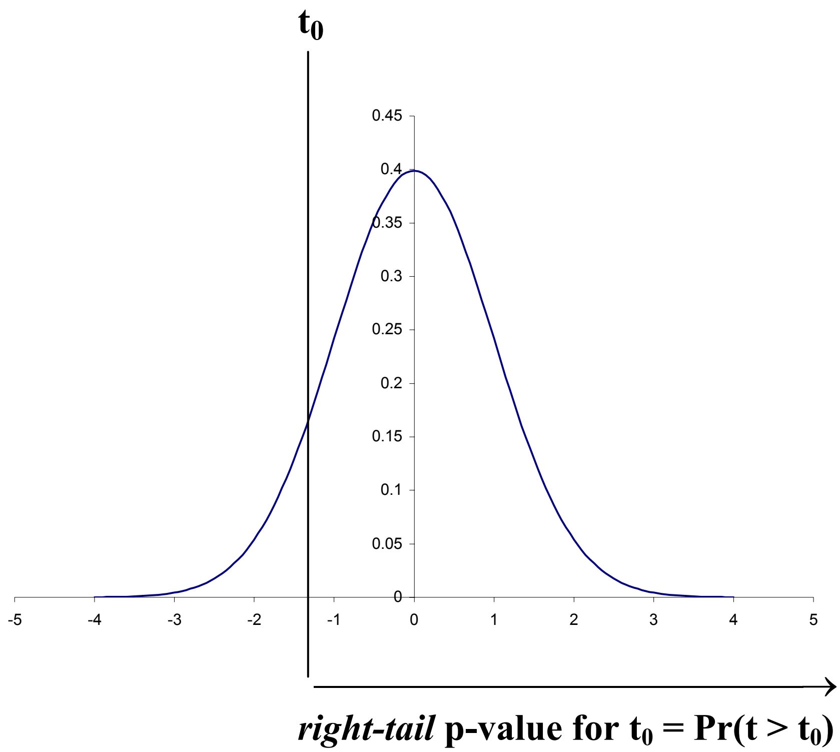
***right-tail* p-value for t_0 = Pr($t > t_0$)**

$\Pr(t > t_0) = \text{ttail}(df, t_0)$

***right-tail* p-value for t_0** $= \Pr(t > t_0) = \text{ttail}(df, t_0)$

Right-tail p-values for sample t-statistic t_0

Case 2: $t_0 < 0$



$\Pr(t > t_0) = \text{ttail}(df, t_0)$

right-tail **p-value for t_0** $= \Pr(t > t_0) = \text{ttail}(df, t_0)$

***Left-tail* t-tests**

- ***Null* and *Alternative* Hypotheses**

  $\mathbf{H_0}$**:** $\boldsymbol{\beta_2 = b_2}$
  $\mathbf{H_1}$**:** $\boldsymbol{\beta_2 < b_2}$ a ***one-sided left-sided* alternative hypothesis**.

- **Definition of *left-tail* p-value for t_0**

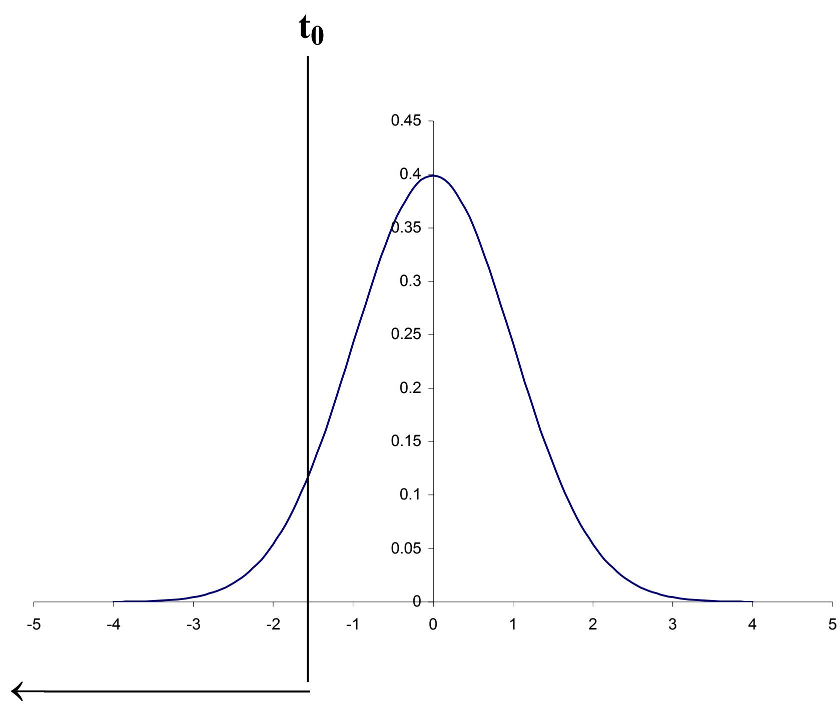
  For a ***left-tail*** **t-test**, the **p-value for t_0** is the probability that the null distribution of the test statistic takes a value ***less than*** **the calculated *sample value*** t_0 -- i.e.,

  left-tail **p-value for t_0 =** $\Pr(t < t_0)$.

  Left-tail **p-value of t_0** is **the *probability* of obtaining a t value *less than* the *sample value*** t_0 if the null hypothesis H_0: $\beta_2 = b_2$ is in fact true.

Left-tail p-values for sample t-statistic t_0

Case 1: $t_0 < 0$



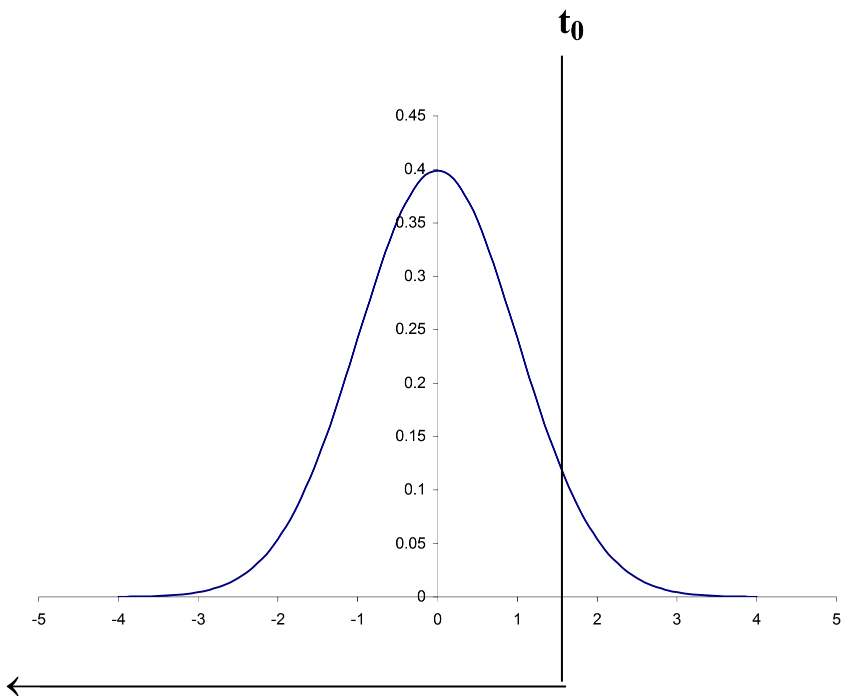
***left-tail* p-value for $t_0 = Pr(t < t_0)$**

$Pr(t < t_0) = 1 - Pr(t > t_0) = 1 - ttail(df, t_0)$

***left-tail* p-value for t_0** $= Pr(t < t_0) = 1 - Pr(t > t_0) = 1 - ttail(df, t_0)$

Left-tail p-values for sample t-statistic t_0

Case 2: $t_0 > 0$



left-tail **p-value for t_0 = Pr(t < t_0) = 1 – Pr(t > t_0)**

Pr(t < t_0) = 1 – Pr(t > t_0) = 1 – ttail(df, t_0)

left-tail **p-value for t_0** = Pr(t < t_0) = 1 – Pr(t > t_0) = 1 – ttail(df, t_0)

❑ Computing *one-tail* critical values and p-values for t-statistics in *Stata 7*

Example 2: One-tail t-tests -- right-tail t-tests

- The following are the ***upper one-tail (right-tail)*** **critical values $t_{\alpha/2}[72]$** of the t[72] distribution, where α is the chosen significance level for the t-test.

  $\alpha = 0.01$: $t_{\alpha}[72] = t_{0.01}[72] = 2.379$;
  $\alpha = 0.05$: $t_{\alpha}[72] = t_{0.05}[72] = 1.666$;
  $\alpha = 0.10$: $t_{\alpha}[72] = t_{0.10}[72] = 1.293$.

- The following commands use the **invttail(*df*, *p*)** statistical function with $p = \alpha$ to display these ***upper one-tail (right-tail)*** **critical values of the t[72] distribution** at the 1%, 5% and 10% significance levels, i.e., for α = 0.01, 0.05 and 0.10:

```
display invttail(72, 0.01)
display invttail(72, 0.05)
display invttail(72, 0.10)
```

```
. display invttail(72, 0.01)
2.3792621

. display invttail(72, 0.05)
1.6662937

. display invttail(72, 0.10)
1.2934205
```

- Now use the **ttail(*df*, *t₀*)** statistical function to display the ***right-tail*** **p-values** of the sample t-values t_0 = 2.379, 1.666, and 1.293, which you already know equal the corresponding values of α, namely 0.01, 0.05, and 0.10, respectively:

```
display ttail(72, 2.379)
display ttail(72, 1.666)
display ttail(72, 1.293)
```

```
. display ttail(72, 2.379)
.01000658

. display ttail(72, 1.666)
.05002935

. display ttail(72, 1.293)
.10007231
```

Example 3: One-tail t-tests -- left-tail t-tests

- The following are the ***lower one-tail (left-tail)*** **critical values $t_{\alpha/2}[72]$** of the t[72] distribution, where α is the chosen significance level for the one-tail t-test; they are taken from a published table of percentage points of the t distribution.

  $\alpha = 0.01$: $t_{\alpha}[72] = t_{0.01}[72] = -2.379$;
  $\alpha = 0.05$: $t_{\alpha}[72] = t_{0.05}[72] = -1.666$;
  $\alpha = 0.10$: $t_{\alpha}[72] = t_{0.10}[72] = -1.293$.

- The following commands use the **invttail(*df*, *p*)** statistical function to display the ***lower one-tail (left-tail)*** **critical values of the t[72] distribution**:

```
display -1*invttail(72, 0.01)
display -1*invttail(72, 0.05)
display -1*invttail(72, 0.10)
```

```
. display -1*invttail(72, 0.01)
-2.3792621

. display -1*invttail(72, 0.05)
-1.6662937

. display -1*invttail(72, 0.10)
-1.2934205
```

- Now use the **ttail(*df*, t_0)** statistical function to display the ***left-tail*** **p-values** of the sample t-values $t_0 = -2.379$, -1.666, and -1.293, which you already know equal the corresponding values of α, namely 0.01, 0.05, and 0.10, respectively:

```
display 1 - ttail(72, -2.379)
display 1 - ttail(72, -1.666)
display 1 - ttail(72, -1.293)
```

```
. display 1 - ttail(72, -2.379)
.01000658

. display 1 - ttail(72, -1.666)
.05002935

. display 1 - ttail(72, -1.293)
.10007231
```

Examples of One-Tail Hypothesis Tests

The Model:

DATA: **auto1.dta** (a Stata-format data file)

MODEL: $\textbf{price}_i = \beta_1 + \beta_2 \textbf{ weight}_i + \textbf{u}_i \quad (\textbf{i} = 1, ..., \textbf{N})$

```
. regress price weight

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  1,    72) =   29.42
   Model |   184233937     1   184233937               Prob > F      =  0.0000
Residual |   450831459    72  6261548.04               R-squared     =  0.2901
---------+------------------------------               Adj R-squared =  0.2802
   Total |   635065396    73  8699525.97               Root MSE      =  2502.3

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
  weight |   2.044063   .3768341      5.424   0.000       1.292858    2.795268
   _cons |  -6.707353    1174.43     -0.006   0.995       -2347.89    2334.475
------------------------------------------------------------------------------
```

$N = 74 \qquad N - 2 = 74 - 2 = \mathbf{72}$

$\hat{\beta}_2 = \mathbf{2.0441} \qquad \hat{se}(\hat{\beta}_2) = \mathbf{0.376834}$

$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$

$t_{\alpha/2}[N-2] = t_{0.025}[72] = \mathbf{1.993}$

$t_{\alpha}[N-2] = t_{0.05}[72] = \mathbf{1.666}$

<u>Test 4 – A Left-Tail Test</u>: Test the proposition that **weight$_i$ has a *negative* effect on price$_i$**.

- ***Null* and *Alternative* Hypotheses**

  H_0: $\beta_2 = 0$
  H_1: $\beta_2 < 0$ a ***one-sided <u>left-sided</u>* alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H_0.
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\hat{se}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- ***P-Value Decision Rule -- Left-Tail Test:***

  If ***left-tail* p-value of $t_0 < \alpha$**, ***reject* H_0** at **significance level α**;
  If ***left-tail* p-value of $t_0 \geq \alpha$**, ***retain* H_0** at **significance level α**.

- Use *Stata 7* function **ttail(*df*, *t$_0$*)** to compute the left-tail p-value of $t_0 = 5.424$, with ***df*** = 72. Enter the following *Stata* command:

```
display 1 - ttail(72, 5.424)
.99999963
```

- ***Inference:***

  ***left-tail* p-value of t_0 = <u>0.9999</u> >> 0.10** $\Rightarrow$ ***retain* H_0** at **α = 0.10**

  ***Retain* H_0: $\beta_2 = 0$** against **H_1: $\beta_2 < 0$** at ***any* significance level**.

Test 5 – A Right-Tail Test: Test the proposition that **$weight_i$ has a *positive* effect on $price_i$.**

- ***Null* and *Alternative* Hypotheses**

  H_0: $\beta_2 = 0$
  H_1: $\beta_2 > 0$ a ***one-sided right-sided* alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the *sample value* of $t(\hat{\beta}_2)$** under the null hypothesis H_0.
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 0$ and $\hat{se}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- ***P-Value Decision Rule -- Right-Tail Test:***

  If ***right-tail* p-value of t_0 < α**, ***reject*** **H_0** at **significance level α**;
  If ***right-tail* p-value of $t_0 \geq \alpha$**, ***retain*** **H_0** at **significance level α**.

- Use *Stata 7* function **ttail(*df*, *t_0*)** to compute the *right-tail* p-value of $t_0 = 5.424$, with ***df*** = 72. Enter the following *Stata* command:

```
display ttail(72, 5.424)
3.713e-07
```

- ***Inference:***

  right-tail* p-value of t_0 = 0.0000 << 0.01** $\Rightarrow$ ***reject **H_0** at **$\alpha = 0.01$**

  Reject **H_0: $\beta_2 = 0$** against **H_1: $\beta_2 > 0$** at the ***1 percent*** **significance level**.

  The **sample evidence *favours*** the alternative hypothesis **H_1: $\beta_2 > 0$.**

<u>**Test 6 – A Right-Tail Test:**</u> Test the proposition that a **1-pound increase in weight$_i$** is associated with an **increase in average price$_i$ of *more than* 1 dollar**.

- ***Null* and *Alternative* Hypotheses**

  $\mathbf{H_0: \ \beta_2 = 1}$
  $\mathbf{H_1: \ \beta_2 > 1}$ a ***one-sided <u>right-sided</u>* alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_2$:

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} \sim t[N-2] = t[72].$$

- **Compute the *sample value* of $\mathbf{t(\hat{\beta}_2)}$** under the null hypothesis H_0.
  Set $\hat{\beta}_2 = 2.0441$, $\beta_2 = 1$ and $\hat{se}(\hat{\beta}_2) = 0.376834$ in the formula for $t(\hat{\beta}_2)$:

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - \beta_2}{\hat{se}(\hat{\beta}_2)} = \frac{2.0441 - 1}{0.376834} = \frac{1.0441}{0.376834} = \mathbf{2.771}$$

- ***P-Value Decision Rule -- Right-Tail Test:***

  If ***right-tail* p-value of $\mathbf{t_0} < \alpha$**, ***reject* $\mathbf{H_0}$** at **significance level α**;
  If ***right-tail* p-value of $\mathbf{t_0} \geq \alpha$**, ***retain* $\mathbf{H_0}$** at **significance level α**.

- Use *Stata 7* function **ttail(*df*, *t$_0$*)** to compute the *right-tail* p-value of $t_0 = 2.771$, with ***df*** = 72. Enter the following *Stata* command:

```
display ttail(72, 2.771)
.00355314
```

- ***Inference:***

  ***right-tail* p-value of $\mathbf{t_0}$ = <u>0.0036</u> < 0.01** $\Rightarrow$ ***reject* $\mathbf{H_0}$** at **α = 0.01**

  ***Reject* $\mathbf{H_0: \beta_2 = 1}$** against **$\mathbf{H_1: \beta_2 > 1}$** at the ***1 percent* significance level**.

  The **sample evidence *favours*** the alternative hypothesis **$\mathbf{H_1: \beta_2 > 1}$.**

P-values for *two-tail* F-tests

For an **F-test**, let the calculated sample value of the F-statistic for a given null hypothesis be F_0.

Definition of p-value for F_0: Then the p-value associated with the sample value F_0 is the probability that the null distribution of the test statistic takes a value greater than the calculated sample value F_0 -- i.e.,

$$\textbf{p-value for } F_0 = \Pr(F > F_0).$$

Note that the F-distribution is defined only over non-negative values that are greater than or equal to zero.