A Guide to Hypothesis Testing in Linear Regression Models

Tests of *One* Coefficient Restriction: *One* Restriction on *One* Coefficient

H_0 specifies ***only one equality*** **restriction** on ***one coefficient***.

❑ *Two-Tail* Tests of *One* Restriction on *One* Coefficient

- ***Example:*** H_0: $\beta_j = b_j$ versus H_1: $\beta_j \neq b_j$ where b_j is a specified constant.

- ***Use:*** either a ***two-tail*** **t-test** or an **F-test**.

- ***Test Statistics:***

⇨ $t(\hat{\beta}_j) = \dfrac{\hat{\beta}_j - \beta_j}{\hat{se}(\hat{\beta}_j)} \sim t[N-k]$ ⇒ ***sample value*** $= t_0(\hat{\beta}_j) = \dfrac{\hat{\beta}_j - b_j}{\hat{se}(\hat{\beta}_j)}$.

⇨ $F(\hat{\beta}_j) = \dfrac{\left(\hat{\beta}_j - \beta_j\right)^2}{\hat{Var}(\hat{\beta}_j)} \sim F[1, N-k]$ ⇒ ***sample value*** $= F_0(\hat{\beta}_j) = \dfrac{\left(\hat{\beta}_j - b_j\right)^2}{\hat{Var}(\hat{\beta}_j)}$.

Note: $\left[t(\hat{\beta}_j)\right]^2 = F(\hat{\beta}_j)$ or $t(\hat{\beta}_j) = \sqrt{F(\hat{\beta}_j)}$ and $\left[t_{\alpha/2}[N-k]\right]^2 = F_\alpha[1, N-k]$.

- ***Decision Rules:***

Reject **H_0** if $|t_0| > t_{\alpha/2}[N-k]$ *or* ***two-tail p-value*** for $t_0 = \Pr\left(|t| > |t_0|\right) < \alpha$;
$F_0 > F_\alpha[1, N-k]$ *or* ***p-value*** for $F_0 = \Pr(F > F_0) < \alpha$.

Retain **H_0** if $|t_0| \leq t_{\alpha/2}[N-k]$ *or* ***two-tail p-value*** for $t_0 = \Pr\left(|t| > |t_0|\right) \geq \alpha$;
$F_0 \leq F_\alpha[1, N-k]$ *or* ***p-value*** for $F_0 = \Pr(F > F_0) \geq \alpha$.

❑ ***One-Tail*** **Tests of *One* Restriction on *One* Coefficient**

- ***Examples:*** H_0: $\beta_j = b_j$ (or $\beta_j \geq b_j$) versus H_1: $\beta_j < b_j$ a ***left-tail*** **test**
  H_0: $\beta_j = b_j$ (or $\beta_j \leq b_j$) versus H_1: $\beta_j > b_j$ a ***right-tail*** **test**

- ***Use:*** a ***one-tail*** **t-test**.

- ***Test Statistic:***

⇨ $t(\hat{\beta}_j) = \dfrac{\hat{\beta}_j - \beta_j}{\hat{se}(\hat{\beta}_j)} \sim t[N-k] \quad \Rightarrow \quad$ ***sample value*** $= t_0(\hat{\beta}_j) = \dfrac{\hat{\beta}_j - b_j}{\hat{se}(\hat{\beta}_j)}$.

- ***Decision Rules -- left-tail t-test:***

Reject **H_0** if $t_0 < -t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t < t_0) < \alpha$;

Retain **H_0** if $t_0 \geq -t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t < t_0) \geq \alpha$.

- ***Decision Rules -- right-tail t-test:***

Reject **H_0** if $t_0 > t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t > t_0) < \alpha$;

Retain **H_0** if $t_0 \leq t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t > t_0) \geq \alpha$.

Tests of *One* Linear Restriction on *Two or More* Coefficients

H_0 specifies ***only one*** linear restriction on ***two or more*** regression coefficients.

❑ *Two-Tail* Tests of *One* Linear Restriction on *Two* Coefficients

- ***Example:*** H_0: $c_j\beta_j + c_h\beta_h = c_0$ versus H_1: $c_j\beta_j + c_h\beta_h \neq c_0$.
- ***Use:*** either a ***two-tail*** **t-test** or an **F-test**.
- ***Test Statistics:***

⇨ $$t\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \frac{(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - (c_j\beta_j + c_h\beta_h)}{\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)} \sim t[N-k] = t[N-k_1]$$

where: $\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h) = \sqrt{\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)}$

$$\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h) = c_j^2\hat{Var}(\hat{\beta}_j) + c_h^2\hat{Var}(\hat{\beta}_h) + 2c_jc_h\hat{Cov}(\hat{\beta}_j, \hat{\beta}_h).$$

sample value $= t_0\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \dfrac{(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - c_0}{\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)}$.

⇨ $$F\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \frac{\left[(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - (c_j\beta_j + c_h\beta_h)\right]^2}{\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)} \sim F[1, N-k] = F[1, N-k_1]$$

where: $\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h) = c_j^2\hat{Var}(\hat{\beta}_j) + c_h^2\hat{Var}(\hat{\beta}_h) + 2c_jc_h\hat{Cov}(\hat{\beta}_j, \hat{\beta}_h)$.

sample value $= F_0\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \dfrac{\left[(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - c_0\right]^2}{\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)}$

- ***Decision Rules:***

Reject* H₀** if $|t_0| > t_{\alpha/2}[N-k]$ *or* ***two-tail p-value for $t_0 = \Pr(|t| > |t_0|) < \alpha$;
$F_0 > F_\alpha[1, N-k]$ *or* ***p-value*** for $F_0 = \Pr(F > F_0) < \alpha$.

Retain* H₀** if $|t_0| \le t_{\alpha/2}[N-k]$ *or* ***two-tail p-value for $t_0 = \Pr(|t| > |t_0|) \ge \alpha$;
$F_0 \le F_\alpha[1, N-k]$ *or* ***p-value*** for $F_0 = \Pr(F > F_0) \ge \alpha$.

❑ ***One-Tail*** **Tests of** ***One*** **Linear Restriction on** ***Two*** **Coefficients**

- ***Examples:*** H_0: $c_j\beta_j + c_h\beta_h = c_0$ vs. H_1: $c_j\beta_j + c_h\beta_h < c_0$ a ***left-tail*** **test**
  H_0: $c_j\beta_j + c_h\beta_h = c_0$ vs. H_1: $c_j\beta_j + c_h\beta_h > c_0$ a ***right-tail*** **test**

- ***Use:*** a ***one-tail*** **t-test**.

- ***Test Statistic:***

$$\Rightarrow \quad t\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \frac{(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - (c_j\beta_j + c_h\beta_h)}{\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)} \sim t[N-k] = t[N-k_1]$$

where $\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h) = \sqrt{\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)}$

$$\hat{Var}(c_j\hat{\beta}_j + c_h\hat{\beta}_h) = c_j^2\hat{Var}(\hat{\beta}_j) + c_h^2\hat{Var}(\hat{\beta}_h) + 2c_jc_h\hat{Cov}(\hat{\beta}_j, \hat{\beta}_h).$$

sample value $= t_0\left(c_j\hat{\beta}_j + c_h\hat{\beta}_h\right) = \dfrac{(c_j\hat{\beta}_j + c_h\hat{\beta}_h) - c_0}{\hat{se}(c_j\hat{\beta}_j + c_h\hat{\beta}_h)}$.

- ***Decision Rules --*** ***left-tail*** ***t-test:***

Reject $\mathbf{H_0}$ if $t_0 < -t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t < t_0) < \alpha$;

Retain $\mathbf{H_0}$ if $t_0 \ge -t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t < t_0) \ge \alpha$.

- ***Decision Rules --*** ***right-tail*** ***t-test:***

Reject $\mathbf{H_0}$ if $t_0 > t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t > t_0) < \alpha$;

Retain $\mathbf{H_0}$ if $t_0 \le t_\alpha[N-k]$ *or* ***one-tail p-value*** for $t_0 = \Pr(t > t_0) \ge \alpha$.

Tests of *Two or More* Linear Coefficient Restrictions

H_0 specifies ***two or more* linear coefficient restrictions**.

- ***Example:*** H_0: $\beta_2 = \beta_4$ *and* $\beta_3 = \beta_5$
  H_1: $\beta_2 \neq \beta_4$ *and/or* $\beta_3 \neq \beta_5$

- ***Use:*** a **general F-test**; only an F-test can be used to test jointly *two or more* coefficient restrictions.

- ***Test Statistics:*** Either of the following two general F-statistics.

$$\Rightarrow \quad F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} = \frac{(RSS_0 - RSS_1)/(k - k_0)}{RSS_1/(N - k)}.$$

$$\Rightarrow \quad F = \frac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} = \frac{(R_U^2 - R_R^2)/(k - k_0)}{(1 - R_U^2)/(N - k)}.$$

Null distribution: $F \sim F[df_0 - df_1, df_1] = F[k - k_0, N - k]$.

- ***Sample value*** of F-statistic under $H_0 = F_0$.

- ***Decision Rules:***

***Reject* H_0** if $F_0 > F_\alpha[df_0 - df_1, df_1] = F_\alpha[k - k_0, N - k]$ *or*
p-value for $F_0 = \Pr(F > F_0) < \alpha$.

***Retain* H_0** if $F_0 \leq F_\alpha[df_0 - df_1, df_1] = F_\alpha[k - k_0, N - k]$ *or*
p-value for $F_0 = \Pr(F > F_0) \geq \alpha$.