Chapter 6

Continuous Random Variables and Probability Distributions

6.1 Continuous Random Variables

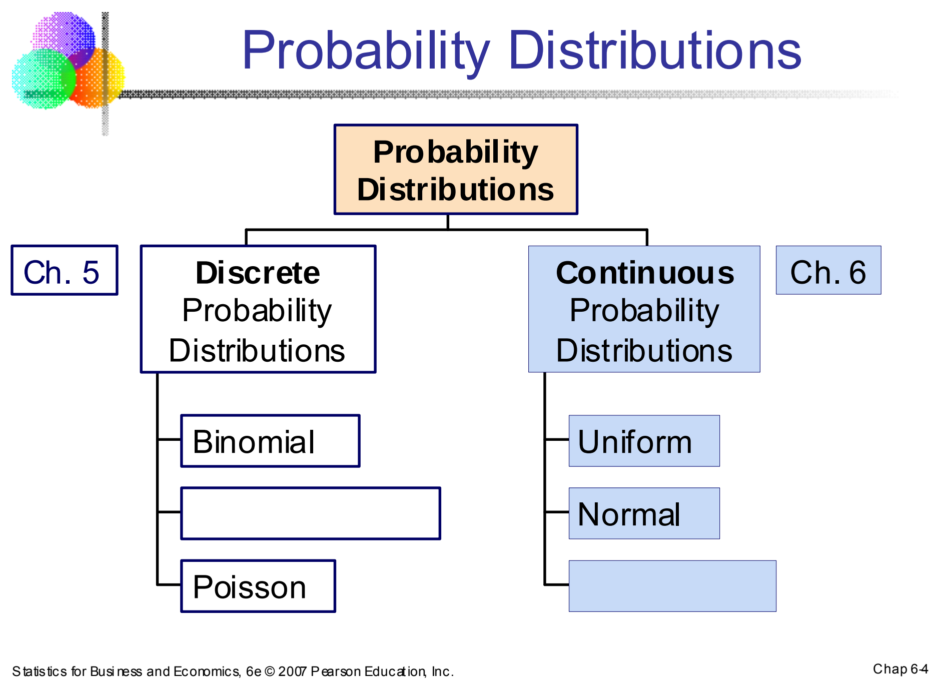

Figure 6.1:

Recall that a continuous random variable, X, can take on any value in a given interval.

- Cumulative Distribution Function (CDF) :

  The cumulative distribution function, $F(x)$, for a continuous random variable X expresses the probability that$X$ does not exceed the value of x, as a function of x

$$F(x) = P(X \leq x)$$

- Because X can take on an infinite number of possible values, its distribution is given by a curve called a **probability density curve** denoted $f(x)$.

Notes

1. For a continuous random variable, the probability is represented by the area under the probability density curve.
2. The probability of any specific value of X is therefore zero, i.e. $P(X = a) = 0$ for a continuous random variable.
3. $f(x) \geq 0$
4. $f(x)$ can be greater than one (why).

- Since probabilities are represented by areas under $f(x)$, they are calculated by integral calculus. i.e. (for $a \leq b$)

$$P(a \leq X \leq b) = P(a < X < b) = \int_a^b f(x)dx.$$

- The cumulative distribution function denoted $F(x_0)$, measures the probability that X is less than or equal to a given value x_0, i.e..:

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)dx$$

-

$$\int_x f(x)dx = 1.$$

$$E[X] = \mu = \int_x xf(x)dx$$

$$V[X] = \sigma^2 = \int_x (x-\mu)^2 f(x)dx$$

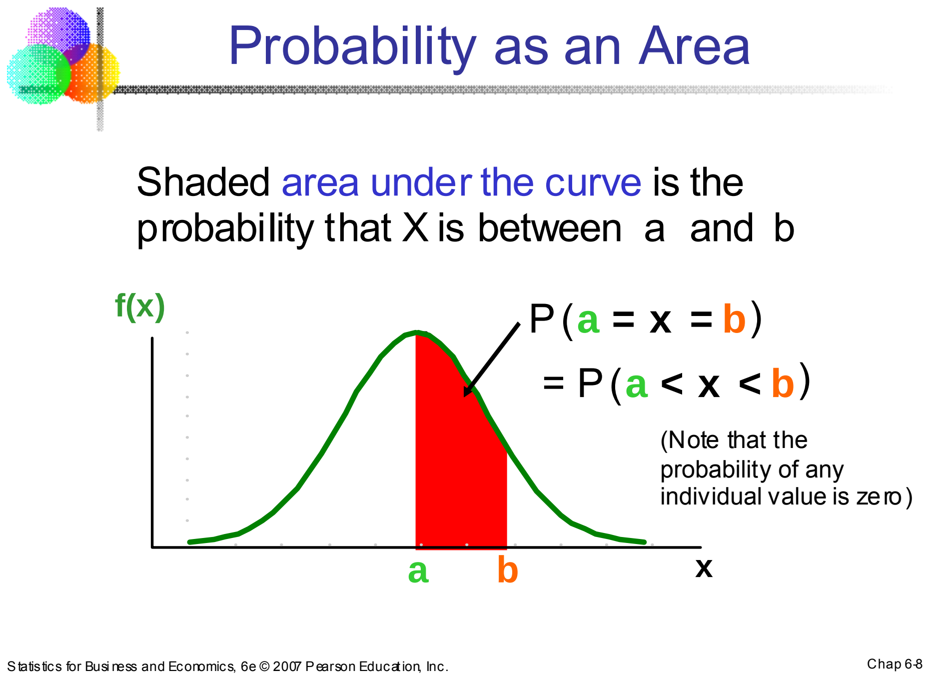

Figure 6.2:

6.2 The Uniform Distribution

The uniform distribution is applicable to situations in which all outcomes are equally likely.

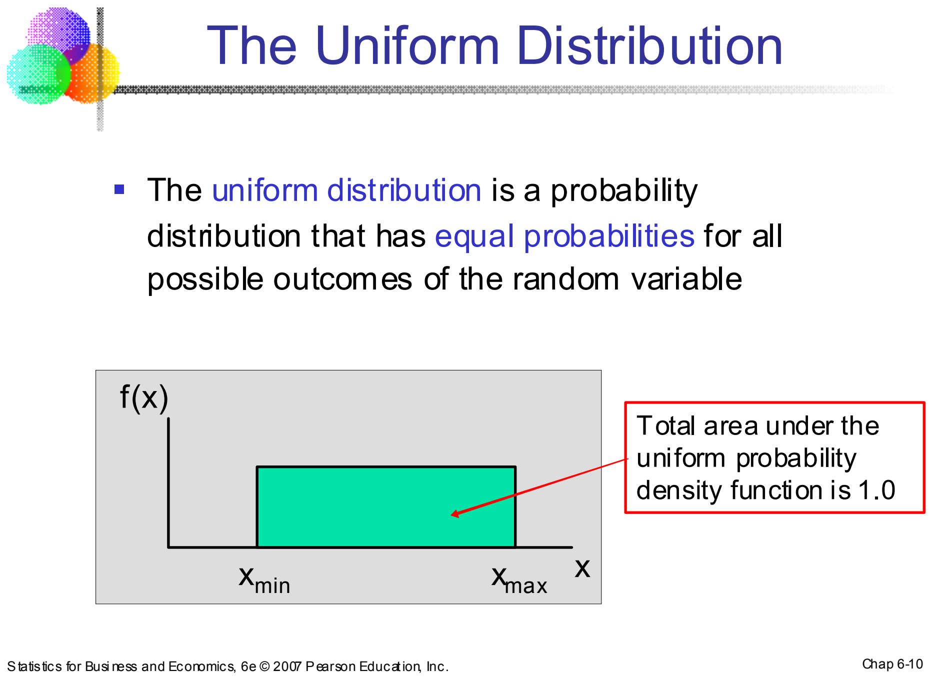

Figure 6.3:

$$f(x) = \frac{1}{b-a} \text{ where } a \leq x \leq b.$$

$$P(X \leq c) = \int_a^c \frac{1}{b-a} dx = \frac{c-a}{b-a}$$

$$P(c \leq X \leq d) = P(X \leq d) - P(X \leq c)$$

$$= \frac{d-a}{b-a} - \frac{c-a}{b-a} = \frac{d-c}{b-a}.$$

$$E[X]) = \frac{(b+a)}{2}$$

$$V[X] = \frac{(b-a)^2}{12}$$

Example of Uniform Distribution:

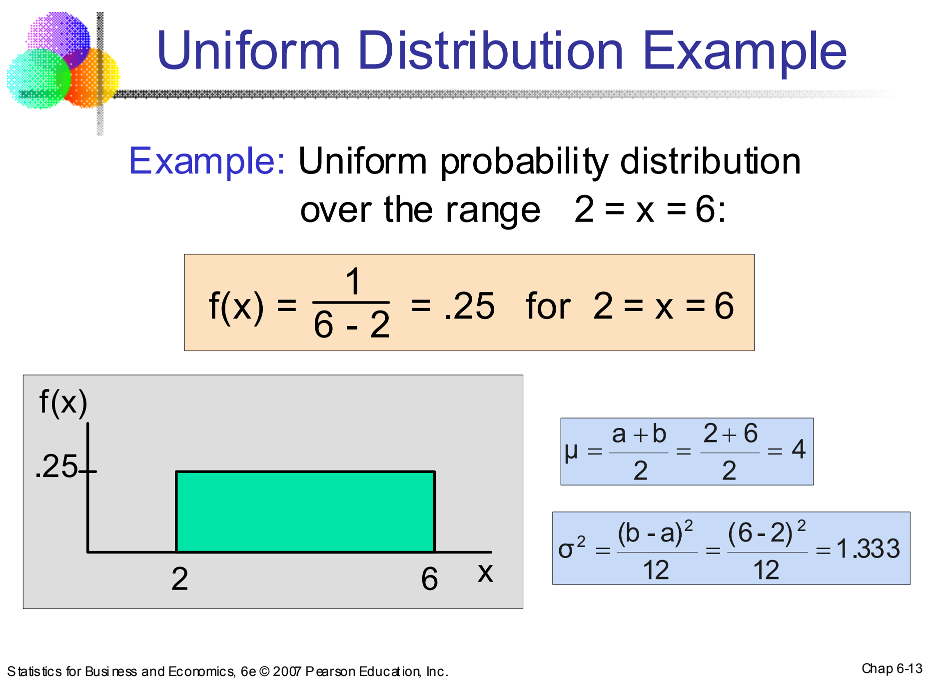

Figure 6.4:

Buses are supposed to arrive in front of Dunning Hall every half hour. At the end of the day, however, buses are equally likely to arrive at any time during any given half hour. If you arrive at the bus stop after a hard day of attending lectures, what is the probability that you will have to wait more than 10 minutes for the bus?

Answer

X is distributed uniformly on [0,30].

$$f(x) = \frac{1}{30}$$

$$P(X > 10) = 1 - P(X < 10) = 1 - \frac{10 - 0}{30 - 0} = \frac{2}{3}.$$

Note the expected amount of time that you will wait for a bus is:

$$E[X] = \frac{0 + 30}{2} = 15.$$

6.3 The Normal Distribution

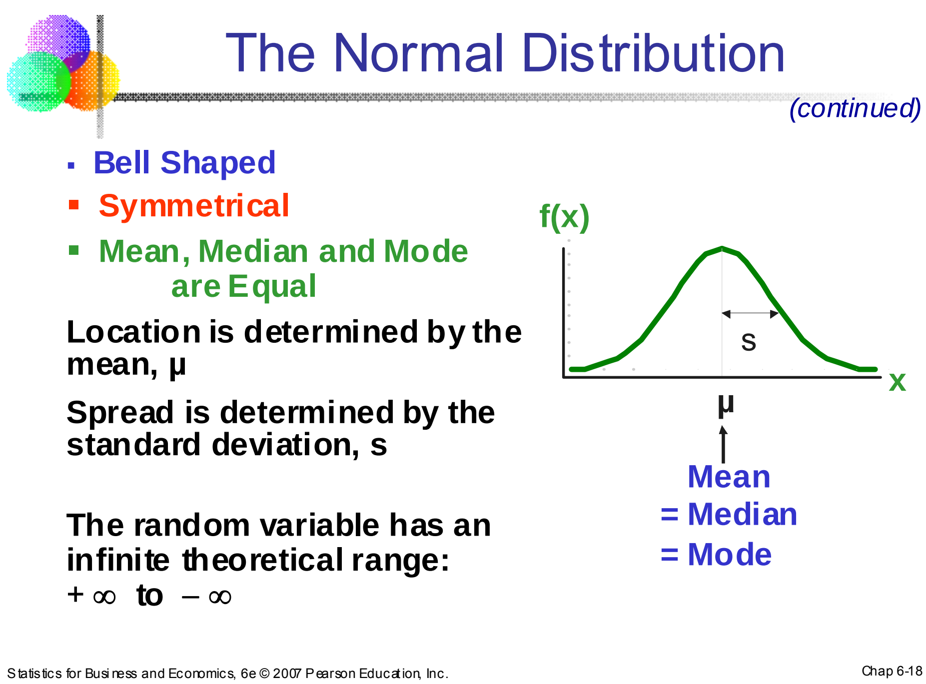

Figure 6.5:

- A random variable X is said to be normally distributed if:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad -\infty < x < \infty$$

Notes:

$$E[X] = \mu.$$

$$V[X] = \sigma^2$$

- The parameters of the normal distribution are μ and σ.

- $e = 2.71828$ and $\pi = 3.14159$

- If X is normally distributed with mean μ and variance σ, then we write $X \sim N(\mu, \sigma^2)$.

- The normal distribution is bell shaped and symmetrical around the value $X = \mu$,

$$ie.\ P(X \geq \mu) = 1 - P(X \leq \mu) = 0.5.$$

The mean, the median, and the mode are all equal and denoted by μ

- Cumulative distribution function $F(x_0) = P(X \leq x_0)$, this is the area under the normal probability density functionn to the left of x_0.As for any proper density function, the total area under the curve is 1; that is $F(\infty) = 1$.

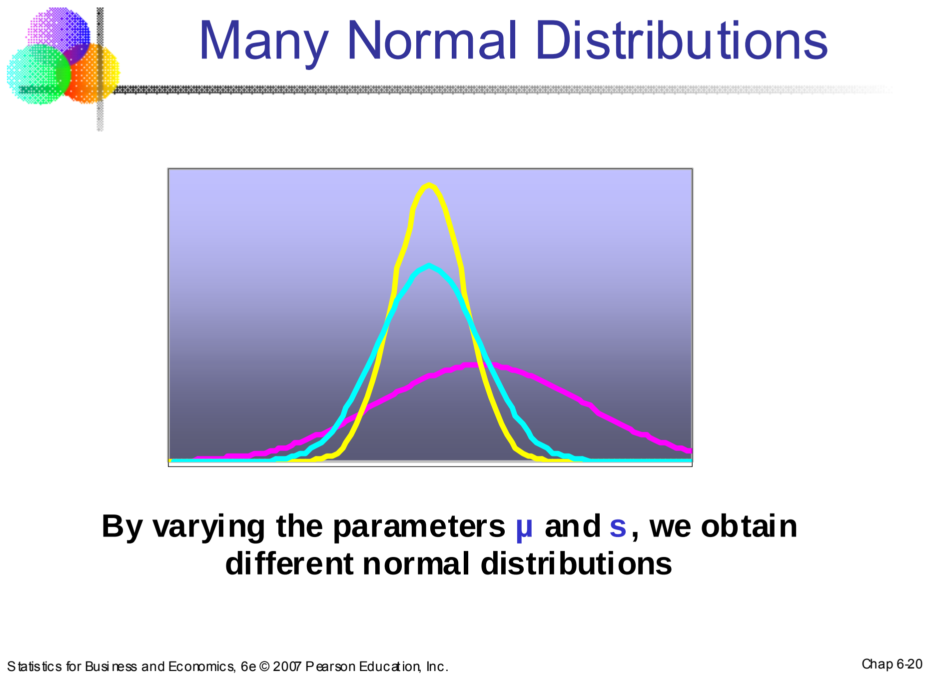
Many Normal Distributions
By varying the parameters μ and s, we obtain different normal distributions
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Figure 6.6:

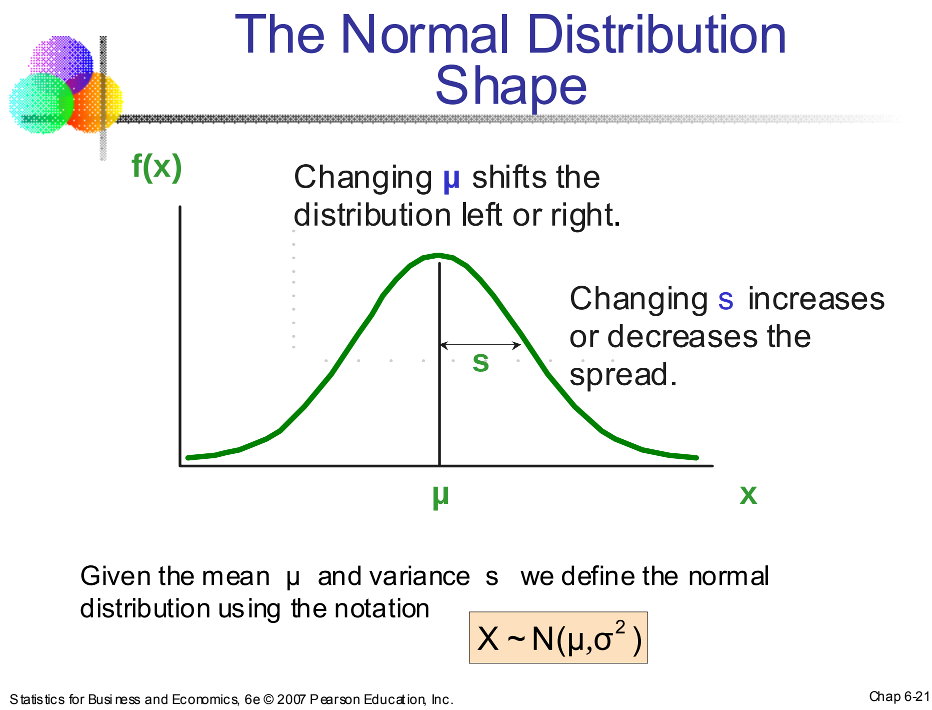

Figure 6.7:

Note s should be σ in above

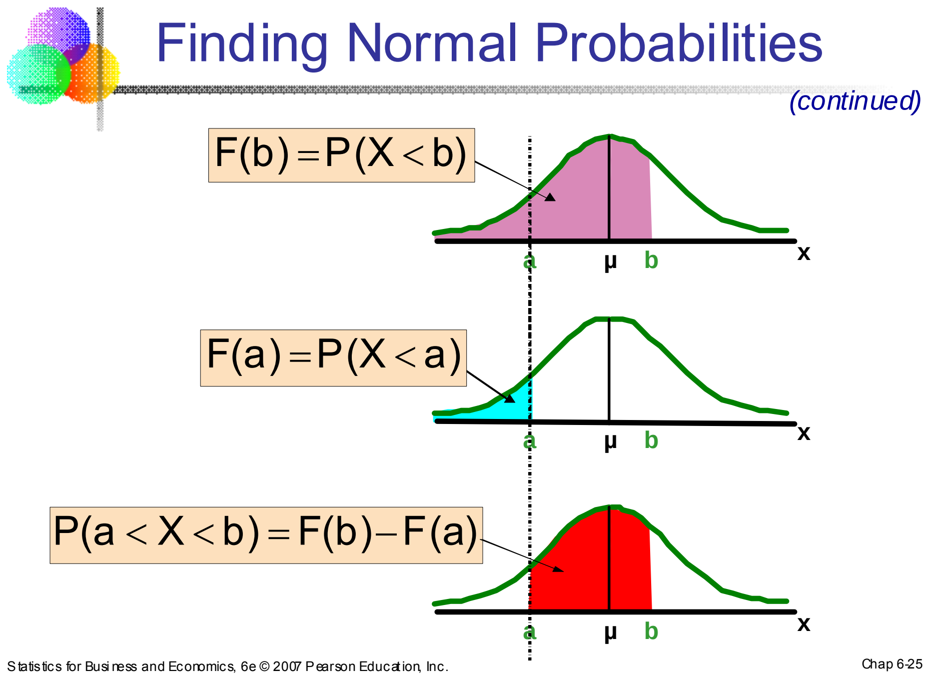
Finding Normal Probabilities
(continued)
F(b) = P(X < b)
F(a) = P(X < a)
P(a < X < b) = F(b) − F(a)
a
μ
b
x
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Figure 6.8:

6.4 The Standard Normal Distribution

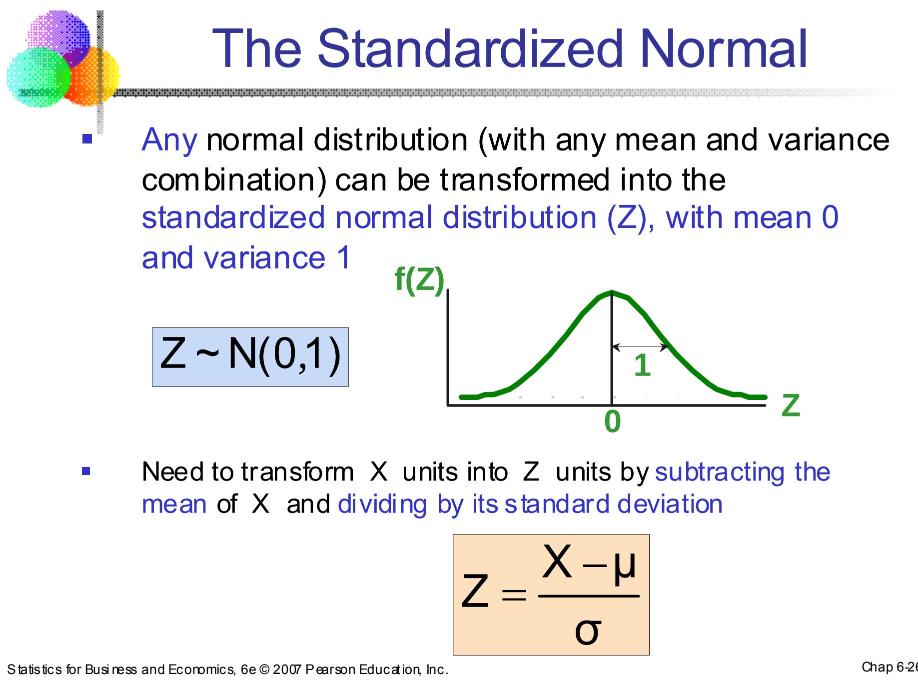

Figure 6.9:

- If X is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, then it is said to have the standard normal distribution.

- A random variable with the standard normal distribution is denoted by the letter Z, i.e. $Z \sim N(0, 1)$.

- Let $F_z(z_0)$ denote the cumulative distribution for the standard normal:

$$F(z_0) = P(Z < z_0)$$

- It is easy to see that for 2 constants a and b such that $a < b$

$$P(a < Z < b) = F_z(b) - F_z(b)$$

6.5 Calculating Areas Under the Standard Normal Distribution

- Table 1 (page 780-781) in NCT Appendix Tables. can be used to calculate areas under the standard normal distribution.

- Note that Table 1 gives areas between -∞ and a positive number, i.e.. $P(-\infty < Z < c)$ for some $c > 0$ (it starts at 0.5 why)

- Recall that symmetry of the normal distribution about its mean implies that $P(Z \geq 0) = 1 - P(Z \leq 0)$.

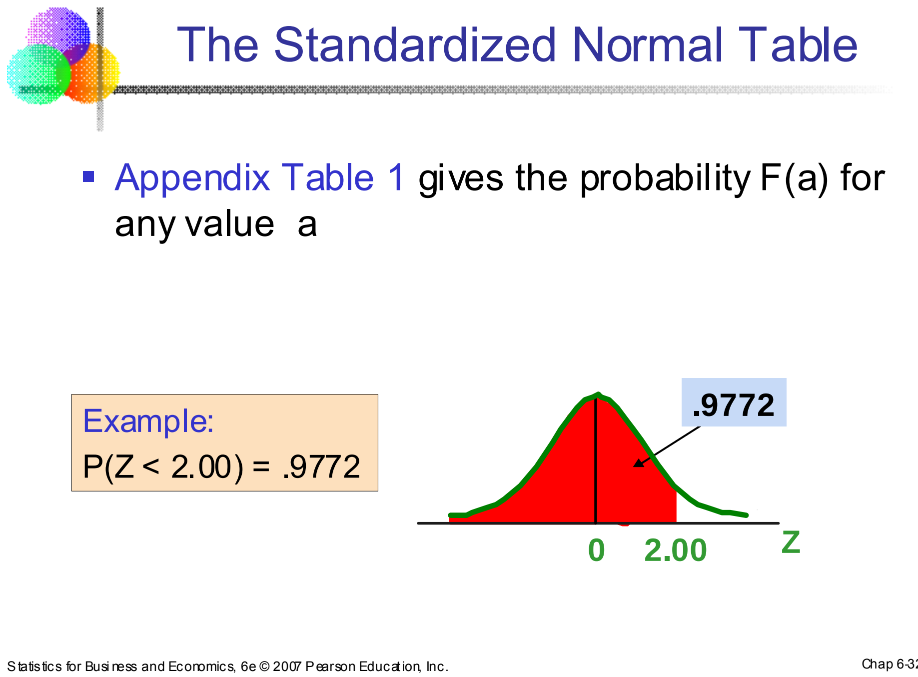
The Standardized Normal Table
Appendix Table 1 gives the probability F(a) for any value a
Example:
P(Z < 2.00) = .9772
.9772
0
2.00
Z
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Figure 6.10:

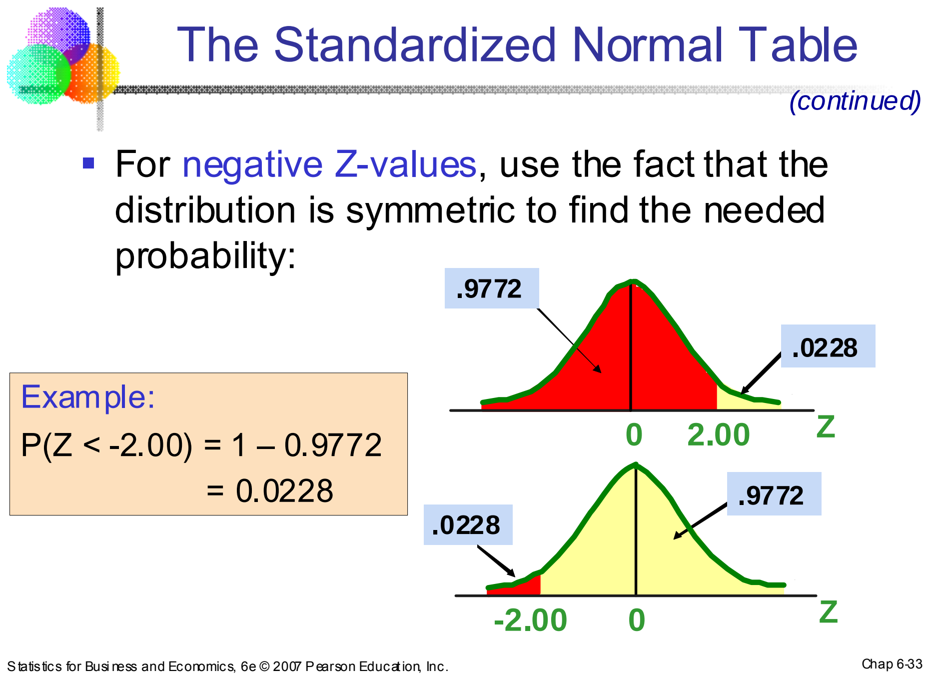
The Standardized Normal Table
(continued)
For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:
Example:
P(Z < -2.00) = 1 – 0.9772
= 0.0228
.9772
.0228
0
2.00
Z
.0228
.9772
-2.00
0
Z
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Figure 6.11:

6.5.1 Examples: of Calculating Standard Normal

- It is a good idea to draw a picture to make sure you are calculating the right area

- $P(0 < Z < 2.42) = P(-\infty < Z < 2.42) - F_Z(0) = F_z(2.42) - F_z(0) = 9922 - .5000 = ..4922$.

- $P(.53 < Z < 2.42) = P(-\infty < Z < 2.42) - P(-\infty < Z < .53) = .9922 - .7019 = .2903$.

- $P(Z > 1.09) = 1 - P(-\infty < Z < 1.09) = 1 - .8621 = .1379$.

- $P(Z > -.36) = P(-\infty < Z < .36) = .6406$.

- $P(-1.00 < Z < 1.96) = (P(-\infty < Z < 1.96) - .5) - (P(-\infty < Z < 1.00) - .5) = .9750 - (1 - .8413) = .8163$.

- Find c such that $P(Z > c) = .0250$.

- If the area to the right of c is .025,

  then the area between $-\infty$ and c must be 1-.025=.975. From Table 1 we see that

- $P(-\infty < Z < 1.96) = .975$, so $c = 1.96$.

6.6 Linear Transformation of Normal Random Variables Theorem

- Linear combinations of normally distributed random variables are normally distributed.

if $X_1, X_2, \ldots, X_n$ are normally distributed and Y

$$Y = A_1X_1 + A_2X_2 + \cdots + A_nX_n$$

- Y is normally distributed.
- Linear combinations of normal variables are normally distributed!

- If X_1 and X_2 are independently normally distributed with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, then if

  $$Y = X_1 + X_2$$

  is normally distributed and:

$$E[Y] = E[X_1] + E[X_2] = \mu_1 + \mu_2$$

- The mean of the linear combination $\mu_y = E[Y]$ is equal to the sum of the individual means $\mu_1 + \mu_2$

$$V[Y] = V[X_1] + V[X_2] = \sigma_1^2 + \sigma_2^2$$

- The variance of the linear combination of independent variables $\sigma_y^2 = V[Y]$ is equal to the sum of the individual variances $\sigma_1^2 + \sigma_2^2$

- So that: if $X \sim N(\mu, \sigma^2)$ and $Y = A + BX$, then:

$$Y \sim N(A + B\mu, B^2\sigma^2)$$

6.7 Standardizing Transformation

Claim: **We can use the standard normal to calculate areas under any normal curve through the use of a simple linear transformation.**

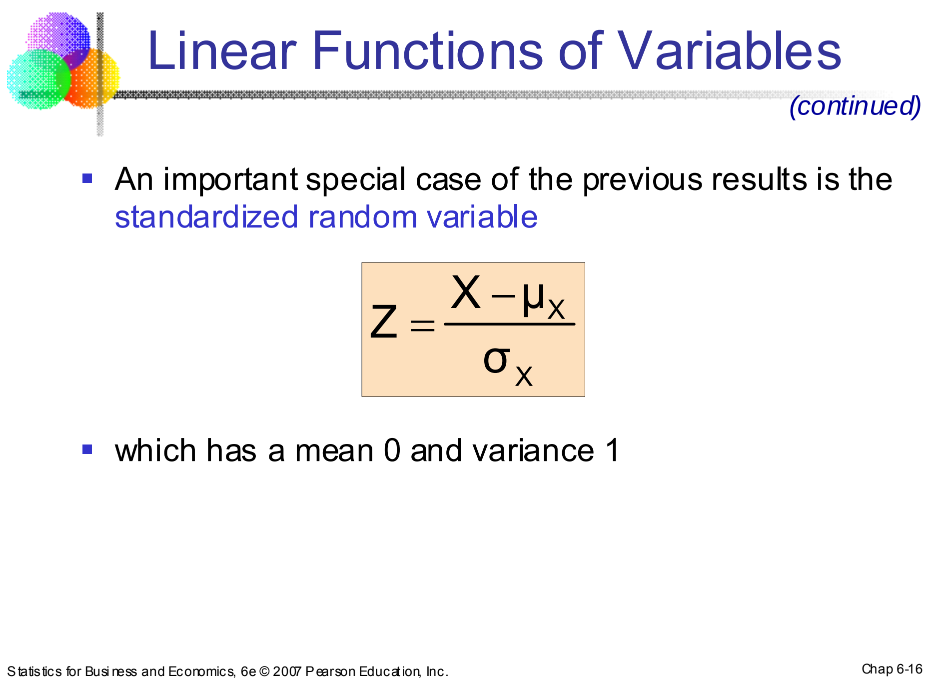

Figure 6.12:

- If $X \sim N(\mu, \sigma^2)$, then:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

- This is called the **standardizing transformation.**

- Note this is simply an application of the linear transformation theorem.

$$Z = \frac{X - \mu}{\sigma} = \frac{X}{\sigma} + \frac{-\mu}{\sigma}$$

- Since $Z = A + BX$ where $B = \frac{1}{\sigma}$ and $A = \frac{-\mu}{\sigma}$. It then follows:

$$E(Z) = A + B\mu = \frac{-\mu}{\sigma} + \frac{\mu}{\sigma} = 0,$$

$$V(Z) = B^2\sigma^2 = \frac{\sigma^2}{\sigma^2} = 1.$$

- By applying the standardizing transformation we can find areas under any normal distribution.
- If $X \sim N(\mu, \sigma^2)$, then:

$$P(a < X < b) = P(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma})$$

$$= P(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma})$$

$$= F_z(\frac{b - \mu}{\sigma}) - F_z(\frac{a - \mu}{\sigma})$$

which can be found from Table 1?.

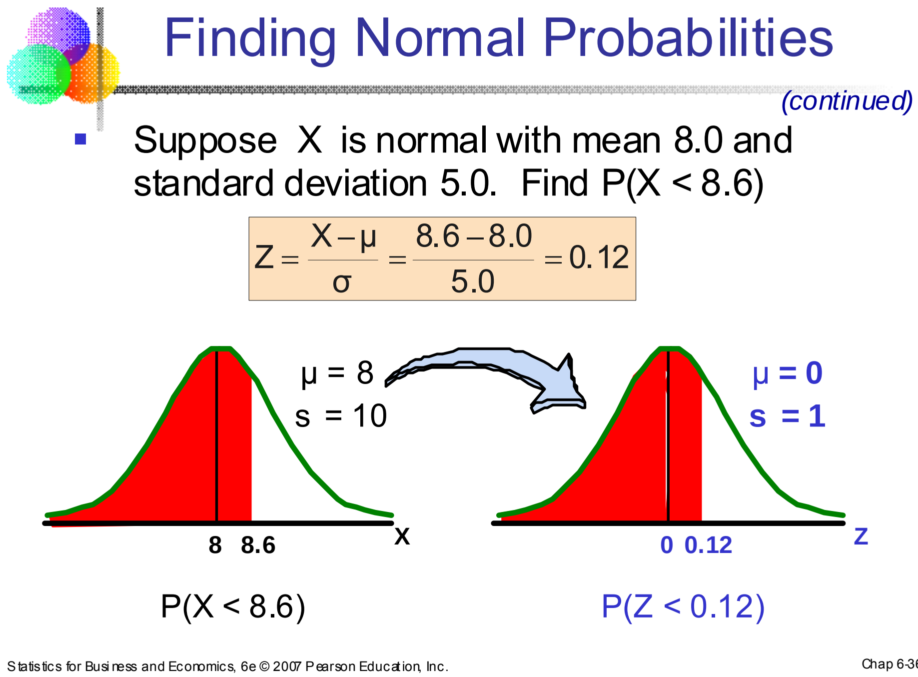
Finding Normal Probabilities
(continued)
Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)
$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$
μ = 8
s = 10
μ = 0
s = 1
8 8.6
X
0 0.12
Z
P(X < 8.6)
P(Z < 0.12)
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Figure 6.13:

6.7.1 Examples of Standardizing Normal Transformations:

Let $X \sim N(25, .5^2)$ and find $P(X < 24)$.

$$P(X < 24) = P(Z < \frac{24 - 25}{.5})$$

$$= P(Z < -2) = 1 - F_z(2) = 1 - .9772 = .0228.$$

Let $X \sim N(1500, 100^2)$ and find $P(1450 < X < 1600)$.

$$P(1450 < X < 1600)$$

$$\begin{aligned} &= P(\frac{1450 - 1500}{100} < Z < \frac{1600 - 1500}{100}) \\ &= P(-.5 < Z < 1.0) \\ &= (F_z(1.0) - .5) + (F_z(.5) - .5) = .5328 \end{aligned}$$

Questions on Standardizing Transformation (see Examples 6.4-6.6 and try problems 6.9,6.12,

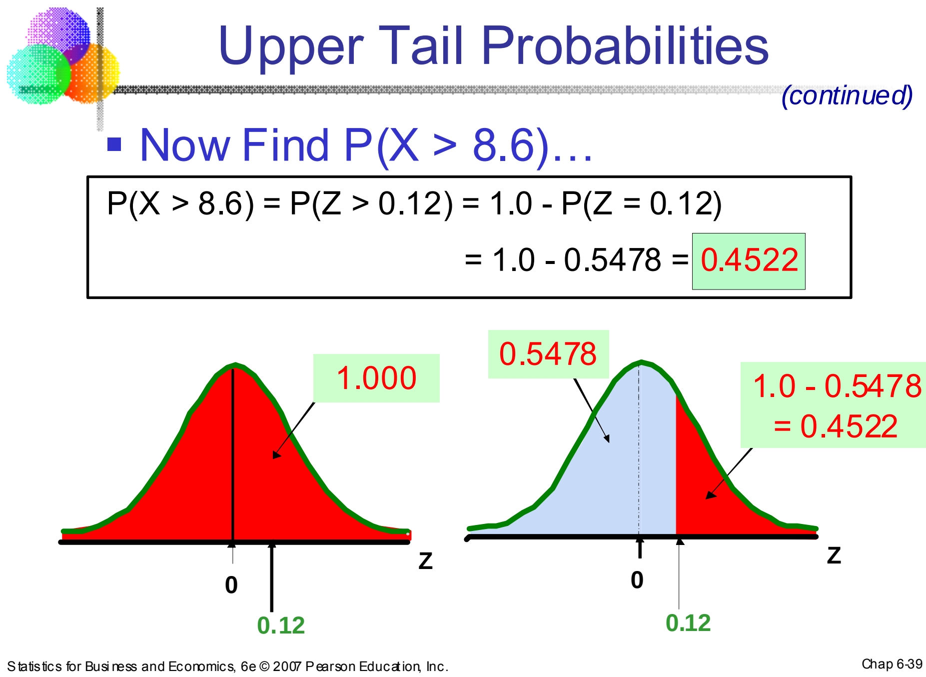

Figure 6.14:

6.16, 6.25,6.26)

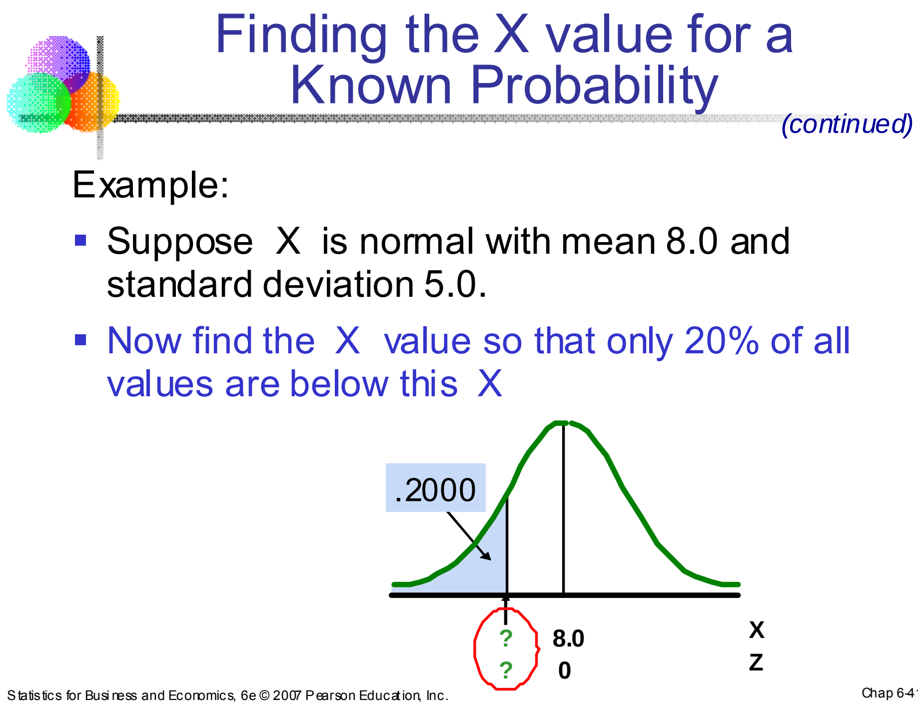
Finding the X value for a Known Probability
(continued)
Example:
Suppose X is normal with mean 8.0 and standard deviation 5.0.
Now find the X value so that only 20% of all values are below this X
.2000
?
?
8.0
0
X
Z
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Figure 6.15:

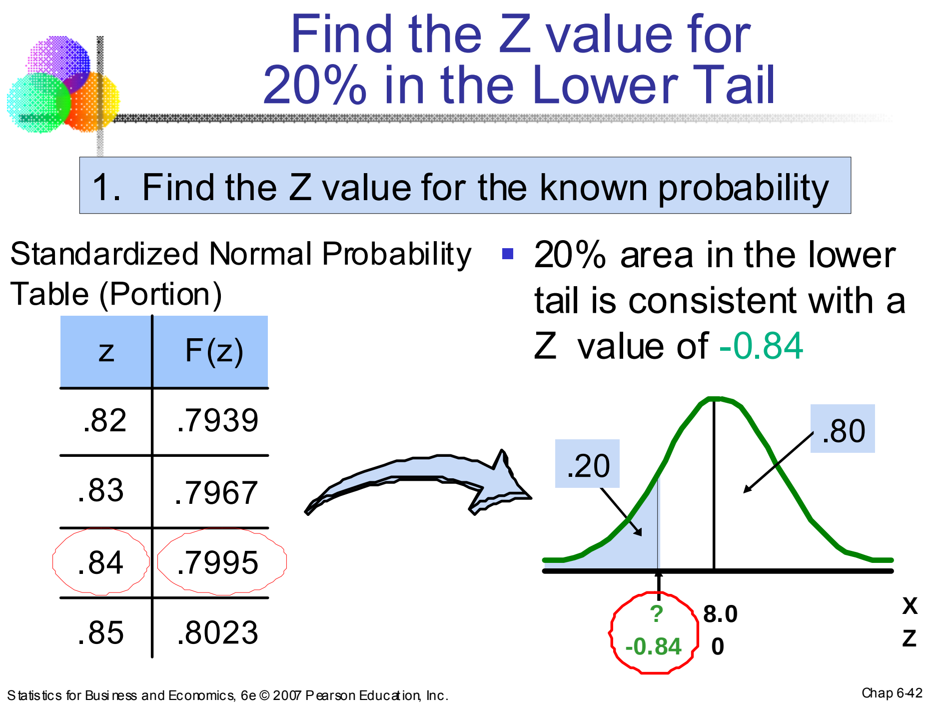
Find the Z value for
20% in the Lower Tail
1. Find the Z value for the known probability
Standardized Normal Probability Table (Portion)
z
F(z)
.82
.7939
.83
.7967
.84
.7995
.85
.8023
20% area in the lower tail is consistent with a Z value of -0.84
.20
.80
?
-0.84
8.0
0
X
Z
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Figure 6.16:

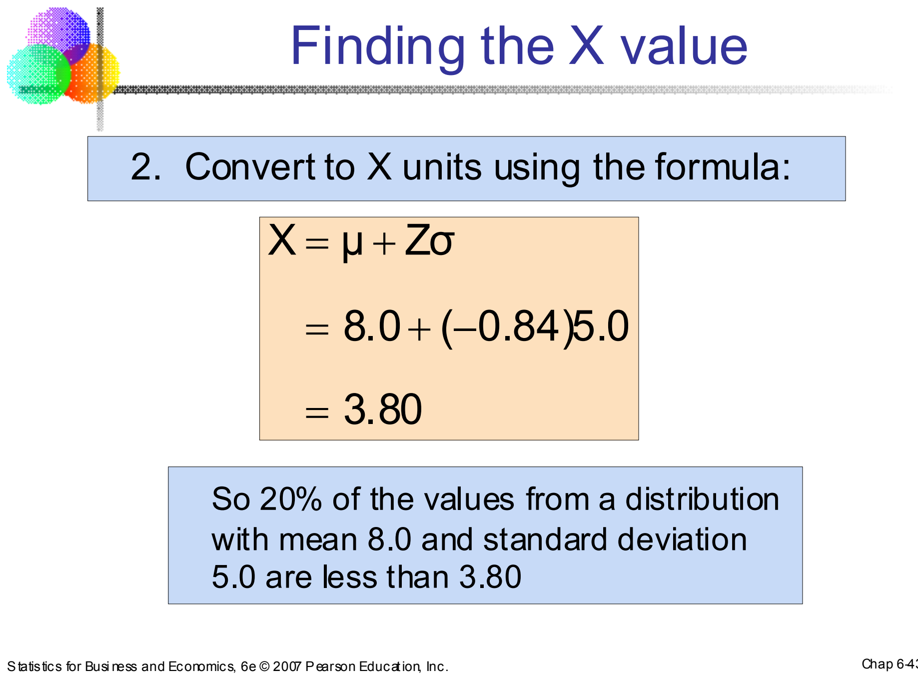
Finding the X value
2. Convert to X units using the formula:
$X = \mu + Z\sigma$
$= 8.0 + (-0.84)5.0$
$= 3.80$
So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80
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Figure 6.17:

6.8 Normal Distribution Approximation for Binomial Distribution

- Let $X =$ the number of sucessesin n independent trilas with probability of success π

- As n gets big the binomial distribution converges into a normal distribution

- This is an example of the **Central Limit Theorem**

- If n is large and π is close to .5 then we can use the normal distribution to get a good approximation to the binomial distribution by setting

$$E[X] = \mu = n\pi \text{ and } Var[X] = \sigma^2 = n\pi(1 - \pi).$$

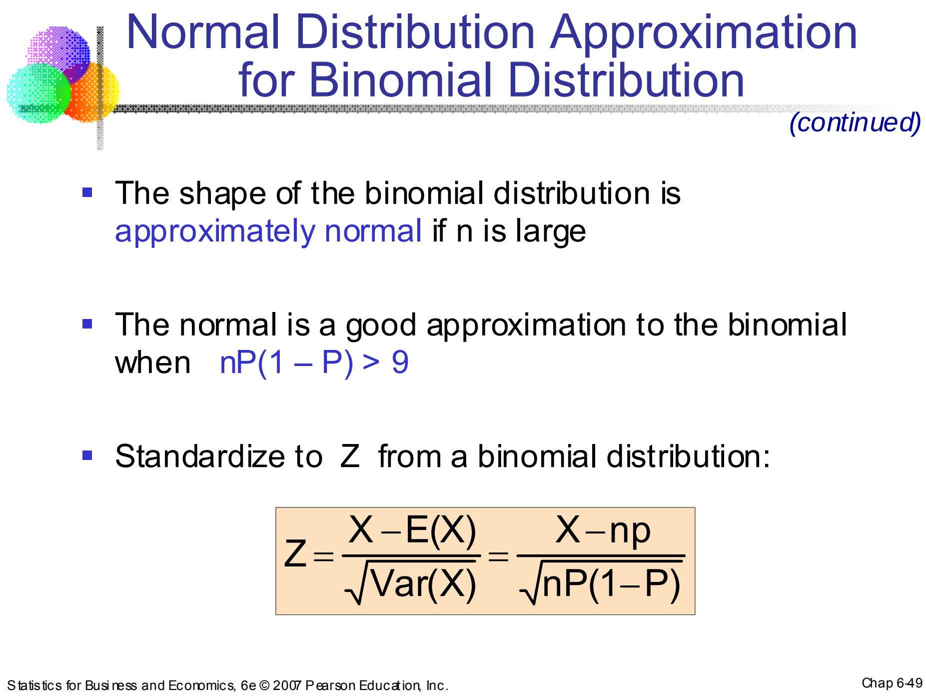
Normal Distribution Approximation for Binomial Distribution
(continued)
The shape of the binomial distribution is approximately normal if n is large
The normal is a good approximation to the binomial when nP(1 – P) > 9
Standardize to Z from a binomial distribution:
$Z=\frac{X-E(X)}{\sqrt{Var(X)}}=\frac{X-np}{\sqrt{nP(1-P)}}$
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Figure 6.18:

- A general guideline is the normal distribution approximates the binomial distribution well whenever: $n\pi(1-\pi) \geq 9$.

- The equation for the standardized binomial variable (where X is the number of successes) is:

$$Z = \frac{X - E[X]}{\sqrt{Var[X]}} = \frac{X - n\pi}{\sqrt{n\pi(1-\pi)}}$$

6.9 Continuity Correction for the Binomial

- We can improve the accuracy of the approximation by applying acorrection for continuity.

- This adjusts the probability to account for the fact that we are approximating a **discrete** distribution with a **continuous** one.

- The rule is to subtract .5 from the lower value of the number of successes, and add .5 to the upper value.

  i.e. we approximate $P(X = x)$ by

$$P(x - .5 < X < x + .5)$$

- and approximate $P(a \leq X \leq b)$ by

$$P(a \leq X \leq b) \cong P(\frac{a - 0.5 - n\pi}{\sqrt{n\pi(1-\pi)}} \leq Z \leq \frac{b + 0.5 - n\pi}{\sqrt{n\pi(1-\pi)}})$$

- The text NCT, makes a distinction as to when the continuity corrections can be applied if $(5 < n\pi(1-\pi) < 9)$—this is really not necessary since continuity corrections will always improve accuracy

- In some cases the difference with or without corrections is minimal $(n\pi(1-\pi) > 9)$

6.10 Example of Continuity Correction for Binomial Approximation

- Suppose 36% of a companies workers belong to unions and the personnel manager samples 100 employees. We are asked to find the probability that the number of unionized workers in the sample is between 24 and 42 inclusive, so $\pi = .36, n = 100$

6.11 Exact Probability Using the Binomial Distribution

- Using the **binomial distribution** which gives the **exact** probability

$$\begin{aligned} P(24 &\leq X \leq 42) = \sum_{k=24}^{42} \binom{100}{k} (.36)^k (.64)^{100-k} \\ &= .9074 \end{aligned}$$

6.12 Normal Approximation without Continuity Correction

- Without continuity correction (an approximation):

$$\mu = n\pi = .36 \times 100 = 36,$$

$$\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{(100 \times .36 \times .64)} = 4.8$$

$$\begin{aligned} P(24 &\leq X \leq 42) \approx P(\frac{24-36}{4.8} \leq Z \leq \frac{42-36}{4.8}) \\ &\approx P(-2.5 \leq Z \leq 1.25) = .8882 \end{aligned}$$

6.13 Normal Approximation with Continuity Correction: A Better Aproximation

$$\begin{aligned} P(24-.5 &\leq X \leq 42+.5) = P(23.5 \leq X \leq 42.5) \\ &= P(\frac{23.5-36}{4.8} \leq Z \leq \frac{42.5-36}{4.8}) \\ &\approx P(-2.60 \leq Z \leq 1.35) = .9068 \end{aligned}$$

6.14 QUESTIONS:

1. 6.29, 6.32, 6.33, 6.50, 6.59,
2. Suppose a golfer is on the first tee of the Kingston Golf and Country Club. His first shot is taken with a driver and his second shot is taken with a three iron.

Let X_1 be the length with his driver and X_2 be the length with his three iron.
Assume that X_1 and X_2 are independently normally distributed, with: and

$$\begin{aligned} \mu_1 &= 200 \text{ yards}, \quad \sigma_1 = 20 \text{ yards} \\ \mu_2 &= 150 \text{ yards}, \quad \sigma_2 = 10 \text{ yards} \end{aligned}$$

What is the probability that his first two shots travel more than 400 yards?

3. The distance a discus thrower can throw a discus is normally distributed with mean 100 meters, and variance 25 meters. If his throws are independent, what is the probability that 2 out of 3 throws travel more than 105 meters?

- Skip the exponential Distribution

6.15 Jointly Distributed Continuous Random Variables

- Let $X_1, X_2, \ldots X_K$ be continuous random variables
- The **joint cumulative distribution function** $F(x_1, x_2, \ldots x_K)$ defines the probability that **simulataneously** X_1 is less than x_1, X_2 is less than x_2, and so on:

$$F(x_1, x_2, \ldots x_K) = P(X_1 < x_1 \cap X_2 < x_2 \cap \ldots X_K < x_K)$$

- The marginal distribution functions are: $F(x_1), F(x_2), \ldots, F(x_K)$
- If $X_1, X_2, \ldots X_K$ are **independent:**

$$F(x_1, x_2, \ldots x_K) = F(x_1) \times F(x_2) \times \ldots \times F(x_K)$$

6.15.1 Covariance

- As with discrete random variables, we can define covariances for continuous random variables:

$$\begin{aligned} C[X,Y] &= E[X - \mu_X][Y - \mu_Y] \\ &= E[XY] - \mu_X \mu_Y \end{aligned}$$

- If X and Y are independent then

$$C[X,Y] = 0 \Rightarrow E[XY] = \mu_X \mu_Y$$

6.15.2 Correlation

- Let $X_1, X_2, \ldots X_k$ be continuous random variables, then the **correlation between** X and Y is

$$\rho_{XY} = Corr[X,Y] = \frac{C[X,Y]}{\sigma_X \sigma_Y}$$

6.15.3 Sum of Random Variables

- Let $X_1, X_2, \ldots X_K$ be continuous random variables with means $\mu_1, \mu_2, \ldots \mu_K$ and variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_K^2$ then

$$E[X_1 + X_2 + \ldots + X_K] = \mu_1 + \mu_2 + \ldots + \mu_k$$

and if $X_1, X_2, \ldots X_K$ are independent (or have zero covariance–a weaker condition) the variance of the sum of contiuous random variables

$$V[X_1 + X_2 + \ldots + X_K] = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_K^2$$

and if the covariances are **NOT** zero, the variance of the sum of continuous random variables is:

$$V[X_1 + X_2 + \ldots + X_K] = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_K^2 + 2\sum_{i=1}^{K-1}\sum_{j=i+1}^{K} Cov[X_i, X_j]$$

6.15.4 Differences Between A Pair of Random Variables (Special case: $K = 2$)

- Let X and Y be two contiuous random variables

$$E[X - Y] = \mu_X - \mu_Y$$

- If X and Y are independent (or have zero covariance—a weaker condition)

$$V[X - Y] = \sigma_X^2 + \sigma_Y^2$$

- If covariance of X and Y is NOT zero then

$$V[X - Y] = \sigma_X^2 + \sigma_Y^2 - 2Cov[X,Y]$$

6.15.5 Linear Combinations of Random Variables

- Let X and Y be two contiuous random variables, and a, b are two constants

$$W = aX + bY$$

- $$\mu_W = E[W] = a\mu_X + b\mu_Y$$

- $$\begin{aligned}\sigma_W^2 &= V[W] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov[X,Y] \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCorr[X,Y]\sigma_X\sigma_Y\end{aligned}$$

- Exercise 6.39-6.45

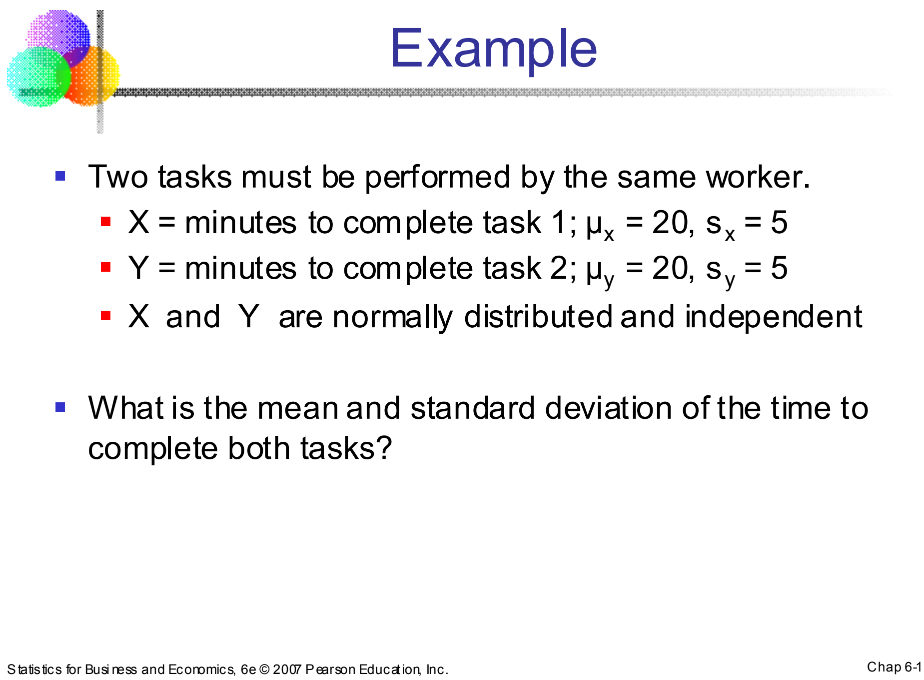

Figure 6.19:

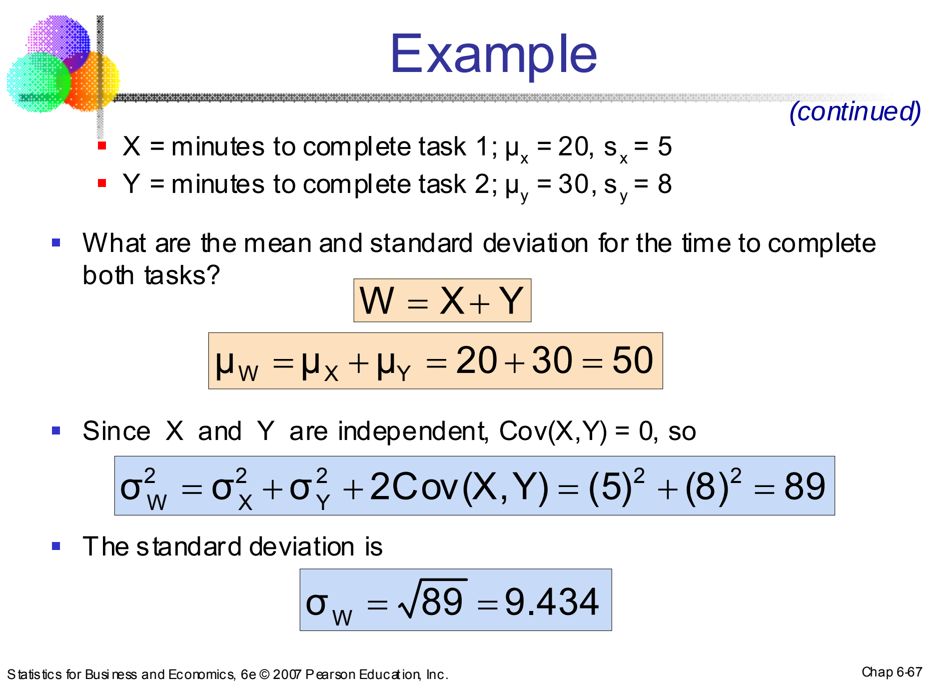

Figure 6.20: