Chapter 7

Estimation:Single Population

7.1 Introduction

- So far we have seen how to study the characteristics of samples (sampling distributions)
- Now we can formalize that by discussing *statistical inference*, how to learn about populations from random samples.

1. *Estimation* (Chapter 8-9)–Using observed data to make informed "guesses" about unknown parameters
2. *Hypothesis Testing* (Chapter 10)– Testing whether a population has some property, given what we observe in a sample.

7.2 Some Principles

- Suppose that we face a population with an unknown parameter.
- A sample statistic which we use to estimate that parameter is called an *estimator*, and when we apply this rule to the sample we have an *estimate* or a *point estimate*. [See *Transparency 8.1*]
- A simple example: Estimate μ by $\bar{X}$.
- The **estimator** is $\bar{X}$ and the **estimate** is a specific number we get when we calculate the sample mean.
- Note the actual value we calculate for the sample mean (like 4.2) is a **realization of a random variable** and is called the **estimate**
- The estimator, $\bar{X}$ is a random variable (i.e. **it has a distribution**).

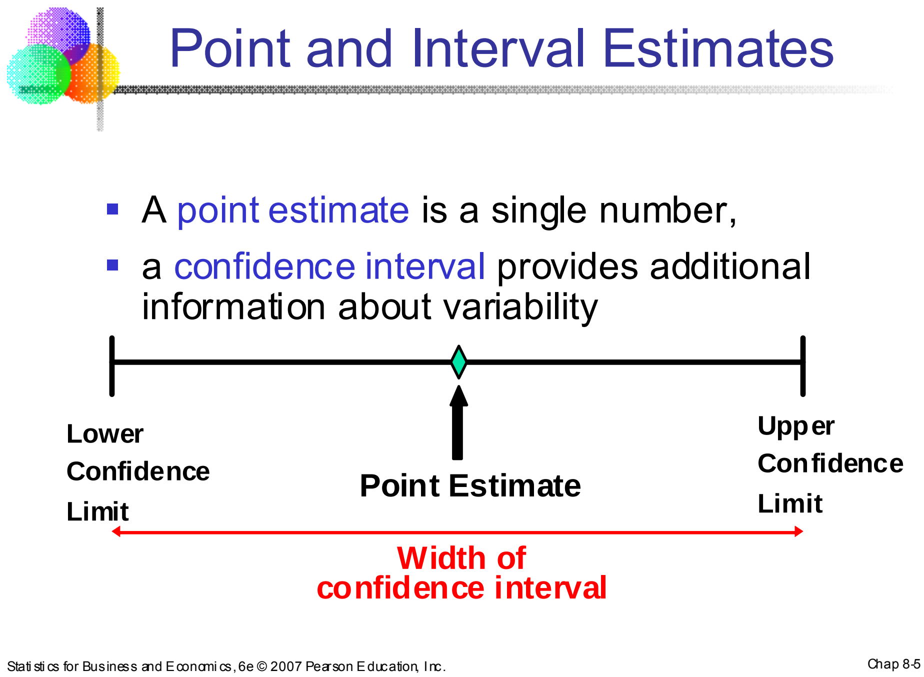


Figure 7.1:

7.3 Desirable Properties in Choosing Estimators

7.3.1 Unbiasedness

- An estimator is unbiased if its expectation equals the population parameter.
- for instance, denote the true population parameter by θ and the estimator by $\hat{\theta}$, we say $\hat{\theta}$ is an **unbiased estimator** of θ

$$E[\hat{\theta}] = \theta$$

- Accordingly, we can define **bias** as

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta$$

- We have seen that:

$$E[\bar{X}] = \mu \qquad and \qquad E[s^2] = \sigma^2. \tag{7.1}$$

- Clearly **the sample mean and sample variance are unbiased estimators.**

Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Proportion	P	$\hat{p}$

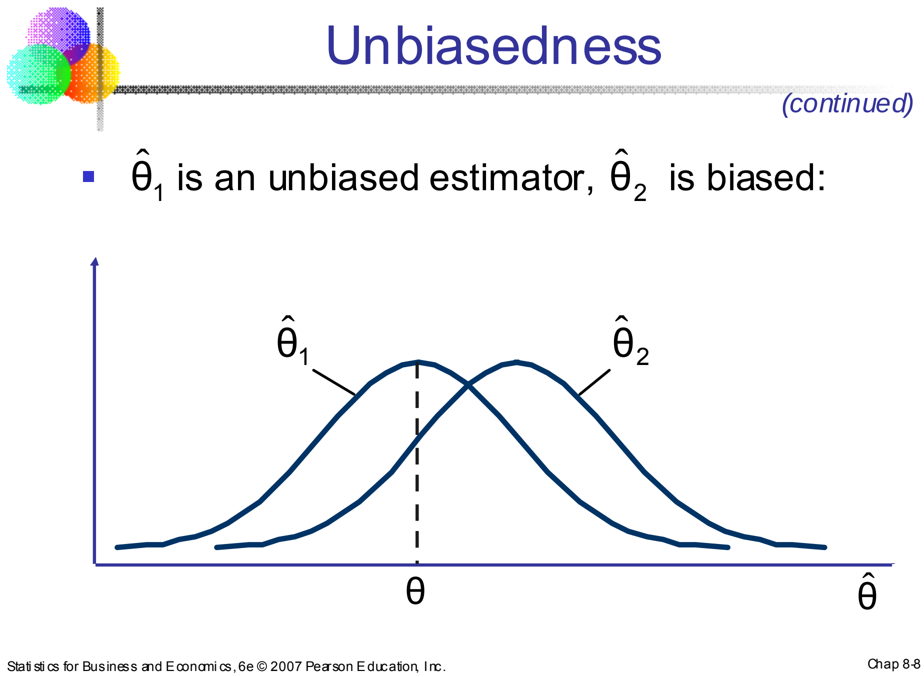
 Chap 8-6

Figure 7.2:

- The point of unbiasedness is not that we can check this directly, for we do not know the true values of μ and σ^2.

- The point is that whatever values they take, the **average** of our estimators will equal those values.

- The *sampling distribution* of the estimator is centered over the population parameter.


Unbiasedness
(continued)
$\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:
$\hat{\theta}_1$
$\hat{\theta}_2$
θ
$\hat{\theta}$
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Chap 8-8

Figure 7.3:

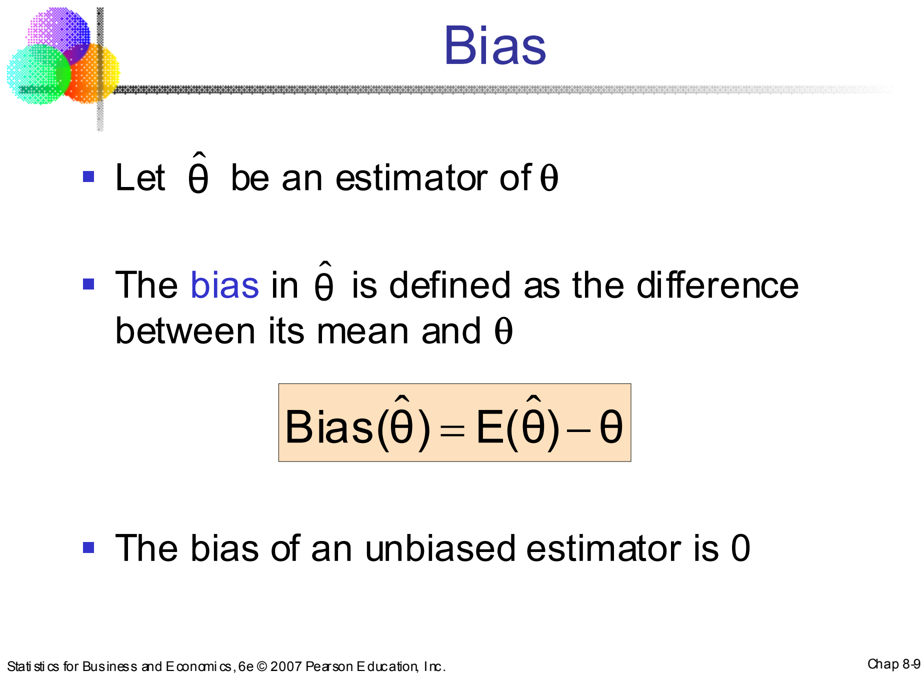
Bias
Let $\hat{\theta}$ be an estimator of θ
The bias in $\hat{\theta}$ is defined as the difference between its mean and θ
$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$
The bias of an unbiased estimator is 0
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Figure 7.4:

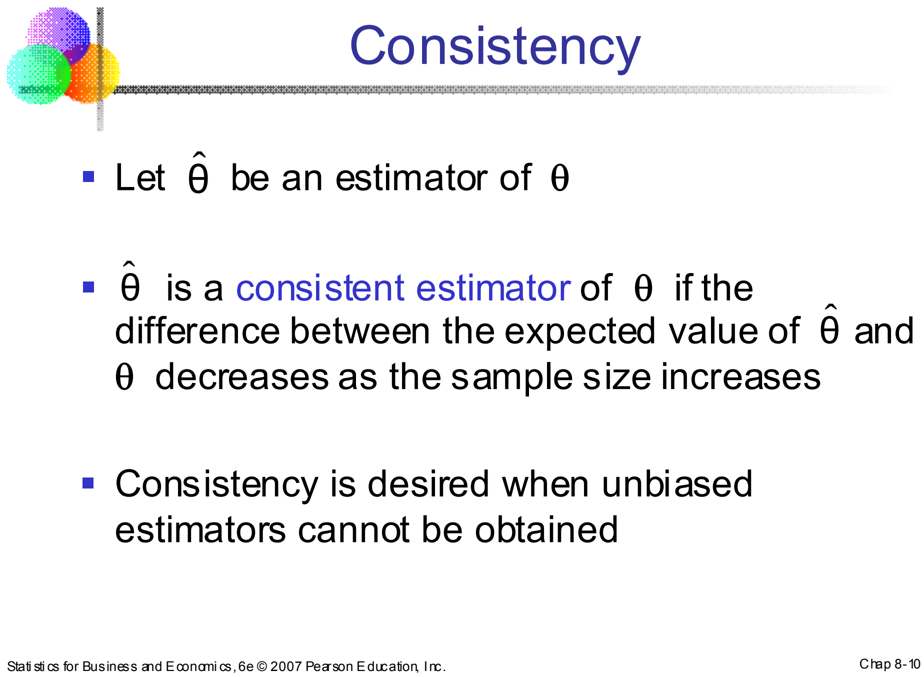

Figure 7.5:

7.3.2 Efficiency: Minimum Variance

- A second criterion to apply in choosing an estimator is that it should have as **small a sample variance** as possible.

Example:

- Suppose we want to estimate μ and we have two samples to choose from, one with 100 observations and one with 200 observations.
- Because the variance of $\bar{X}$ is σ^2/n we will have a smaller variance by using the larger sample, though both are unbiased.
- We say that the estimator using the larger number of observations is more *efficient*.

7.4 Minimum Variance Unbiased Estimator

- Let $\hat{\theta}$ be an unbiased estimator and let $\tilde{\theta}$ be any other unbiased estimator of θ.

- If $V[\hat{\theta}] \leq V[\tilde{\theta}]$, for any $\tilde{\theta}$, then $\hat{\theta}$ is a **minimum variance unbiased estimator** of θ.

- We can show that $\bar{X}$ **is a minimum variance unbiased estimator** of μ.

7.4.1 Relative Efficiency

- The relative effiiciency of $\hat{\theta}$ to $\tilde{\theta}$

$$\text{Relative Efficiency} = \frac{V[\tilde{\theta}]}{V[\hat{\theta}]}$$

- For the minimum variance unbiased estimator this ratio is always greater than 1.

- The median is less efficient than the sample mean in estimating the population mean

7.5 Confidence Interval Estimator

- So far we have seen simple examples of **point estimates**.

- But often we would like to **estimate a range** which **might** bracket the true parameter.

- These ranges are called **interval estimates or confidence intervals**. [See *Transparency 8.4*].

- A confidence interval extimator for a population parameter is a rule for determining (based on sample information) a range, or interval that is likely to include the parameter.

- The corresponding estimate is called a confidence interval estimate.

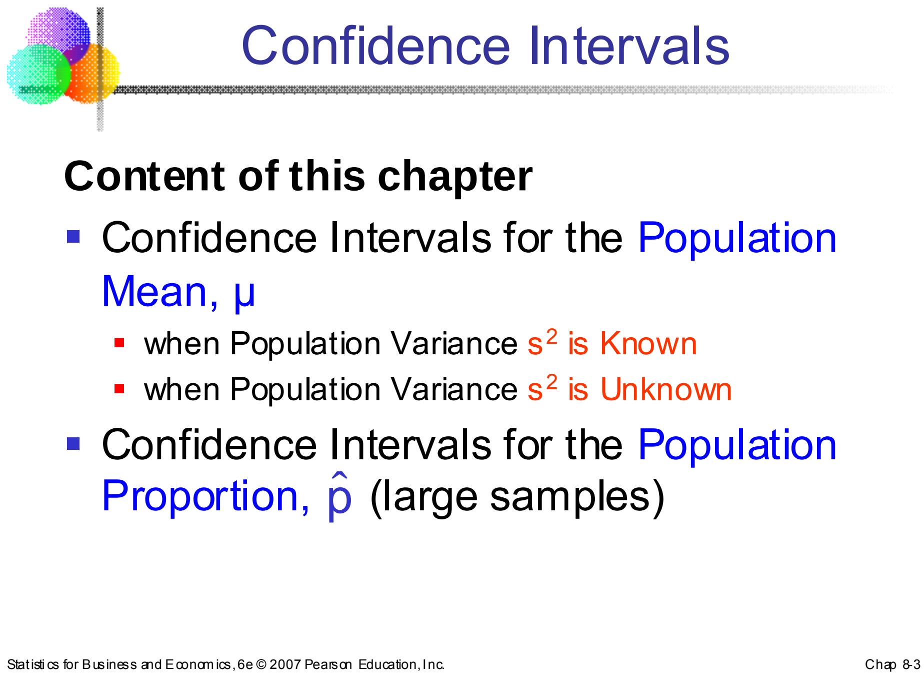
Confidence Intervals
Content of this chapter
Confidence Intervals for the Population Mean, μ
when Population Variance s² is Known
when Population Variance s² is Unknown
Confidence Intervals for the Population Proportion, p̂ (large samples)
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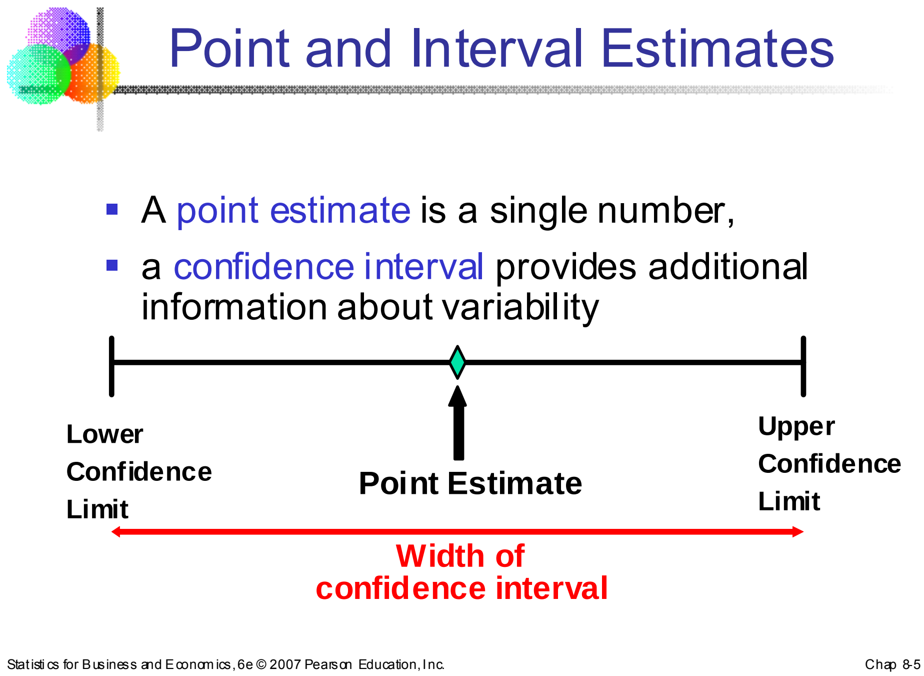
Figure 7.6:


Point and Interval Estimates
A point estimate is a single number,
a confidence interval provides additional information about variability
Lower Confidence Limit
Point Estimate
Upper Confidence Limit
Width of confidence interval
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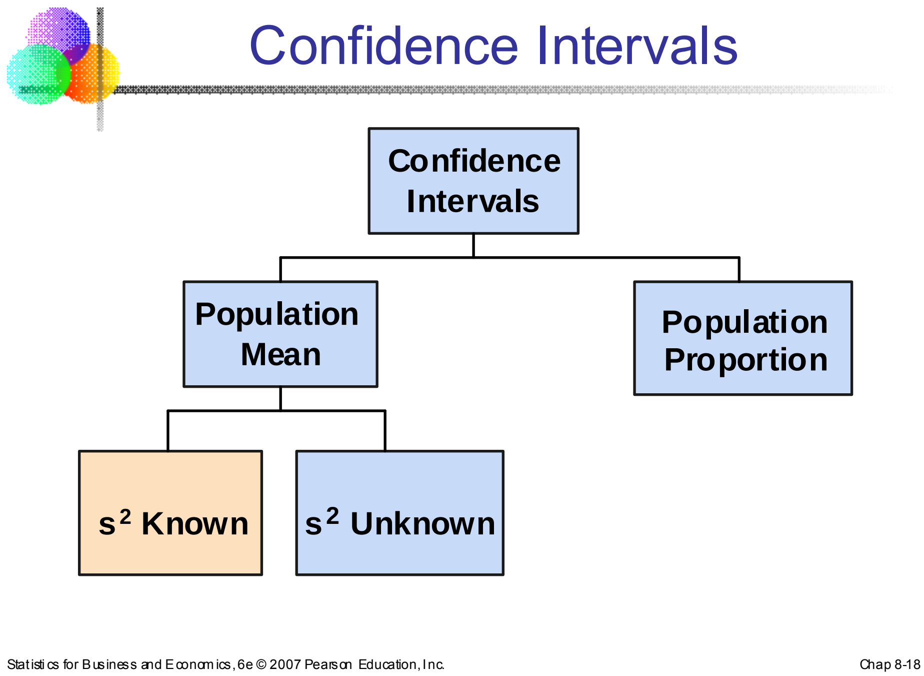
Chap 8-5

Figure 7.7:

Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Proportion	P	$\hat{p}$

Figure 7.8:


Confidence Intervals
Confidence Intervals
Population Mean
Population Proportion
s2 Known
s2 Unknown
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Chap 8-18

Figure 7.9:

7.6 Example: Population Variance σ^2 Known

- Let us take an unrealistic but simple example in which we know σ^2 but do not know μ.

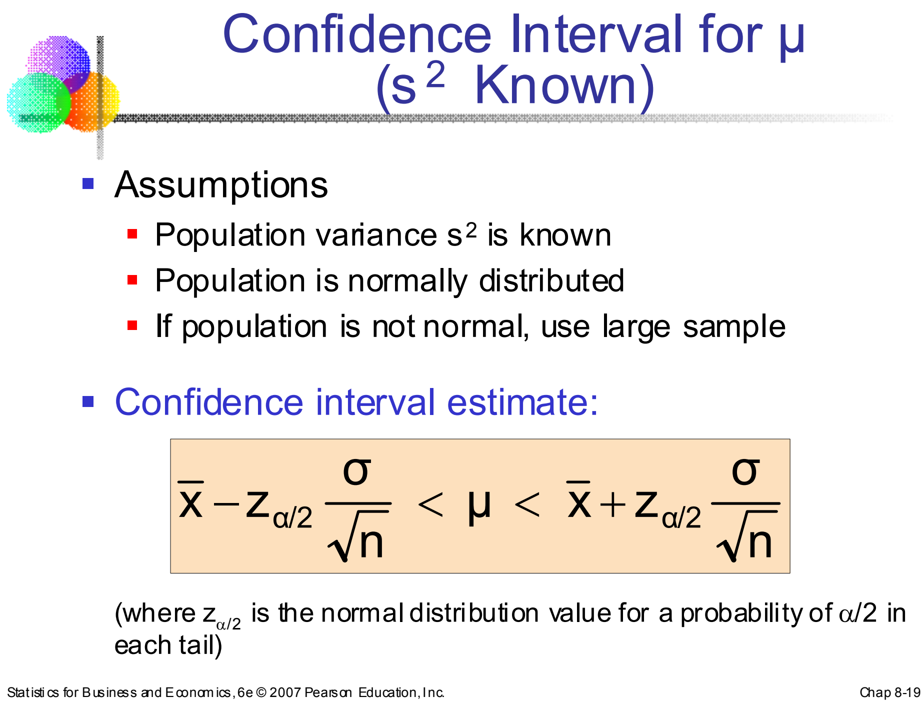
Confidence Interval for μ
(s 2 Known)
Assumptions
Population variance s2 is known
Population is normally distributed
If population is not normal, use large sample
Confidence interval estimate:
$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
(where $z_{\alpha/2}$ is the normal distribution value for a probability of $\alpha/2$ in each tail)
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Figure 7.10:

- Then we know that

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1). \tag{7.2}$$

- Because the estimator $\bar{X}$ is unbiased, this statistic has a mean of zero.
- We can see from the tabulated, standard normal distribution that there is a probability of .025 that $Z < -1.96$ and a probability of .025 that $Z > 1.96$. [See *Transparency 8.3*].
- Let us call the sum of those two cut-off probabilities α.
- And let us call the cut-off points $-Z_{\alpha/2}$ and $Z_{\alpha/2}$.
- Then the area between these points is 0.95 and $\alpha = .05$ (so $Z_{\frac{.05}{2}} = 1.96$)

$$P(-Z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}) = 1 - \alpha, \tag{7.3}$$

- We can (after some careful thinking about inequalities) obtain:

$$P(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha. \tag{7.4}$$

- This gives us a 100 (1-α) % confidence interval for the population mean μ:

$$\bar{X} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

  [See *Transparencies 8.4 and 8.6*].

- Margin of error (the sampling error, the bound, or the interval half width) is given by

$$ME = Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

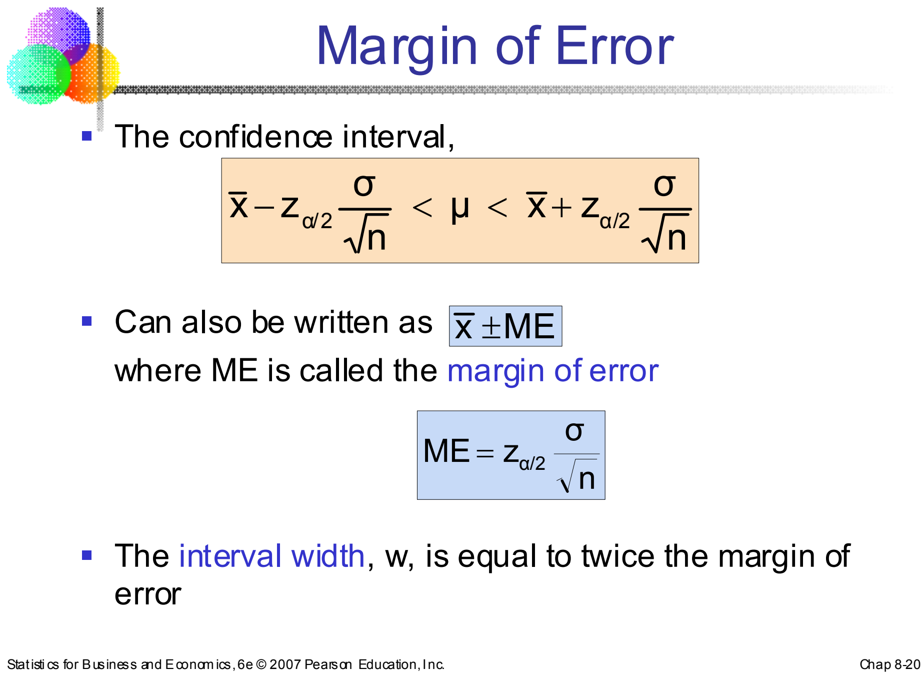
Margin of Error
The confidence interval,
$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
Can also be written as $\bar{x} \pm ME$
where ME is called the margin of error
$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
The interval width, w, is equal to twice the margin of error
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Figure 7.11:

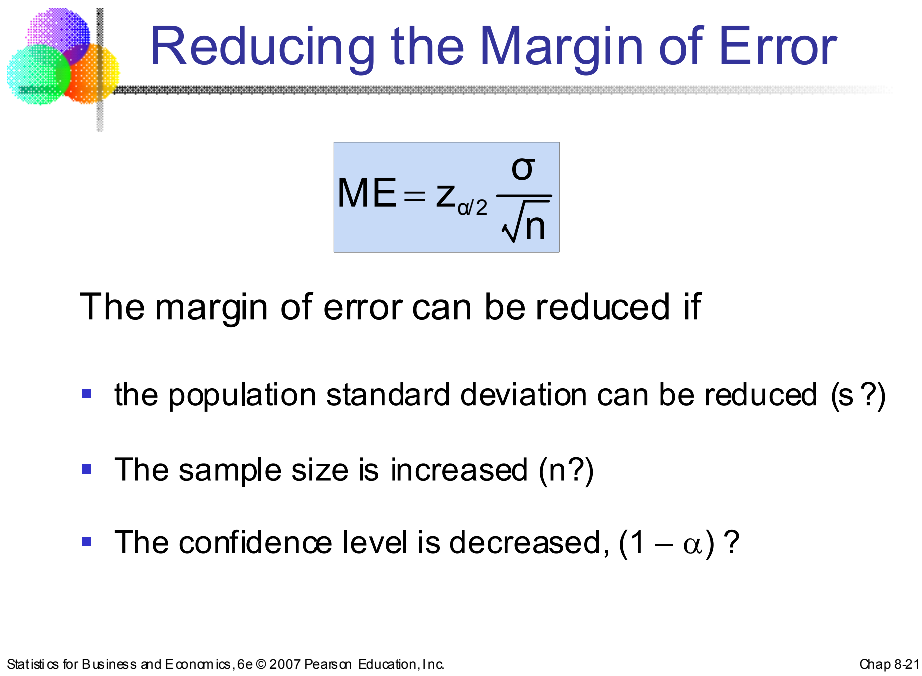

Figure 7.12:

7.7 Example of a Confidence Interval

- Ten patients are given a sleep inducing drug in clinical trials.The average increase in sleep is $\bar{X} = 1.58$ hours.
- Now suppose, unrealistically, that we know that $\sigma^2 = 1.66$.
- Then with $n = 10$ our 95 percent confidence interval for μ is:

$$\bar{X} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 1.58 \pm 1.96\frac{\sqrt{1.66}}{\sqrt{10}} = (.78,\ 2.38) \tag{7.5}$$

7.8 Notes and Interpreting Confidence Intervals

- We know that
-

$$P(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha. \tag{7.6}$$

However from the above example we **cannot** say:

$$P(.78 \leq \mu \leq 2.38) = .95 \tag{7.7}$$

- Once we have calculated the confidence interval (the realization of a random variable) μ is either in or out (ie. **probability is zero or 1**)

7.8.1 Appropriate Interpretation of a Confidence Interval

- Imagine that we select another sample then work out another confidence interval and if we keep taking additional samples (of the same size) then we obtain a set of confidence intervals.

- We can say that 95% of these confidence intervals contain the true μ.

- We do not know whether any particular interval contains μ or not.

7.8.2 Notes on Confidence Intervals

1. Notice that the for a given α, the confidence interval is smaller as n (sample size) increases.

2. If we wish to make a more confident statement (a smaller α) then the confidence interval must be wider (i.e. $Z_{\alpha/2}$ is larger)

3. If σ increases, the confidence interval increases.

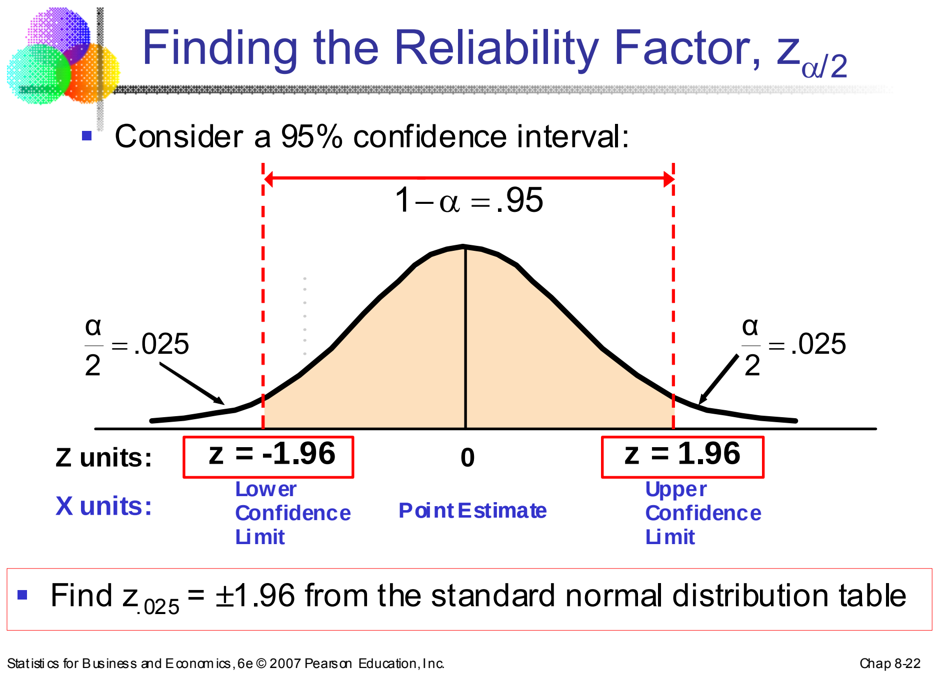
Finding the Reliability Factor, $z_{\alpha/2}$
Consider a 95% confidence interval:
$1-\alpha=.95$
$\frac{\alpha}{2}=.025$
$\frac{\alpha}{2}=.025$
Z units:
z = -1.96
0
z = 1.96
X units:
Lower Confidence Limit
Point Estimate
Upper Confidence Limit
Find $z_{.025}=\pm 1.96$ from the standard normal distribution table
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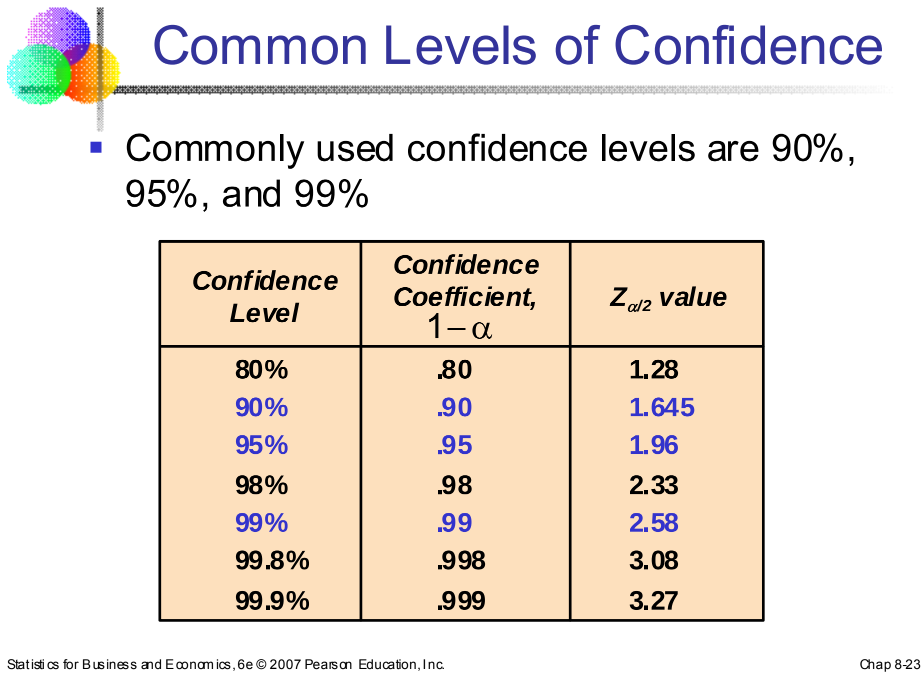
Figure 7.13:



Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, and 99%

Confidence Level	*Confidence Coefficient,* $1-\alpha$	$Z_{\alpha/2}$ *value*
80%	.80	1.28
90%	.90	1.645
95%	.95	1.96
98%	.98	2.33
99%	.99	2.58
99.8%	.998	3.08
99.9%	.999	3.27

 Chap 8-23

Figure 7.14:

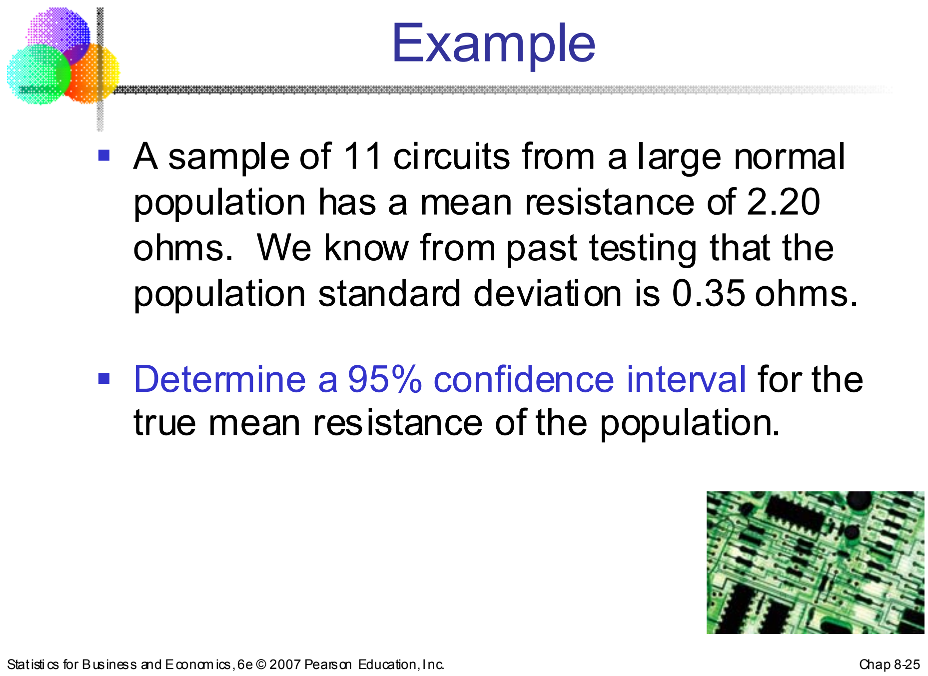

Example
A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms.
Determine a 95% confidence interval for the true mean resistance of the population.
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Figure 7.15:

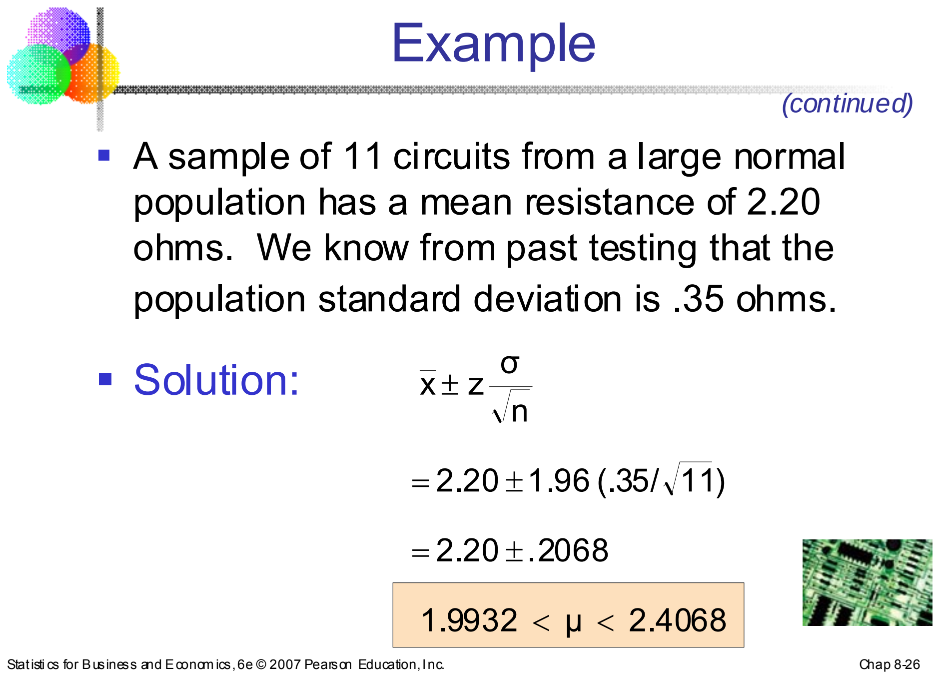

Figure 7.16:

7.9 Unknown Variance and the student's t distribution

- With this background, we can now take the usual applied situation where we do not know σ.

- If we replace σ by an unbiased estimate, the sample standard deviation in our standardized test statistic we get:

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}} \tag{7.8}$$

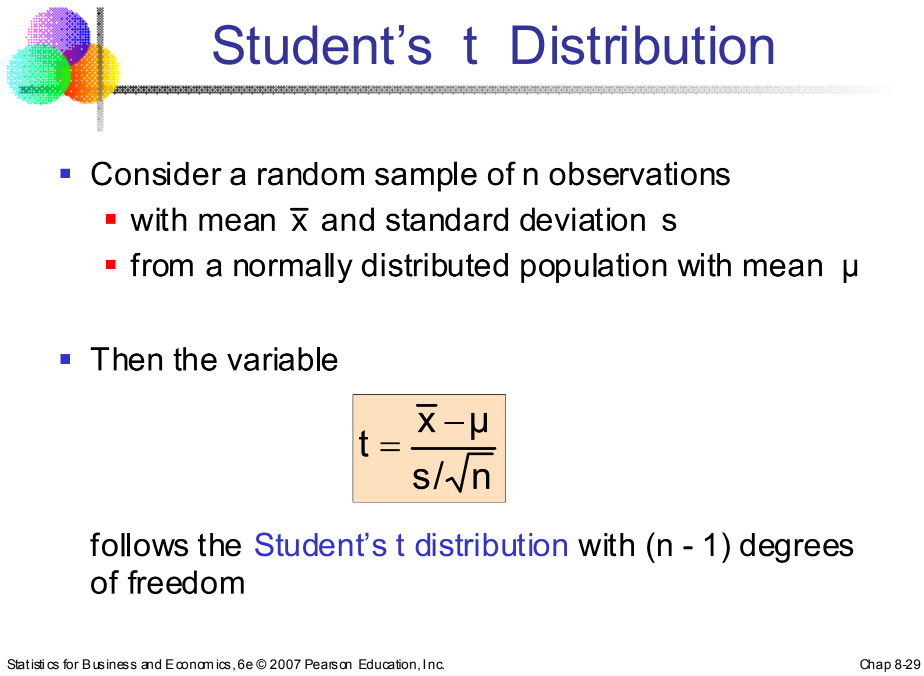
Student's t Distribution
Consider a random sample of n observations
with mean x̄ and standard deviation s
from a normally distributed population with mean μ
Then the variable
$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$
follows the Student's t distribution with (n - 1) degrees of freedom
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Figure 7.17:

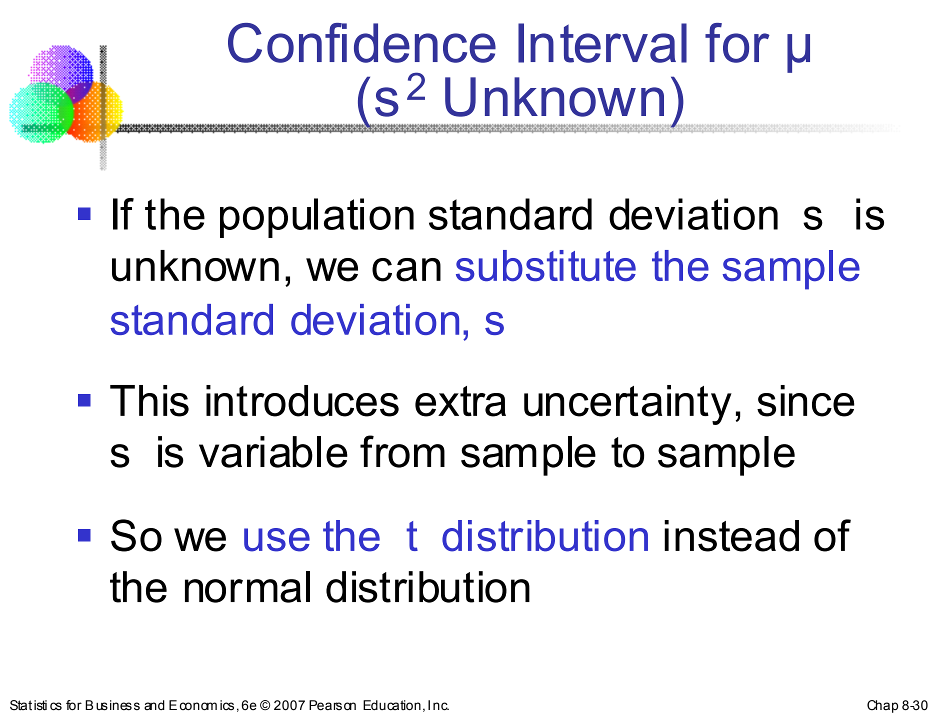

Figure 7.18:

- This statistic is distributed as a t distrbution

7.9.1 Notes on the student's t distribution

1. If we have many samples, this statistic varies across them for two reasons: because $\bar{X}$ and s both will tend to differ from sample to sample.
2. This contrasts with the sample variation in Z which arose only because of variation in $\bar{X}$.
3. This new statistic will be more variable and its distribution will be more dispersed than the normal distribution and it is said to follow student's **t distribution**. [See *Transparency 8.7*].
4. The t-distribution is tabulated (Table 8) just like the normal but depends on the **degrees of freedom**, labelled $\nu = n - 1$ for this problem. Hence we have a different value for each degrees of freedom.
5. The relationship between variables that are t-distributed and normally distributed is:
$$t \Rightarrow Z \quad as \;\; n \rightarrow \infty \tag{7.9}$$
That is the student's t distribution becomes standard normal for large n. In fact

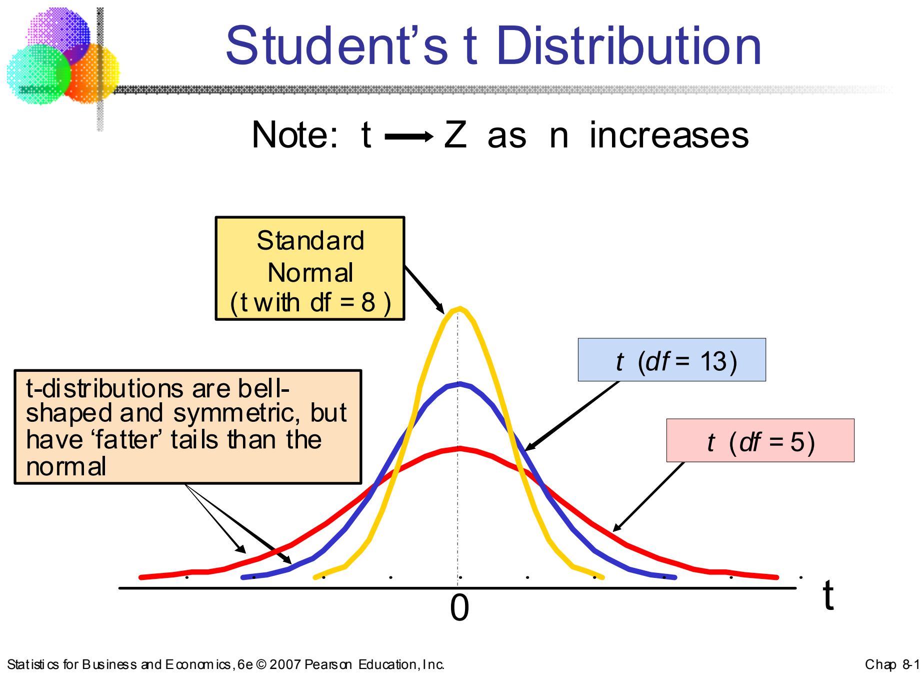

Figure 7.19:

for $n \geq 30$ the student's t distribution is close to the normal.

6. The student's t distribution is symmetric about 0 and like the normal distribution has a single mean, median and mode (at 0).

7. The student's t distribution has **thickertails** reflecting the added uncertainty from estimating the variance.

7.10 Confidence Intervals when the Variance is Unknown

- Now we can form confidence intervals for the population mean when we do not know the population variance.

$$P(\bar{X} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}) = 1 - \alpha. \qquad (7.10)$$

- The confidence interval is:

t distribution values

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z
.80	1.372	1.325	1.310	1.282
.90	1.812	1.725	1.697	1.645
.95	2.228	2.086	2.042	1.960
.99	3.169	2.845	2.750	2.576

Note: t → Z as n increases

 Chap 8-35

Figure 7.20:

$$\bar{X} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \tag{7.11}$$

7.11 Notes on Confidence Intervals with student's t distribution

1. The width of the confidence interval will vary with the sample because s varies.

2. We have indexed the t as $t_{\alpha/2,n-1}$ to remind you that this value depends on both the confidence level $100(1-\alpha)$ and the degrees of freedom n-1.

7.11.1 Confidence Interval Example with Unknown Variance

- Let us return to the example of drugs.

- Suppose that in our sample of ten trials $\bar{X} = 1.58$ and that $s = 1.23$. Then $s/\sqrt{n} = .389$.

- The value for the degrees of freedom is $\nu = n - 1 = 9$.

- A 95 percent confidence interval implies $\alpha = .05$, and $\alpha/2 = .025$ and from Table VI we find $t_{.025,9} = 2.262$.
- Thus the confidence interval is $(.70,\ 2.46)$.

Questions and Points to Note

- Why is this wider even though the sample standard deviation is *smaller* than the population standard deviation which we used in the previous example? [See *Transparency 8.8*].

- Suppose that we had $n = 20$ clinical trials on the drug with the same sample standard deviation.
- Calculate what our confidence interval be? Why should it be smaller, why?
- As we get more observations (n gets bigger) the fact that we do not know the variance and need to estimate it becomes unimportant

- The interpretation of the confidence interval is the same as before: **If we construct a large number of confidence intervals then we would expect that 95% of the intervals will bracket the true (unknown) population mean μ.**