Economics 212B
Microecoomics Theory I
Fall 2008

Note 7: Inputs and Production Functions

Introduction to Inputs and Production Functions

Inputs or Factors of Production: the productive resources such as labour, raw material and capital equipments (machines) that a firm uses to manufacture goods and services.

Production: a process by which inputs are transformed into finished (goods) or output. Firms can often choose several combinations of inputs to produce a given level of output.

Production Function: a mathematical representation that shows the maximum quantity of output a firm can produce given the quantities of inputs that it may employ. Formally,

$$Q = f(K, L)$$

where Q is the output and K and L are inputs in the production process.

Production set: set of technically feasible combinations of inputs and outputs.

Technically inefficient: set of points in the production set at which the firm is getting less output from its input (e.g labour) than it could.

Technically efficient: set of points in the production set at which the firm is producing as much output as it possibly can given the amount of labour it employs.

Labour requirement function: a function that indicates the minimum amount of labour required to produce a given amount of output.

Production function with a single input

Total product function: a production function with a single input.

Marginal product of labour: rate at which total output changes as the quantity of labour the firm uses is changed.

Increasing marginal returns to labour: the region along the total product function where output rises with additional labour at an increasing rate.

Diminishing marginal returns to labour: the region along the total product function in which output rises with additional labour but at decreasing rate.

Diminishing total returns to labour: the region along the total product function where output decreases with additional labour.

Average product: the average amount of output per unit of labour. It characterizes the productivity of workers. It is calculated

$$AP_L = \frac{\text{total product}}{\text{quantity of labour}} = \frac{Q}{L}$$

Marginal product of labour: the rate at which total output changes as the quantity of labour changes. It is calculated

$$MP_L = \frac{\text{change in total product}}{\text{change in quantity of labour}} = \frac{\Delta Q}{\Delta L}$$

- In the region of increasing marginal returns, the marginal product function is increasing.
- In the region of diminishing marginal returns, the marginal product function is decreasing.
- In the region of diminishing total returns, the marginal product function cuts through the horizontal axis and becomes negative.

Law of diminishing marginal returns: principle that as the usage of one input increases (the quantities of other inputs being held fixed) a point will be reached beyond which the marginal product of the variable input will decrease.

Relationship between marginal and marginal product:

- when $AP_L \uparrow$, must have $MP_L > AP_L$,
- when $AP_L \downarrow$, must have $MP_L < AP_L$,
- when AP_L is at its maximum, must have $MP_L = AP_L$.

Production functions with more than one input

Case of two inputs labour, L, and capital, K; $Q = f(L, K)$.

Isoquant: a curve that shows all of the combinations of labour and capital that can produce a given level of output.

- Any production function has an infinite number of isoquants, each one corresponding to a particular level of output.

- Given specific functional form production function, we can find equation for isoquant by solving for K. e.g., $Q = KL \Rightarrow$ isoquant: $K = \frac{\bar{Q}}{L}$ where $\bar{Q}$ is a constant level of output.

Economic region of production: the region where isoquants are downward sloping.

Uneconomic region of production: the region of upward sloping or backward bending isoquants. In the uneconomic region, at least one input has a negative marginal product.

- The marginal product of labour is given by

$$MP_L = \frac{\textit{change}\text{ in quantity of output Q}}{\textit{change}\text{ in quantity of labour L}}|_{\text{K is held constant}} = \frac{\Delta Q}{\Delta L}|_{\text{K is held constant}}$$

- The marginal product of capital is given by

$$MP_K = \frac{\textit{change}\text{ in quantity of output Q}}{\textit{change}\text{ in quantity of capital K}}|_{\text{L is held constant}} = \frac{\Delta Q}{\Delta K}|_{\text{L is held constant}}$$

Marginal rate of technical substitution of labour for capital: the rate at which the quantity of capital can be reduced for every one unit increase in the quantity of labour, holding the quantity of output constant. It is calculated as

$$MRTS_{L,K} = -\frac{\Delta K}{\Delta L} = \frac{MP_L}{MP_K}$$

Diminishing marginal rate of technical substitution: a production function exhibits diminishing marginal rate of technical substitution of L for K if $MRTS_{L,K}$ diminishes as the quantity of labour increases along an isoquant.

Capital-labour ratio: the ratio of the quantity of capital to the quantity of labour, $\frac{K}{L}$.

Elasticity of substitution: a measure of how easy it is for a firm to substitute labour for capital. Denote by σ the elasticity of substitution

$$\sigma = \frac{\text{percentage change in capital-labour ratio}}{\text{percentage change in } MRTS_{L,K}} = \frac{\%\Delta(\frac{K}{L})}{\%\Delta MRTS_{L,K}}$$

$\sigma \geq 0$.

Specific production functions

Linear production function (perfect substitutes): a production function of the form $Q = aL + bK$ where $a, b \geq 0$.

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{a}{b}$$

Isoquants are straight lines. Thus, the slope of any isoquant is constant and the $MRTS_{L,K}$ does not change. So, as we move along an isoquant, $\Delta MRTS_{L,K} \Rightarrow \sigma = \infty$: inputs are infinitely (perfect) substutable for each other.

Fixed-proportions (Leontief) production function (perfect complements): a production function where the inputs must be combined in a constant ration to one other. Note that K and L are perfect complements and that the elasticity of substitution is zero. Note also that the $MRTS_{L,K}$ is undefined at kinks. It typically has the form

$$Q = min\{aL; bK\}$$

Cobb-Douglas production function: intermediate between a linear production function and a fixed-proportions production function. It typically has the form
$Q = AL^a K^b$ where $A, a, b > 0$. $MRTS_{L,K} = \frac{aK}{bL}$ and $\sigma = 1$

Constant elasticity of substitution (CES) production function: it has a constant elasticity of substitution, σ, and the form

$$Q = [aL^{\frac{\sigma-1}{\sigma}} + bK^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$$

where a, b, and σ are positive constants.
Each of the three production functions discussed above (linear, fixed-proportions and Cobb-Douglas production functions) is a special case of the CES production function:

- if $\sigma = 0 \Rightarrow$ perfect complements
- if $\sigma = 1 \Rightarrow$ Cobb-Douglas
- if $\sigma = \infty \Rightarrow$ perfect substitutes

Returns to scale

The concept that tells us the percentage (%) by which output will increase when all inputs are increased by a given percentage (%). It is measured as

$$\text{returns to scale} = \frac{\%\Delta(\text{quantity of output })}{\%\Delta(\text{quantity of } \textit{all} \text{ inputs })}$$

Assume $Q = f(K, L)$. Suppose that all inputs (K and L) are "scaled up" by the same amount $\lambda > 1$ (i.e., $\uparrow L$ to λL, $\uparrow K$ to λK, with $\lambda L > L$ and $\lambda K > K$). Let α represent the resulting proportionate increase in output, Q (i.e., $\uparrow Q$ to αQ). Need to compare αQ to $f(\lambda L, \lambda K)$:

- *Increasing returns to scale*: a proportionate increase in all input quantities results in a greater than proportionate increase in output. That is, $\alpha > \lambda$ or $\alpha Q > f(\lambda L, \lambda K)$.

- *Constant returns to scale*: a proportionate increase in all input quantities results in the same percentage increase in output. That is, $\alpha = \lambda$ or $\alpha Q = f(\lambda L, \lambda K)$.

- *Decreasing returns to scale*: a proportionate increase in all input quantities results in a less than proportionate increase in output. That is, $\alpha < \lambda$ or $\alpha Q < f(\lambda L, \lambda K)$.

Technological progress:

A change in a production process that enables a firm to achieve more output from a given combination of inputs or, equivalently, the same amount of output from less inputs.

- *Neutral technological progress*: a technological progress that decreases the amounts of labour and capital needed to produce a given output without affecting the $MRTS_{L,K}$.

- *Labour-saving technological progress*: a technological progress that causes the marginal product of capital, MP_K, to increase relative to the marginal product of labour, MP_L. At same $\frac{K}{L}$ ratio substitute more labour L to compensate for given $\downarrow K$. $\Rightarrow$ capital K more productive ($MP_K \uparrow$ faster than $MP_L \Rightarrow \downarrow MRTS$). That is, $MRTS_{post-change} < MRTS_{pre-change}$

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