Assignment 3
Economics 872 - Risk Management
Due: Tuesday June 17, 12:00 noon
Assignment Worth: 10%

In this assignment we will investigate the consequences of liquidity risk. Please document all code (i.e. use comments liberally). You need to hand in your output and write-up **as well as your code** on paper under my door (MC A421) before the due date and time. As well, you need to **email me** your code before the deadline (thompsonj@econ.queensu.ca). Your code must run the results you handed in on paper. You can use loops if you want for this assignment. This assignment is LONGER and more DIFFICULT than the previous two, so please give yourself PLENTY of time to work on it.

We are going to look at a trader who is forced to liquidate a risky position when that position is subject to liquidity risk. For simplicity, we assume that there is one risky asset x, and one riskless bond y. The riskless bond pays r at each date, where r is the risk-free rate of interest. The risky asset will be affected by liquidity risk, but for simplicity, the riskless asset will not. The underlying variation of the price of the risky asset will be described by the setup of Cox, Ross and Rubinstein (CRR) (1979), choosing the probabilities to be 1/2 you can producing the following up and down movements,

$$u = \exp[\,(r - (1/2)\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\,]$$
$$d = \exp[\,(r - (1/2)\sigma^2)\Delta t - \sigma\sqrt{\Delta t}\,]$$

This setup approximates the well known continuous brownian motion,

$$\frac{dS}{S} = \mu dS + \sigma dZ$$

where S is the stock, μ is the mean of the stock, σ is its volatility, and dZ is the brownian motion.

In addition to the exogenously given price movements above, the trader has the power to move the price themselves. An individual chooses to buy, sell or hold the stock at each time and state according to the following price impact function taken directly from Cetin, Jarrow and Protter (2002):

$$P(\omega, \Delta a(\omega), t) = e^{\alpha \cdot \Delta a(\omega)}\, price(\omega, t)$$

where ω is the state, t is the time, σ is the volatility underlying the stock, and α is the liquidity parameter used to adjust the amount of liquidity risk faced. Finally, $price(\omega, t)$ is the price derived from the CRR binomial process described above at each time and state. It is assumed that to purchase stock the appropriate amount of bond must be sold. Similarly, when selling stock, the appropriate amount of bonds are bought.

An individual is defined by a utility function: $U(\cdot)$, that is strictly increasing, quasi-concave, twice differentiable and continuous. Utility is a function of the value of the liquidated stock and bond in the final period plus accumulated interest on the bond. In the three date model, expected utility can now be defined as follows:

$$E[U(X_T)] = \pi_{21}U(X_{21}) + \pi_{22}U(X_{22}) + \pi_{23}U(X_{23}) + \pi_{24}U(X_{24})$$

Where for example, π_{21} is the associated state probability at time $t = 2$, state 1. To illustrate the idea of a portfolio manager liquidating his clients positions, the assumption is made that all stock and bonds must be liquidated in the final date. It is also assumed that there are sufficiently many small traders to take the other end of all trades requested. Essentially, the portfolio manager acts as a monopolist in the market. For simplicity, we will set the bond price to 1 in all states and periods. In what is to follow, we will consider utility function from the HARA class of functions. It follows that the amount traded in the final time and state, $X_{2\omega}$ is:

$$X_{21} = (1+r)^2(y_{11}) + \left[e^{\alpha(x_{11})}(price_s(21)) + \Delta P(11)\right](x_{11})$$

$$\vdots$$

$$X_{24} = (1+r)^2(y_{14}) + \left[e^{\alpha(x_{14})}(price_s(24)) + \Delta P(12)\right](x_{14})$$

where $x_{2\omega}$ is the amount of the stock held and liquidated in the final date in state ω, $y_{2\omega}$ is the amount of bond held and liquidated in the final date in state ω, and where $price_s(\cdot)$ is the underlying price obtained from the CRR binomial process. Finally, define the amount the price is moved in time $t = 0$ and $t = 1$ to be:

$$\begin{aligned}
\Delta P(0) &= e^{\alpha(\overline{x}-x_0)}price_s(0) - price_s(0) \\
\Delta P(11) &= \Delta P(0) + e^{\alpha(x_0-x_{11})}price_s(11) - price_s(11) \\
\Delta P(12) &= \Delta P(0) + e^{\alpha(x_0-x_{12})}price_s(12) - price_s(12)
\end{aligned}$$

One can easily show from the definition of the CRR process that $\pi_{21} = \pi_{22} = \pi_{23} = \pi_{24} = \frac{1}{4}$ in the three period model. (you don't need to show this)

We now set up the problem:

$$\begin{aligned}
\max_x E[U(X_T)] = 1/4 \; \{ \; & U[(1+r)^2(y_{11}) + \left[e^{\alpha(x_{11})}(price_s(21)) + \Delta P(11)\right](x_{11})] + \\
& U[(1+r)^2(y_{11}) + \left[e^{\alpha(-x_{11})}(price_s(22)) + \Delta P(11)\right](x_{11})] + \\
& U[(1+r)^2(y_{12}) + \left[e^{\alpha(-x_{12})}(price_s(23)) + \Delta P(12)\right](x_{12})] + \\
& U[(1+r)^2(y_{12}) + \left[e^{\alpha(-x_{12})}(price_s(24)) + \Delta P(12)\right](x_{12})] \; \}
\end{aligned}$$

Subject to:

$$(\overline{y} - y_0) + \left[e^{\alpha(\overline{x} - x_0)} price_s(0)\right](\overline{x} - x_0) = 0$$

$$(1 + r)(y_0 - y_{11}) + \left[e^{\alpha(x_{11} - x_0)}(price_s(11)) + \Delta P(0)\right](x_{11} - x_0) = 0$$

$$(1 + r)(y_0 - y_{12}) + \left[e^{\alpha(x_{12} - x_0)}(price_s(12)) + \Delta P(0)\right](x_0 - x_{12}) = 0$$

Where $(\overline{x}, \overline{y})$ represent the endowments of the risky and risk-free assets respectively. As you can see, even with just 3 dates, this is a fairly complicated non-linear program. Instead of attempting to use a non-linear algorithm to try to find a solution, we are going to simplify the algorithm a lot so that we can get a feel for the optimal strategy. In particular, we will restrict the trades of the risky asset to be an integer value. Furthermore, we will restrict the set of possible integer values. In this way, we can use the brute force method of plugging in every integer value allowable for the stock trades, and seeing which give us the optimal solution (i.e. yields the most utility).

1. For this question, I want you to write two functions: a main function, and a get_prices function that is used to get the risky asset prices. You can use more functions if you wish to make your code more modular. We begin by estimating the effects of varying the liquidity parameter α. We assume that the stock holding can take any number in the interval $[-8, 8]$. If the stock holding is -8, it means that the trader is short selling the stock. Let the initial price of both the bond and stock be 1, and the initial endowment of each, $(\bar{x}, \bar{y}) = (3, 3)$. Let the interest rate $r = 1.05$, and the volatility $\sigma = 0.2$. Finally, let our utility function be logarithmic (i.e $ln(X_T)$). Use the values of α given, and reproduce this table:

	alpha				
	0.0	0.01	0.05	0.1	0.2
x_0	sell 11	sell 5			buy 5
x_{11}	trade 0	sell 2			trade 0
x_{12}	trade 0	sell 2			sell 7
x_{21}	buy 8	buy 4			sell 8
x_{22}	buy 8	buy 4			sell 8
x_{23}	buy 8	buy 4			sell 1
x_{24}	buy 8	buy 4			sell 1
Utility	2.056081	1.948532	1.448464		1.355211

Interpret the strategy that the trader is using, and why the results come out how they do.

2. Now repeat the same exercise with the exponential utility function. Note that $\lim_{X \to \infty} -e^{-X} = 0$, so you can simply add 1 to your utility to achieve a positive number. The following table will help you.

	alpha				
	0.0	0.01	0.05	0.1	0.2
x_0	sell 5				sell 1
x_{11}	sell 5				sell 1
x_{12}	sell 6				sell 1
x_{21}	buy 7				sell 1
x_{22}	buy 7				sell 1
x_{23}	buy 8				sell 1
x_{24}	buy 8				sell 1
Utility	0.999315	0.999034			0.996297

Interpret the strategy that the trader is using, and why the results come out how they do. Also, compare the results with what you obtained in question (1). Why and how do they differ (the why is the key part of this question).

3. ***BONUS*** This question is not required, and you will not lose any marks if you don't attempt it. The most you can receive on this assignment is 100%, so this question can help offset any marks lost above if you decide to do it. In order to receive marks for this, you have to make a substantial attempt (i.e. you need to set up the optimization problem on paper for me, and you need to get results that are reasonably close). Perform question (1) with 4 dates instead of 3. In the interest of computational time, please do this on the interval $[0, 8]$ instead of $[-8, -8]$. In other words, short sales are no longer allowed.