Endogenous probability of financial crises, lender of last resort, and the accumulation of international reserves

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(Chapter 2 of dissertation)

Abstract

In this paper, we use the global game approach to build a model of bank runs caused by aggregate liquidity shocks. We then use the model to analyze the motive of the central bank to build up international reserves as self-insurance against financial crisis. In the model, domestic banks collect deposits from foreign creditors and divide them between monetary reserves and long-term investments. Banks face stochastic aggregate withdrawal demand before the long-term projects are completed. If banks use up their own reserves and the reserves borrowed from the central bank, they must liquidate assets to meet the withdrawals. Since liquidation is costly, bank runs can happen because of self-fulfilling panics. We show that higher central bank reserves can reduce the likelihood of bank runs. We also find that the central bank’s incentive to accumulate reserve depends on the reserve held by the private sector. Finally, we show that high external borrowing cost will give central banks strong incentive to accumulate reserves to self-insure.

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Keywords: Financial Crises; Bank Runs; Lender of Last Resort; International Reserves; Global Game;

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“...As I argued in my K.B. Lall Lecture in 2001, following the Asian crisis of the late 1990s it was likely that countries might choose to build up large foreign exchange reserves in order to be able to act as a “do it yourself” lender of last resort in US dollars. It is now clear that this is exactly what many Asian countries have done.... ” — Mervyn King, Governor of the Bank of England, New Delhi, India, 20 February 2006.

1 Introduction

In this paper, we use the global game approach to build a model of bank runs caused by aggregate liquidity shocks. We then use the model to analyze the central bank’s motive to build up international reserves as self-insurance against financial crisis.

During the past decade, there has been a rapid build-up of international reserves in developing countries (see Figure 1), especially in the East Asia. The causes and the consequences of international reserve accumulation has become an important policy issue. Among the many motivations for reserve accumulation, one that is shared by most countries is to insure against financial crises. This paper analyzes one particular channel of self-insurance: the central bank holds international reserves in order to act as the lender of last resort during liquidity shocks.

We first show that, if financial crises can lead to social costs such as unemployment and output losses, the central bank would want to hold reserves and act as the lender-of-last-resort in order to reduce those costs. We then show that higher central bank reserves can reduce not only the costly liquidations and output losses during liquidity shocks, but also the likelihood of financial crises. We also analyze how low reserves held by the private sector and high external borrowing costs can give the central bank higher incentive to accumulate reserves.

Traditional theories of international reserves focus on the role of reserves as a buffer stock. Countries need reserves to smooth the adjustments that they must make when they have current account imbalances. A rule-of-thumb for reserve adequacy was that reserves should be sufficient to pay for about three to four months of imports. Traditional theories also predict that free flow of capital between countries should reduce the level of reserves because countries can borrow more easily from the international capital market. But now we realize that free flow of capital may also increase financial instability. In particular, the damage caused by capital flight during

1The Appendix provides a detailed table for the reserve levels of the largest reserve holding countries. According to “International Financial Statistics” of the IMF, international reserve includes Gold holdings, Reserve Position in the IMF, Special Drawing Rights and the Foreign Exchange holdings of the monetary authority.
the Asian financial crises came as a shock to most people. As a result, after the Asian Financial Crises, people realized that the high mobility of capital is becoming increasingly important in evaluating the adequacy of international reserves.

For example, Greenspan(1999) proposed a new rule-of-thumb of capital adequacy: countries should hold reserves at least equal to their short-term external debt, defined as debt with a remaining maturity of less than one year. Calvo(1996) argues that since capital flight can also happen among domestic residents, indicators such as the ratio between reserves and broad money supply(M2) should also be used to measure the probability of crisis. Studies of early warning system usually find that the ratio between international reserves and the short-term debt is a strong indicator of the probability of financial crises.\(^2\) Some other studies also find the ratio between reserves and the broad money supply is a predictor of financial crises.\(^3\)

As summarized by Fisher(2001):

"We have also seen in the recent crises that countries that had big reserves by and large did better in withstanding contagion than those with smaller reserves - to an extent that is hard to account for through our usual analyses of the need for reserves.

It is therefore no surprise that the traditional current account approach has been viewed more skeptically in recent years. There has been a growing conviction that emerging market countries with open capital accounts need more reserves rather

\(^2\)For example, Bussiére and Mulder(1999), Rodrik and Velasco(1999), Berg, Borensztein, Milesi-Ferretti and Patillo(1999)

than less, and that we should look to the capital account in determining a country’s need for reserves.”

In this paper, we build a model to analyze the effects of the central bank’s reserve on financial crises. We only focus on liquidity risks, and assume away exchange rate risks. We use a simple two-generation model in which the central bank collects taxes from the first generation to build up reserves for the second generation. The second generation faces a three-period bank run situation. In period 1, banks collect deposits from foreign depositors and divide them between liquid monetary reserves and illiquid long-term investments. In period 2, banks face stochastic aggregate withdrawal risks. Banks can borrow from the central bank to meet part of the withdrawal needs. If the withdrawal demand is high, banks need to liquidate long-term investments. Bank runs may happen in this period. In period 3, returns from banks’ remaining investments are realized.

The main results are as follows:

First, we use this model to explain why the central bank will have incentive to hold reserves. We assume that when private banks make investments, all domestic workers are hired by the new projects. If a project is liquidated, its workers will be unemployed. As a result, liquidation leads to unemployment and lower income of workers. However, when commercial banks make their optimal choice of reserves, they do not take into account the costs of liquidation to workers. Their optimal reserve level is thus lower than the socially optimal level. We assume that the central bank acts as a social planner and tries to maximize social welfare. As a result, the central bank has an incentive to accumulate reserves and then make loans to banks in order to reduce the social welfare costs of asset liquidation.

Second, we analyze the effects of central bank reserves on financial crisis. We model both bank runs with and without self-fulfilling panics. One important lesson from the Asian financial crisis is that crisis can happen due to self-fulfilling panics even if the fundamentals are not very bad. Although the countries which had crises did have some weaknesses in their fundamentals, it is hard to justify the scale and the damage of the crisis purely by fundamental reasons.\footnote{For example, Krugman(1999) argues that “the Asian crisis has settled some disputes... it decisively resolves the argument between “fundamentalist” and “self-fulfilling” crisis stories.(I was wrong; Maury Obstfeld was right).” Note: The first generation model of currency crisis such as Krugman(1979) and Flood and Garber(1984) model currency crisis as caused by week fundamentals, such as unsustainable budget deficit under fixed exchange rate. The second generation model, such as Obstfeld(1994) argues that currency crisis can be self-fulfilling.}
is also the reason why the crisis came as a surprise to most people. In this paper, we use the global game approach to model bank runs. In equilibrium, bank runs are caused by both the fundamental factor, which is high liquidity shock in our model, and self-fulfilling panics. We also model bank runs without panics, which is used to show more clearly the extra effects of self-fulfilling panics.

We show that under both types of models, the central bank reserve can be used to reduce liquidations of assets. Although higher central bank reserves will endogenously cause commercial banks to optimally choose lower reserves, overall, higher central bank reserves help to reduce the probability of bank runs.

Third, we analyze the central bank’s incentive to build up reserves. We find that the actions of the private sector is important. If the private sector is cautious and holds high reserves, then there is less need for the central bank to hold reserves. If the private sector holds less reserves, then the central bank will have higher incentive to hold reserves.

Finally, we extend the model to allow for external borrowing. We show that if countries can borrow from external sources such as the IMF, but find that the costs are much higher than previously expected, they will try to accumulate more reserves to self-insure.

An interesting finding is that a country may want to accumulate more reserves when it can borrow more from external sources. The reason is that if the private banks expect that the government can borrow more from external sources, and if the government can not commit not to borrow and lend to domestic banks at low costs, then domestic banks will hold less reserves. If the external borrowing cost is very high, the government may find it is better to accumulate more reserves to self-insure than borrowing from external sources at high costs during crisis.

**Related Literature**

Our paper is related to the recent works which model international reserves as an insurance against financial crises. For example, Aizenman and Lee(2005) analyze international reserves using a model of commercial bank reserve management. The commercial bank can make long-term investments but is subject to stochastic aggregate withdrawals before the investments mature. If the withdrawal demand is high, the bank must carry out costly liquidations. In

Some recent examples are Aizenman, Lee and Rhee(2004), Aizenman and Lee(2005), Garcia and Soto(2004), Jeanne and Ranciere(2005), and Kim, Li, Rajan, Sula and Willett(2005). Caballero and Panageas(2004a, 2004b) analyze how state-contingent instruments rather than reserves can be used to insure consumptions when the external borrowing constraint is subject to shocks.
equilibrium, the bank chooses the optimal deposit and reserve level to maximize its expected profit.\footnote{They also test two different views for reserve accumulation. The first view is that countries hold reserves in order to insure against financial crises. The second view is that reserves are accumulated because countries try to maintain an undervalued exchange rate in order to encourage exports. They argue that their evidence supports the self-insurance view.}

We follow Aizenman and Lee (2005) and model liquidity shocks as the need of depositors to withdraw deposits. The main difference is that we focus on the central bank’s motive to hold reserves and how the reserve policy of the central bank will interact with the reserve level chosen by private banks. The reason for the central bank to hold reserves is not to maximize the profit of the commercial banks, but to reduce the social costs of crises. This makes our paper quite different from Aizenman and Lee (2005). Since the IMF defines international reserves as the foreign exchange reserves held by the monetary authority, “central bank reserves” is also more consistent with that definition. Another important difference is that we explicitly model bank runs, so that we can analyze how international reserves could endogenously affect the likelihood of financial crises.

The self-fulfilling bank runs are modelled using the global game method approach. This approach was developed by Carlsson and van Damme (1993). The basic idea is that in a multiple equilibrium game, if we introduce some small noise into the private signal, players would become less certain about other players’ actions, and this may weaken the strategic complementarities of the payoff and lead to a unique equilibrium. Morris and Shin (1998) apply this approach to currency attacks.\footnote{There is a rapidly growing literature on the application of the global game method for analyzing different economic issues. See Morris and Shin (2000) for an introduction and Morris and Shin (2003) for a survey. Some other examples are Morris and Shin (2004) and Corsetti, Guimaraes and Roubini (2004) for debt crises, Dasgupta (2005) for contagion, Corsetti, Dasgupta, Morris and Shin (2004) for large size players, Chamley (1999) for regime switches, Hellwig (2002) and Morris and Shin (2002) for the effects of public information.} Goldstein and Pauzner (2005) and Rochet and Vives (2005) apply it to bank runs caused by stochastic asset returns. We follow the method of Goldstein and Pauzner (2005) and adapt it for our more complicated payoff structure. The advantage of the global game method is that we can get a unique equilibrium for each level of the liquidity shock, and thus, the probability of bank run is endogenously determined by people’s choice and the level of the liquidity shock instead of being exogenously given by the probability of sunspots. As a result, we can explicitly analyze how central bank reserves affect the probability of financial crises and the
social welfare. By applying the global game method to model banks runs caused by aggregate liquidity shocks, we also made a contribution to the bank run literature.

The paper is organized as follows. Section 2 explains the environment of the model. Section 3 analyzes the withdrawal game in period 2, and section 4 analyzes the optimal choice of banks in period 1. Section 5 provides numerical example of the effects of exogenous changes in central bank reserves. Section 6 analyzes the central bank’s incentive to accumulate reserves in the basic cases. Section 7 shows that if the private sector holds less reserves, then the central bank will have higher incentive to hold reserves. Section 8 extends the model to allow for borrowing from external sources. Section 9 summarizes the results and discusses possible future works.

2 The environment

2.1 The basic three-period model

Preferences and Technology

The main part of the model is a three-period bank run model. Denote time as $t = 1, 2, 3$. The economy consists of the domestic market and the international market. There are two types of domestic agents: households and bankers. The total population of each type of agents is normalized to 1. Households and bankers only care about their consumptions in period 3. The utility function of households is $u^h = -\frac{1}{c}$. Bankers are risk neutral and their utility function is $u^b = c$. Each banker is endowed with $e_b$ units of consumption goods in period 1 and each household is endowed with $e_h$ units of consumption goods in period 3. Each banker also owns a bank.

There is a two-period investment technology in the economy. Each unit of consumption goods invested in period 1 will turn into 1 unit of capital goods in period 3, which can be combined with labour to produce new consumption goods in period 3. The production function is

$$y = AK^\alpha L^{1-\alpha}$$ (1)

where $A$ is the productivity factor, $K$ and $L$ is the level of capital and labour input in each firm. We assume that only domestic agents have access to the production technology, and only households can be workers.
Banks collect deposits from foreign creditors, and divide the deposits between liquid reserves and long-term investments.

Two-Stage withdrawal game.
Stage 1: Each agent observes the aggregate liquidity shock with a private noise. Each agent decides whether to withdraw.

Stage 2: The aggregate shock is known. Every depositor also knows whether he is a mover. Movers must withdraw, non-movers also decide whether to withdraw. In both stages, banks can borrow from the central bank before liquidating assets.

Production occurs in remaining projects; Central bank loan is repaid; Remaining depositors are paid; Agents consume.

Figure 2: Timing of events

Events
The timing are shown in Figure 2. We assume that only bankers make investments in period 1. Bankers can borrow additional debt from foreign creditors in the form of deposit. We assume that in period 1, the government exogenously sets a limit to the aggregate foreign debt level, and then each bank competes for the deposits. The size of each foreign creditor is small and we assume every bank borrows from a continuum of foreign creditors of size 1. Denote the deposit by each foreign depositor in a bank as \( d_t \), since the total size of the creditors is normalized to 1, we can also use \( d_t \) to denote the total deposit in the bank. We use \( f \) to denote the debt-equity ratio \( d_t/e_b \). All lendings to domestic banks are denominated in dollars and must be paid in dollars. Foreign creditors are risk neutral.

Each bank then divides the deposits between monetary reserve in dollars and long-term investments. There is no official reserve requirement and banks can freely choose the reserve level. Let \( \gamma \) denote the reserve ratio chosen by the bank. The monetary reserve is \( \gamma d_t \), and the long-term investments is \( e_b + (1 - \gamma) d_t \). We assume that the net real return for monetary reserve is zero.

Let \( R \) denote the gross real return for unliquidated assets in period 3. In order to simplify the analysis, we design the model such that \( R \) is decided in period 1 and is not affected by the liquidations in period 2. We assume that when banks invest in long-term projects, they create a continuum of small firms. Workers are then allocated into those firms such that each firm in the economy has the same capital-labour ratio, \( k \). Workers are hired by firms in period 1 but only work for the firm in period 3. In period 2, if banks need to liquidate long-term investments, they will liquidate assets firm by firm. If a firm is liquidated, the workers hired by that firm will

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8We denote the deposit as \( d_t \) in order to avoid confusions with the “\( d \)” in an integral.
be unemployed in period 3. And we assume that for the remaining firms, workers and capital will both be paid according to their marginal product. Let $\delta$ denote the depreciation rate of capital, and we assume the value of un-depreciated capital in period 3 is 1. Then we have

$$R = A\alpha K^{\alpha - 1} L^{1 - \alpha} + (1 - \delta) = A\alpha k^{\alpha - 1} + (1 - \delta)$$

(2)

where $K$ and $L$ are capital and labour input in each firm. $R$ is not affected by the liquidations in period 2 because the the capital-labour ratio $k$ is decided in period 1 and is not affected by the liquidations. And since every firm has the same $k$, $k$ is also equal to the ratio between aggregate investments in period 1 and the aggregate labour force.

After the investments are made and workers are hired, we enter period $t=2$, where we will have a two-stage withdrawal game. (In this paper, we use “period” to denote the three periods $t = 1, 2, 3$, and “stage” to denote the two stages of the game in period $t=2$.) Right before the game starts, the nature decides the value of the aggregate liquidity shock. A random fraction $\pi(\theta)$ of foreign creditors will be chosen as ‘movers’ who must withdraw their funds and move the funds out of the country. The reinvestment return for the funds withdrawn is assumed to be zero. $\theta$ is uniformly distributed between $[0, 1]$, and $\pi(\theta) \in [0, 1]$ is strictly increasing and continuous in $\theta$. The value of $\theta$ is chosen by the nature and is independently and identically distributed, and the probability for each creditor to be a mover is $\pi(\theta)$.

We assume that after the nature chooses the value of $\theta$, each agent is not notified immediately whether he is chosen to be a mover or not. Instead, in the first stage of the game, everyone will have a private estimation of the aggregate liquidity shock that is going to happen. We model this private estimation by assuming that every agent will receive a private signal $\theta_i$, which is drawn uniformly from the interval $[\theta - \epsilon, \theta + \epsilon]$ for some small $\epsilon$. The distribution of $\theta_i$ is common knowledge. Based on the private signal $\theta_i$, each agent decides whether to withdraw his deposit right away.

Immediately after the first-stage withdrawal, the second stage of the game begins. Every agent is notified whether he is chosen to be a mover. In addition, the true value of $\theta$ becomes common knowledge, and people also observe the number of withdrawers in the first stage. If an agent is a mover but he hasn’t withdrawn in the first stage, then he must withdraw his deposit in the second stage. The amount of movers who still need to withdraw can be inferred from the value of $\theta$ and the total withdrawers in the first stage. So it is common knowledge how many resources the bank will have after serving the withdrawal of the remaining movers. Since all
information is known, we assume that non-movers in the second stage will coordinate for the better equilibrium, they will only withdraw if waiting gives them a lower payoff.

Banks are subject to the sequential-service-constraint. In either of the two stages, banks can borrow from the central bank if they have used up all their own reserves.

The central bank holds international reserves and makes loans to banks. We assume that the central bank follows the a simple policy rule, which is publicly announced in period 1. The central bank allocates a borrowing quota to each bank, and promises to lend as long as the limit is not reached. The quota is decided as follows: the ratio between bank $i$’s quota and the aggregate quota is equal to the ratio between bank $i$’s deposit balance and the aggregate deposit balance. The lending rate is normalized to zero and is fixed. With this particular assumption, we can focus on how the quantity of international reserves affect the equilibrium.

If a bank has used up the central bank borrowing quota, then it needs to liquidate the long-term projects. Each investment project can be transformed into $\lambda$ units of consumption goods and sold in the international market for $\lambda$ dollars, where $\lambda < 1$. The cash is then used to pay the depositors. We assume that the bank must make sure that the central bank loan will be repaid, so the liquidation will stop when the value of the remaining capital is just enough to repay the central bank loan. If a bank runs out of all resources in period 2, then the remaining depositors who haven’t withdrawn will lose all their deposits.

In the basic model, we assume that no other alternative methods can be used to insure against liquidity shocks. When the liquidity shock hits, neither the domestic banks nor the central bank can borrow additional money from the international market, and loans from international institutions are not available. This assumption will be modified in section 8.

In period $t = 3$, the remaining assets are used to produce consumption goods. Workers in unliquidated firms are be paid according to the marginal product of labour and the wage rate is $w = (1 - \alpha)Ak^\alpha$. Let $\phi$ denote the share of long-term projects that is liquidated in period 2, since the aggregate size of households is 1, then the aggregate wage income can be written as $(1 - \phi)w$. We assume that workers in the economy fully share their income.

After the production, consumption goods and un-depreciated capital goods are sold in the international market. We assume that consumption goods and capital goods are traded on the international market and their prices are always 1. After paying international creditors, remaining consumption goods are consumed by domestic agents.
Bank contract

The deposit contract takes the following form. Banks promise to pay gross rate \( r_m = 1 \) to anyone who withdraws in \( t = 2 \), and gross rate \( r_n \) to anyone who waits until period 3. \( r_m \) and \( r_n \) are denominated in dollars.\(^9\) Both \( r_m \) and \( r_n \) are fixed in period 1 and are not contingent on the realized value of the liquidity shock \( \pi(\theta) \).

Banks are subject to the sequential-service constraint, they must meet the withdrawal demand on a first-come-first-served basis, until they run out of all resources.

We assume banks act competitively on the deposit side, at the beginning of each period, each bank announces the deposit contract, and accept all deposits offered by depositors. In the equilibrium, each bank chooses the interest rate \( r_n \) and reserve ratio \( \gamma \) to maximize the expected utility of depositors. Since both bankers and depositors are risk neutral, banks simply maximizes the expected payoff of depositors subject to the zero expected profit condition.

Banks also act competitively on the investment side, they will take the return to long-term investments, \( R \), as given.

2.2 The central bank’s decision to build up reserves

In order to analyze the central bank’s optimal choice of international reserves, we need to know both the benefits and costs of accumulating reserves. The above three-period bank run model describes the benefits of accumulating international reserves. And we use a very simple way to model the costs for building up reserves.

We assume that there are two successive generations, the first generation lives in \( T = 1 \) and the second generation in \( T = 2 \). The income and initial international reserves for the first generation are exogenously given, but the second generation will experience the three-period bank run model. The central bank can collect taxes from the first generation to build up reserves for the second generation. The central bank will use the reserves to make loans to banks in \( T = 2 \), and at the end of \( T = 2 \), the reserve will be consumed by the second generation.

For simplicity, we assume the central bank only collects taxes from and make transfers to households, and the social welfare function is defined as the sum of the expected utility of the.

\(^9\) Although we assume \( r_m = 1 \) in this paper, we can also generalize the model to cases in which \( r_m \neq 1 \). But assuming \( r_m = 1 \) simplifies the numerical analysis of the equilibrium. We will continue to write the short-term interest rate as \( r_m \) instead of 1 because this makes the equations in this paper easier to understand.
households in $T = 1$ and $T = 2$. \(^{10}\)

Note that we are not trying to compute the optimal central bank reserve in an infinite-horizon model. We use the two-generation model because it provides a transparent and simply way to compare the incentive of the central bank to build up reserves under different situations.

3 The two-stage withdrawal game in period $t = 2$

In this section, we analyze the two-stage withdrawal game in $t = 2$. The commercial banks’ choice of $\tilde{r}$ and $\gamma$ in period $t = 1$ will be analyzed in section 4.

We will analyze two cases, each case has a different information structure in the first stage of the game. In the homogenous information case, there is no noise in the private signal and all information are common knowledge. We assume that people will coordinate for the no-run equilibrium whenever it is a feasible choice. As a result, bank runs are only caused by fundamentals (the liquidity shock) and are never caused by self-fulfilling panics. We then use the heterogenous information structure to model the possibility of self-fulfilling panics. In this case, since people are no long sure about other people’s actions, bank runs can happen because of self-fulfilling panics.

The main results can be summarized as follows: when there are no self-fulfilling panics, bank runs only happen when the liquidity shock is higher than a level $\theta_e$ above which waiting is strictly worse than withdrawing right away. Given the same choices of commercial banks and the central bank, when there are self-fulfilling panics, bank runs will happen when the aggregate liquidity shock is higher than a threshold $\theta^*$, with $\theta^* < \theta_e$. So self-fulfilling panics tend to increase the probability of bank runs.

3.1 Homogenous information

Suppose the private noise is zero and all agents have $\theta_i = \theta$, so $\theta$ and $\pi(\theta)$ are common knowledge. In stage 1, even if each person still does not know whether he will be a mover or not, it is already a common knowledge how many resources the banks will have at the end of stage 2 after paying

\(^{10}\)Since we use different utility functions for bankers and households, adding them together may distort the result. This assumption will not greatly affect the result because in the equilibrium, the expected profit for bankers is always zero, and bankers’ expected utility is always $e_b R$. The effects of central bank reserve on $R$ is usually very small.
the movers. We assume that people always coordinate for the no-run equilibrium whenever they can. As a result, if $\theta$ is low such that the non-movers can get at least $\overline{r_m}$ in period 3, then all depositors will choose not to withdraw in stage 1, and only movers withdraw in stage 2. If $\theta$ is high such that $\overline{r_m}$ in period 3 is not feasible, then all depositors will run the bank in stage 1.

Let $\kappa_e$ denote the highest level of $\pi(\theta)$ at which there is no bank run. $\kappa_e$ can be solved from:

\[
\kappa_e \overline{r_m} = \gamma d_t + \overline{b} + \lambda \phi_e [(1 - \gamma) d_t + e_b] \tag{3}
\]

\[
(1 - \kappa_e) \overline{r_m} = (1 - \phi_e) [(1 - \gamma) d_t + e_b] R - \overline{b} \tag{4}
\]

Equation (3) means that the withdrawal by movers at $\pi(\theta) = \kappa_e$ is met by the reserve of the bank $\gamma d_t$, the maximum loan from the central bank $\overline{b}$, and the proceeds by liquidating $\phi_e$ of the long-term investments. Equation (4) means that the remaining resources are just enough to give the remaining depositors the return $\overline{r_m}$ in period 3. $1 - \phi_e$ is the proportion of unliquidated investments and $\overline{b}$ is the repayment for central bank loan.

The solution of $\phi_e$ and $\kappa_e$ is

\[
\phi_e = \frac{\gamma d_t + [(1 - \gamma) d_t + e_b] R - \overline{d_t \overline{r_m}}}{(R - \lambda) [(1 - \gamma) d_t + e_b]} \tag{5}
\]

\[
\kappa_e = \frac{\gamma d_t + \overline{b} + \lambda \phi_e [(1 - \gamma) d_t + e_b]}{d_t \overline{r_m}} \tag{6}
\]

Note that if we have $\kappa_e \geq 1$, then the bank will always be able to pay $\overline{r_m}$, and no bank run will happen. Also, we have $\partial \kappa_e / \partial \overline{b} > 0$, so higher central bank reserve tends to increase $\kappa_e$ and reduce the probability of bank runs.

Suppose $\kappa_e < 1$. If $\pi(\theta) > \kappa_e$, then all depositors will run the bank in stage 1. Let $n_f$ denote the amount of the depositors who withdraw successfully in the bank run. We have

\[
n_f d_t \overline{r_m} = \gamma d_t + \overline{b} + \lambda \phi_f [(1 - \gamma) d_t + e_b] \tag{7}
\]

\[
0 = (1 - \phi_f) [(1 - \gamma) d_t + e_b] R - \overline{b} \tag{8}
\]

Equation (7) means that the maximum cash that can be paid by the bank is equal to the reserve of the bank, plus the loan from the central bank and the income from liquidating $\phi_f$ of the long-term assets. Equation (8) means that the liquidation will stop at $\phi_f$ when the value of the unliquidated assets is just enough to pay back the central bank loan.
And the solution for $\phi_f$ and $n_f$ is

$$\phi_f = 1 - \frac{\bar{b}}{[(1 - \gamma)d_t + e_b]R}$$

$$n_f = \frac{\gamma d_t + \bar{b} + \lambda \phi_f [(1 - \gamma)d_t + e_b]}{d_t r_m}$$

(9)

In summary, if $\pi(\theta) \leq \kappa_e$, no bank run happens. Only movers withdraw in stage 2. If $\pi(\theta) > \kappa_e$, all people run the bank in stage 1, even if they still do not know whether they will be movers. Only $n_f$ of the depositors can get their money back.

3.2 Heterogenous Information

3.2.1 The payoff differential function

Now we assume that the private signal $\theta_i = \theta + \epsilon_i$ is uniformly distributed over $[\theta - \epsilon, \theta + \epsilon]$ with $\epsilon$ being a small positive number.

We will focus on the stage 1 of the game. Let $n$ denote the amount of depositors who decide to withdraw in stage 1.

If $n \geq n_f$, since the maximum amount of withdrawal that the bank can meet in period $t=2$ is $n_f$, then only $n_f$ depositors can get the money, and the probability for successful withdrawal is $\frac{n_f}{n}$. And the bank fails in stage 1.

If $n < n_f$, the bank does not fail in stage 1. Since every depositor is equally likely to become movers, then in stage 2, among the remaining depositors, $\pi(\theta)$ of them will turn out to be movers, who must withdraw. So the minimum withdrawal in stage 2 is $(1 - n)\pi(\theta)$, and the minimum total withdrawal after two stages is $n + (1 - n)\pi(\theta)$. Denote it as $\kappa$:

$$\kappa \equiv n + (1 - n)\pi(\theta)$$

(10)

If $\kappa = \kappa_e$, then the remaining resources will be just enough to give depositors $\widehat{r_m}$ in period 3. (see equation 3 and 4). Since we assume that both the values of $\theta$ and $n$ become common knowledge in stage 2, people commonly known the value of $\kappa$ and the resources that banks will have in period 3. We assume that non-movers who haven’t withdraw in stage 1 will choose to wait if they can get at least $\widehat{r_m}$ by waiting. Otherwise, all remaining depositors withdraw in stage 2. As a result, bank run does not happen in stage 2 if and only if $\kappa \leq \kappa_e$. Let $n_e$ denote the level
Table 1: The withdrawal decisions and the expected utilities

<table>
<thead>
<tr>
<th>withdraw in stage 1</th>
<th>do not withdraw in stage 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq n \leq n_e$:</td>
<td>$\overline{r}m d_t$</td>
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</tr>
<tr>
<td>$n_e &lt; n &lt; n_f$:</td>
<td>$\overline{r}m d_t$</td>
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<tr>
<td>$n_f \leq n \leq 1$:</td>
<td>$\overline{r}m d_t$ : with probability $\frac{n_f}{n}$</td>
</tr>
<tr>
<td></td>
<td>0 : with probability $1 - \frac{n_f}{n}$</td>
</tr>
</tbody>
</table>

$Eu^W (Eu^{NW})$ is the expected utility if withdraw(not withdraw) in stage 1. The payoff deferential function is defined as $v(\theta, n) = \frac{Eu^W - Eu^{NW}}{d_t}$.

<table>
<thead>
<tr>
<th>withdraw in stage 1</th>
<th>do not withdraw in stage 1</th>
<th>payoff deferential function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq n \leq n_e$:</td>
<td>$Eu^W = \overline{r}m d_t$</td>
<td>$Eu^{NW} = \pi(\theta)\overline{r}m d_t + (1 - \pi(\theta)) r^n d_t$</td>
</tr>
<tr>
<td></td>
<td>$v(\theta, n) = (1 - \pi(\theta)) (\overline{r}m - r^n)$</td>
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</tr>
<tr>
<td>$n_e &lt; n &lt; n_f$:</td>
<td>$Eu^W = \overline{r}m d_t$</td>
<td>$Eu^{NW} = \frac{n_f-n}{1-n} \overline{r}m d_t$</td>
</tr>
<tr>
<td></td>
<td>$v(\theta, n) = \frac{1-n_f}{1-n} \overline{r}m$</td>
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</tr>
<tr>
<td>$n_f \leq n \leq 1$:</td>
<td>$Eu^W = \frac{n_f}{n} \overline{r}m d_t$</td>
<td>$Eu^{NW} = 0$</td>
</tr>
<tr>
<td></td>
<td>$v(\theta, n) = \frac{n_f}{n} \overline{r}m$</td>
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</tr>
</tbody>
</table>

of $n$ at which $\kappa = \kappa_e$,

$$
\begin{align*}
\kappa_e &= n_e + (1 - n_e) \pi(\theta) \\
\Rightarrow \quad n_e(\theta) &= \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)}
\end{align*}
$$

Note that $n_e(\theta)$ is positive only if $\pi(\theta) < \kappa_e$. If $\pi(\theta) \geq \kappa_e$, the amount of movers, $\pi(\theta)$, is higher than $\kappa_e$. Since movers must withdraw, so the bank must fail during the two-stage game even if the withdrawal in stage 1 is zero. In that case, we define $n_e(\theta) = 0$. $n_e$ is smaller than 1 only if $\kappa_e < 1$. If $\kappa_e \geq 1$, no bank run would happen and we define $n_e(\theta) = 1$.

Suppose $n_e(\theta) \in (0, 1)$. If $n \leq n_e$, then $\kappa \leq \kappa_e$, and there is no bank run in stage 2 and only movers withdraw in stage 2. If $n_e < n < n_f$, then $\kappa \geq \kappa_e$, and all remaining depositors will run the banks in stage 2. Since the banks can pay at most $n_f$ depositors after two stages, only $n_f - n$ depositors can get their money back. So if there is a run in stage 2, the probability for successful withdrawal is $\frac{n_f-n}{1-n}$.

The withdrawal decision and the expected payoff are summarized as follows, which is also shown in Table 1.
1. If $0 \leq n \leq n_e$, the bank will survive both stages. All withdrawers in stage 1 get $\overline{r}m d_t$. If a depositor does not withdraw in stage 1, then in stage 2, by probability $\pi(\theta)$, he will be a mover, and he will withdraw $\overline{r}m d_t$ from the bank. By probability $1 - \pi(\theta)$, he will be a non-mover. In that case, he will wait until period 3 and get $r^n d_t$ from the bank, where $r^n \in [\overline{r}m, \overline{r}]$ is the actual interest rate paid in period 3.

2. If $n_e < n < n_f$, the bank does not fail in stage 1, and all people who withdraw in stage 1 can get $\overline{r}m d_t$. In stage 2, all remaining depositors will run the bank, a withdrawer gets $\overline{r}m d_t$ by probability $\frac{n_f - n}{1-n}$, and loses his money by $1 - \frac{n_f - n}{1-n}$.

3. If $n_f \leq n \leq 1$, the bank fails in the first stage. A withdrawer in the first stage will be able to withdraw $\overline{r}m d_t$ by probability $\frac{n_f}{n}$, and will lose his money by probability $1 - \frac{n_f}{n}$. People who do not withdraw in the first stage will lose their deposits.

We define the payoff deferential function $v(n, \theta)$ as $\frac{Eu^W - Eu^{NW}}{d_t}$, where $Eu^W (Eu^{NW})$ are the expected utility for withdrawing (not withdrawing) in stage 1. It is better to withdraw if $v(n, \theta) > 0$. We divide the difference of $Eu^W$ and $Eu^{NW}$ by $d_t$ because $d_t$ is given in period 1 and does not affect the result of the game in period 2.

**The shape of $v(n, \theta)$ given the value of $\theta$**

We first make the following assumption:

**Assumption 1.** The bank will always be able to pay the promised long-term rate $\overline{r}m$ in period 3 if the liquidity shock in period 2 can be met by the bank’s own reserve and the reserve borrowed from the central bank, i.e., when the liquidity constraint is not binding.

The purpose of this assumption is to rule out the special case where the bank can not make the promised payment in period 3 (i.e., $r^n < \overline{r}m$) even when the withdrawal in period 2 is very low.\(^{11}\) With this assumption, the payment in period 3 will be $r^n = \overline{r}m$ as long as the total

\(^{11}\)In our model, the bank’s profit can be increasing in $\pi(\theta)$ when the liquidity shock is very low. The reason is that when the liquidity shock is low, the bank does not need to liquidate assets. At the same time, the payment if the depositor withdraws early, $\overline{r}m$, is lower than the payment if the depositor chooses to wait, $\overline{r}m$, so the bank’s profit may actually increase. So theoretically, there exists a special case where bank’s profit is negative and $r^n < \overline{r}m$ when the liquidity shocks are very low. But we usually do not see these type of contract in reality. In order to simplify the model, we explicitly rule out this possibility. Actually, in our numerical analysis of the model, for the parameters we’ve tried, we find Assumption 1 always holds for bank’s optimal contract.
withdrawal in period 2 is lower than a particular level. We define this level as $\kappa_s$. And we define $n_s(\theta)$ as the level of $n$ at which $\kappa = \kappa_s$. So for $n \leq n_s(\theta)$, we have $r^n = \bar{r}m$. The solution for $\kappa_s$ and $n_s(\theta)$ are as follows:

**Proposition 1.** $n_s(\theta) = \frac{\kappa_s - \pi(\theta)}{1 - \pi(\theta)}$, and the solution for $\kappa_s$ is

$$
\kappa_s = \frac{[(1 - \gamma)d_t + e_b]R - \bar{b} + (\gamma d_t + \bar{b})\frac{R}{\lambda} - d_t \bar{r}m}{\frac{R}{\lambda}d_t \bar{r}m - d_t \bar{r}m}
$$

(12)

Proof: see the Appendix.

Figure 3 shows the shape of $v(\theta, n)$ when $n_e(\theta)$ is between $(0,1)$. The figure shows how the value of $v(n, \theta)$ would change when we fix the value of $\theta$ and vary the value of $n$.

Over $n \in [0, n_s(\theta)]$, $v(\theta, n)$ is equal to $(1 - \pi(\theta))(\bar{r}m - r^n)$, which is a constant. But for very high $\theta$, we may have $\pi(\theta) > \kappa_s$ and we will not have $n_s(\theta) > 0$. In that case, we define $n_s(\theta) = 0$.

Over $(n_s(\theta), n_e(\theta))$, we have the following result:

**Proposition 2.** Over $(n_s(\theta), n_e(\theta))$, $\bar{r}m > r^n > \bar{r}m$, $v(\theta, n) = (1 - \pi(\theta))(\bar{r}m - r^n)$ is negative and increasing in $n$. The solution for $r^n$ is

$$
r^n(\kappa) = \frac{R}{\lambda} \bar{r}m + \frac{1}{1 - \kappa} W
$$

(13)

where $W$ is a constant

$$
W = \frac{[(1 - \gamma)d_t + e_b]R + \frac{R}{\lambda}(\gamma d_t + \bar{b} - d_t \bar{r}m) - \bar{b}}{d_t}
$$

(14)

Proof: see the Appendix.

At $n = n_e(\theta)$, $r^n = \bar{r}m$ and $v(\theta, n) = 0$. 

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Over \( n_e(\theta) < n < n_f \), we have \( v(\theta, n) = \frac{1 - n_f}{1 - n} \), which is positive and strictly increasing in \( n \). Over \( n_f \leq n \leq 1 \), we have \( v(\theta, n) = \frac{n_f}{n} \), which is positive and strictly decreasing in \( n \).

It is decreasing in \( n \) because when \( n > n_f \), the bank fails in stage 1, and increasing \( n \) will only reduce the probability for each withdrawer to get the money.

### 3.2.2 The threshold equilibrium

If \( \kappa_e \geq 1 \), no bank run will happen. So we will focus on the cases where \( \kappa_e < 1 \).

Before we derive the equilibrium, we first add a refinement condition. In our model, bank runs are only possible when \( \kappa_e < 1 \), which also means \( n_e(\theta) < 1 \). From figure 3, we know that as long as \( n > n_e(\theta) \), we have \( v(\theta, n) > 0 \). This implies that if everyone thinks that all other people will withdraw in stage 1, that is \( n = 1 \), then it would be better for everyone to withdraw. This holds for any \( \theta \). But it may be reasonable to think that when the private signal of the liquidity shock is extremely low, people will not withdraw from banks because of self-fulfilling panics. So we impose the following refinement condition:

**Assumption 2.** Agents will coordinate on an equilibrium in which when agents observe signals that are extremely low (\( \theta_i \in [0, \epsilon] \)), they do not run. Agents will coordinate for this equilibrium as long as the following condition is satisfied: given that all agents \( j \neq i \) do not run for \( \theta_j \in [0, \epsilon] \), agent \( i \) also finds it is better not to run for \( \theta_i \in [0, \epsilon] \).

The meaning of this assumption is that people will not run when the private signal is extremely good (so that they know that the fundamental is excessively good).\(^\text{12}\)

If the condition of Assumption 2 is not satisfied, that is, if we can not find an equilibrium in which people do not run the bank when the private signal is extremely good: \( \theta_i \in [0, \epsilon] \), then we simply assume that everyone will run the bank regardless of the private signal.

In the following analysis, we assume that Assumption 2 is satisfied. And we have the following result:

**Proposition 3.** Suppose \( \kappa_e < 1 \), then in the two-stage withdrawal game, there is only one unique equilibrium, and the equilibrium takes the form of threshold equilibrium. Depositors will withdraw in stage 1 if the private signal is higher than a threshold level \( \theta^* \), and will not withdraw if the private signal is lower than \( \theta^* \).

\(^{12}\)See Goldstein and Pauzner(2005) for a discussion of this type of refinement condition in global games.
Proof: see the Appendix

The meaning of the proposition is that the equilibrium takes the form of threshold equilibrium, and the threshold $\theta^*$ is unique. And there is no other non-threshold equilibrium.

Here we will define the equilibrium condition and explain the basic steps of the proof, we will then show the basic results of the equilibrium.

The private signal received by each depositor is the true value of the fundamental variable plus a uniformly distributed noise:

$$\theta_i \in [\theta - \epsilon, \theta + \epsilon]$$ (15)

As a result, in the equilibrium, the expected proportion of depositors who run the bank at each realized value of $\theta$ is

$$n(\theta, \theta^*) = \begin{cases} 
1 & : \text{if } \theta > \theta^* + \epsilon \\
\frac{1}{2} + \frac{\theta - \theta^*}{2\epsilon} & : \text{if } \theta^* - \epsilon \leq \theta \leq \theta^* + \epsilon \\
0 & : \text{if } \theta < \theta^* - \epsilon 
\end{cases}$$ (16)

where $\theta$ in $n(\theta, \theta^*)$ is the realized value of $\theta$, and $\theta^*$ is the threshold level of the equilibrium strategy. When $\theta > \theta^* + \epsilon$, all private signals $\theta_i$ are higher than $\theta^*$, and everyone would withdraw. If $\theta < \theta^* - \epsilon$, all private signal $\theta_i$ are lower than $\theta^*$, and no one would withdraw. When $\theta^* - \epsilon \leq \theta \leq \theta^* + \epsilon$, $n(\theta, \theta^*)$ increases uniformly from 0 to 1.

Denote the candidate for the threshold equilibrium as $\theta'$. And depositor $i$ withdraws if the private signal $\theta_i$ is higher than $\theta'$. Let $\Phi(\theta_i, \theta')$ denote the depositor’s expected utility differential between withdrawing and not withdrawing in stage 1 when the private signal is $\theta_i$. Since $\theta_i$ is uniformly distributed between $[\theta - \epsilon, \theta + \epsilon]$, after the depositor observes $\theta_i$, the posterior belief of $\theta$ is a uniform distribution over $[\theta_i - \epsilon, \theta_i + \epsilon]$. For each value of $\theta$, the payoff differential function is $v(\theta, n(\theta, \theta'))$. Note that here the value of $n(\theta, \theta')$ is affected by $\theta$ according to equation (16). $\Phi(\theta_i, \theta')$ is the average of $v(\theta, n(\theta, \theta'))$ over $[\theta_i - \epsilon, \theta_i + \epsilon]$.

$$\Phi(\theta_i, \theta') = \frac{1}{2\epsilon} \int_{\theta_i - \epsilon}^{\theta_i + \epsilon} v(\theta, n(\theta, \theta')) d\theta$$ (17)

In the equilibrium, depositors will prefer to withdraw if $\theta_i > \theta'$ and will prefer to wait if $\theta_i < \theta'$. Depositors will be indifferent between these two choices if the private signal is equal to the threshold, $\theta_i = \theta'$. So the equilibrium is defined by $\Phi(\theta', \theta') = 0$.

In the Appendix, we prove proposition 3 with the following two steps.
The first step is to show that there is a unique threshold equilibrium. Let $\theta_e$ denote the $\theta$ at which $\pi(\theta) = \kappa_e$. We show that $\Phi(\theta', \theta')$ is negative when $\theta'$ is very low: $\theta' \leq \epsilon$, and is positive when $\theta'$ is very high: $\theta' > \theta_e + \epsilon$. We then show that over $[\epsilon, \theta_e + \epsilon]$, $\Phi(\theta', \theta')$ is continuous and strictly increasing in $\theta'$. So there exists one and only one value of $\theta'$ which satisfies the equilibrium condition $\Phi(\theta', \theta') = 0$. Define this unique value of $\theta'$ as $\theta^*$, and we have

$$
\Phi(\theta^*, \theta^*) = \frac{1}{2\epsilon} \int_{\theta^* - \epsilon}^{\theta^* + \epsilon} v(\theta, n(\theta, \theta^*)) d\theta = 0 \tag{18}
$$

We then prove that if the private signal $\theta_i$ is lower than $\theta^*$, then $\Phi(\theta_i, \theta^*) < 0$ and depositors will choose to wait. And if $\theta_i > \theta^*$, then $\Phi(\theta_i, \theta^*) > 0$ and depositors will choose to withdraw. So depositors will not want to deviate from the equilibrium strategy, and $\theta^*$ is indeed a threshold equilibrium.

In the second step, we show that there is no other non-threshold equilibrium.

At $\theta_i = \theta^*$, depositors are indifferent between the two choices. In this paper, we simply assume that depositors choose to wait at $\theta_i = \theta^*$. This assumption does not affect the result in this paper.

### 3.2.3 The equilibrium when $\epsilon \to 0$

We will focus on the simple case where the private noise is approaching zero. When $\epsilon \to 0$, it is very easy to analyze the number of people who run the banks at each level $\theta$. It turns out that either all depositors run, or no one runs.

#### Decide $\theta^*$

When $\epsilon \to 0$, we can approximate $v(\theta, n(\theta, \theta^*))$ in equation (18) with $v(\theta^*, n(\theta, \theta^*))$. The reason is as follows. In (18), no matter how small $\epsilon$ is, when $\theta$ changes from $\theta^* - \epsilon$ to $\theta^* + \epsilon$, $n(\theta, \theta^*)$ always changes uniformly from 0 to 1 (see equation (16)). Other than affecting $n(\theta, \theta^*)$, $\theta$ only affects the value of $v(\theta, n(\theta, \theta^*))$ by affecting the value of $\pi(\theta)$. This can be seen from the function form of $v(\theta, n)$ in Table 1. Given $n$, $\theta$ only enters $v(\theta, n)$ through $\pi(\theta)$ and $r^n$ (note that $n_f$ is not a function of $\theta$), and $r^n$ is related with $\theta$ only through $\pi(\theta)$. Since $\pi(\theta) \to \pi(\theta^*)$ when $\theta \to \theta^*$, we can approximate the value of $v(\theta, n(\theta, \theta^*))$ with $v(\theta^*, n(\theta, \theta^*))$. As a result, equation (18) becomes the average value of $v(\theta^*, n)$ over $n \in [0, 1]$. \[13\] $r^n$ is related with $\kappa$, which is a function of $\pi(\theta)$. 

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Because \( n \) is uniformly distributed over \([0, 1]\), we have
\[
\Phi(\theta^*, \theta^*) \approx \int_0^1 v(\theta^*, n)dn
\] (19)

This value can be computed with the function form of \( v(\theta, n) \) in Table 1. Since \( \Phi(\theta^*, \theta^*) = 0 \), essentially, we need to find the \( \theta \) at which the average value of \( v(\theta, n) \) over \( n \in [0, 1] \) is zero.

**Proposition 4.** \( \theta^* \) is implicitly defined by
\[
0 = \int_0^1 v(\theta^*, n)dn
\] (20)

If \( n_s(\theta^*) > 0 \) (i.e., \( \kappa_s \geq \pi(\theta^*) \)), \( \int_0^1 v(\theta^*, n)dn \) is equal to
\[
(\kappa_e - \pi(\theta^*))\overline{r} - (\kappa_s - \pi(\theta^*))\overline{m} - \frac{R\overline{r}}{\lambda}(\kappa_e - \kappa_s) + W \ln \left( \frac{1 - \kappa_s}{1 - \kappa_e} \right) + (1 - n_f)\overline{r}[\ln(1 - \kappa_e) - \ln(1 - n_f) - \ln(1 - \pi(\theta^*))] - n_f\overline{m} \ln n_f
\] (21)

If \( n_s(\theta^*) = 0 \) (i.e., \( \kappa_s < \pi(\theta^*) \)), \( \int_0^1 v(\theta^*, n)dn \) is equal to
\[
(\kappa_e - \pi(\theta^*))\overline{r} - \frac{R\overline{r}}{\lambda}(\kappa_e - \pi(\theta^*)) + W \ln \left( \frac{1 - \pi(\theta^*)}{1 - \kappa_e} \right) + (1 - n_f)\overline{r}[\ln(1 - \kappa_e) - \ln(1 - n_f) - \ln(1 - \pi(\theta^*))] - n_f\overline{m} \ln n_f
\] (22)

with \( W \) defined in equation (14).

**Proof:** see the Appendix.

**Bank runs in the 2-stage game**

**Proposition 5.** In the limit when \( \epsilon \to 0 \), all people will run the bank in stage 1 if \( \theta > \theta^* \), and there is no bank run in either stages of the game when \( \theta < \theta^* \). The probability for bank runs is \( 1 - \theta^* \).

**Proof:** We first show that \( \kappa_e > \pi(\theta^*) \). Because the average value of \( v(\theta^*, n) \) is zero over \( n \in [0, 1] \), and because the value of \( v(\theta^*, n) \) is negative over \([0, n_e(\theta^*)]\) and positive over \((n_e(\theta^*), 1]\), we must have \( n_e(\theta^*) > 0 \), otherwise, the average value of \( v(\theta^*, n) \) would be positive. \( n_e(\theta^*) > 0 \) means
\[
\frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)} > 0 \implies \kappa_e > \pi(\theta^*)
\] (23)

In stage 1, according to equation (16), we have \( n = 0 \) if \( \theta < \theta^* - \epsilon \) and \( n = 1 \) if \( \theta > \theta^* + \epsilon \). If \( \epsilon \to 0 \), then \( \theta^* + \epsilon \to \theta^* \), so all people will withdraw when \( \theta > \theta^* \), and will not withdraw
if \( \theta < \theta^* \). Since \( \theta \) is uniformly distributed between \([0,1]\), bank runs in stage 1 happen with probability

\[
\int_{\theta^*}^{1} d\theta = 1 - \theta^*
\]

(24)

If \( \theta < \theta^* \), depositors will not withdraw in stage 1. In stage 2, the minimum withdrawal is the amount of movers \( \pi(\theta) \). Since \( \pi(\theta^*) < \kappa_e \), so when \( \theta < \theta^* \), we have \( \pi(\theta) < \kappa_e \), which means if non-movers choose to wait together, their payment in period 3 is higher than \( \tilde{r}^m \). As a result, non-movers will choose to wait in stage 2, and no bank run will happen.  

3.2.4 Compare with the homogenous information case

Recall that in section 3.1, when there is perfect information, people only withdraw when \( \pi(\theta) > \kappa_e \), that is, when the liquidity shock is large enough to make the bank fail. Recall that at \( \theta_e \), we have \( \pi(\theta) = \kappa_e \). Under heterogenous information, since \( \pi(\theta^*) < \kappa_e \), so \( \theta^* < \theta_e \). This means that people will withdraw before the fundamental factor (i.e., liquidity shock) is high enough to actually make the bank fail. The reason is that with the noisy signals, each person is no longer sure about other people’s actions, and people will no longer be able to coordinate perfectly.

For example, suppose all people now try \( \theta_e \) as the threshold. And suppose agent \( i \) receives private signal \( \theta_i = \theta_e \), so he knows that the true value of \( \theta \) is between \([\theta_e - \epsilon, \theta_e + \epsilon]\). When \( \epsilon \to 0 \), the true value of the liquidity shock is \( \theta \to \theta_e \). In the limit, the fundamental uncertainty (uncertainty concerning the value of the fundamental factor) almost disappears, but there is still the strategic uncertainty (uncertainty concerning the actions of other depositors). Some depositors may receive signals slightly below the threshold and choose not to withdraw, and some may receive private signals slightly above \( \theta_e \) and choose to withdraw. And agent \( i \) would find that it is not optimal to use \( \theta_e \) as the threshold, and so he will choose to deviate. In our model, only at the unique \( \theta^* \), people will choose not to deviate from the equilibrium.

As we can see, on the one hand, bank runs will only happen when the fundamental factor \( \theta \) is high enough (i.e., \( \theta > \theta^* \)). On the other hand, due to uncertainties about the actions of other people, agents will run the bank even when \( \theta \) is still lower than \( \theta_e \) (i.e., when \( \theta^* < \theta \leq \theta_e \)). So bank runs in this case are caused by both fundamental factors and self-fulfilling panics.

---

\[14\text{If the bank does not fail in stage 1, then bank runs in stage 2 are only possible when } n \in [n_e(\theta, \theta^*), n_f], \text{ but } n \text{ is located in } (0, 1) \text{ only when } \theta \in [\theta^* - \epsilon, \theta^* + \epsilon], \text{ this range is small when } \epsilon \to 0. \]
4 Decisions in period \( t = 1 \)

Given the results of the two-stage withdrawal game in period 2, we can now analyze the optimal choice of banks in period 1. Banks choose \( r^m \) and \( \gamma \) to maximize the expected payoff of depositors subject to the zero expected profit condition. What follows are the basic results of the expected utility of depositors and the expected profit of banks. Detailed computations are shown in the Appendix.

4.1 Homogenous information

Depositor’s expected payoff

Recall that when the information is homogenous, depositors will run the bank in \( t = 2 \) only when \( \pi(\theta) > \kappa_e (\theta > \theta_e) \). If a bank run happens, the expected payoff is \( n_f d_t r^m \). When \( \theta \leq \theta_e \), there is no bank run and only movers withdraw from the bank in \( t = 2 \), non-movers will wait until period 3 and get payoff \( d_t r^n \). So we have

\[
Eu^d = \int_{\theta_e}^{1} n_f d_t r^m d\theta + \int_{0}^{\theta_e} [\pi(\theta)d_t r^m + (1 - \pi(\theta))d_t r^n(\theta)]d\theta
\]

\[
\Rightarrow \frac{Eu^d}{d_t} = (1 - \theta_e)n_f r^m + \int_{0}^{\theta_e} [\pi(\theta)r^m + (1 - \pi(\theta))r^n(\theta)]d\theta
\]

where \( \frac{Eu^d}{d_t} \) is the expected return.

Bank’s expected profit

Bank’s constraint is

\[
\pi(\theta)d_t r^m \leq \gamma d_t + b + \lambda \phi[(1 - \gamma)d_t + \epsilon_b]
\]

\( (1 - \pi(\theta))d_t r^n \leq \max(\gamma d_t - \pi(\theta)d_t r^m, 0) + (1 - \phi)[(1 - \gamma)d_t + \epsilon_b]R - b \)

Equation (26) is the liquidity constraint in period \( t = 2 \), and (27) is the resource constraint in period 3. “\( \max(\gamma d_t - \pi(\theta)d_t r^m, 0) \)” means that if bank’s own reserve is higher than withdrawal in period 2, then the unused reserve will be carried to period 3.

Since bank’s gross payment is zero when \( \theta > \theta_s \), we only need to focus on \( \theta \in [0, \theta_s] \). In this range, \( r^n = r^m \) and the expected profit of the bank is

\[
E\Pi = \int_{0}^{\theta_s} \{\max(\gamma d_t - \pi(\theta)d_t r^m, 0) - b + (1 - \phi(\theta))(1 - \gamma)d_t + \epsilon_b)R
-(1 - \pi(\theta))r^m d_t\} d\theta - \epsilon_b R
\]

(28)
which is equal to the expected remaining resources after all payments are made, minus the the opportunity cost of bank’s equity $e_b R$.

4.2 Heterogenous information

Depositor’s expected payoff

Under heterogenous information, depositors will run the bank if $\theta < \theta^*$, and the probability to get paid is $n_f$. If $\theta > \theta^*$, there is no bank run and only movers withdraw in stage 2. So the expected payoff is

$$Eu^d = \int_{\theta^*}^{1} n_f d_t r^m d\theta + \int_{0}^{\theta^*} [\pi(\theta) d_t r^m + (1 - \pi(\theta)) d_t r^n(\theta)] d\theta$$

$$\Rightarrow \frac{Eu^d}{d_t} = (1 - \theta^*) n_f r^m + \int_{0}^{\theta^*} [\pi(\theta) r^m + (1 - \pi(\theta)) r^n(\theta)] d\theta$$

(29)

Bank’s expected profit

If $\theta^* > \theta_s$, since we only need to consider $\theta \in [0, \theta_s]$, the expected profit can be written in the same way as equation (28). If $\theta^* < \theta_s$, since bank’s gross payment is zero above $\theta^*$, bank’s expected profit is

$$E\Pi = \int_{0}^{\theta^*} \{\max(\gamma d_t - \pi(\theta) d_t r^m, 0) - b + (1 - \phi(\theta))(1 - \gamma) d_t + e_b) R
\quad -(1 - \pi(\theta)) r^m d_t\} d\theta - e_b R$$

(30)

which is the same as equation (28) except the upper bound of the integral is changed from $\theta_s$ to $\theta^*$.

In section 5, we will assume a particular function form of $\pi(\theta)$, and $Eu^d$ and $E\Pi$ can then be computed analytically. The details are shown in the Appendix.

4.3 Period 1 equilibrium

We will only focus on the symmetric equilibrium in which the decisions by all banks are the same.

The equilibrium is defined as follows. Each bank accepts the same amount of deposits. And all banks set the same reserve ratio $\gamma$ and deposit rate $r^m$. Given the equilibrium level of $\gamma$, the return rate $R$ is decided by the aggregate investment according to equation (2). Given $R$ and
the central bank reserve level $\bar{b}$, each bank $i$ should find the equilibrium $\gamma$ and $\bar{m}$ maximize the expected utility of its depositors, and bank $i$’s expected profit is zero.

5 Numerical example: basic effects of central bank reserves

In this section, we use a simple numerical example to show the basic effects of central bank reserves. The purpose is not to quantitatively calibrate the optimal international reserves in reality. Instead, we focus some basic qualitative results, such as whether commercial banks will choose to hold more reserves when self-fulfilling panics are possible and whether increases in central bank reserves will lead to lower probability of bank runs.

5.1 The function form and parameter values:

The main steps of the simulation are explained in the Appendix. We use the following function form of $\pi(\theta)$ in the simulation

$$\pi(\theta) = \theta^n$$

The advantage of this function form is that the integral $\int \pi(\theta) d\theta$ can be computed analytically, and as a result, we can compute the depositors’ expected utility and the bank’s expected profit analytically (see the Appendix for the detailed results.) The density function for $\pi(\theta)$ is $f(\pi) = \frac{1}{\eta \pi^{\frac{1}{\eta} - 1}}$.\(^{15}\) We set $\eta = 4$ in our simulation. The reason is as follows: since $\theta$ is distributed over $[0, 1]$, given $\theta$, the value of $\theta^n$ is higher for lower values of $\eta$. So when $\eta$ is low, it is more likely for the liquidity shock $\pi$ to take high values, and banks will hold more reserves. We find that given the values of parameters we chose below, there is no bank runs for $\eta = 1$ or 2. And at $\eta = 3$, bank runs only happen when the central bank reserve is extremely low. In order for the example to be interesting, we choose $\eta = 4$ so that bank runs happen with positive probability for low levels of central bank reserves. Higher $\eta$ will cause $\pi$ to concentrate more on low values, and banks will care less about high values of $\pi$ and will hold lower reserves. We find that this will lead to slightly higher probability of bank runs, and bank runs will also happen for wider

\(^{15}\)In the general case, when we draw a function $\pi(\theta)$ according to a density function $f(\pi)$, we only need to set the distribution function $F(\pi)$ as $\theta$, and $\pi$ is $\pi = F^{-1}(\theta)$, where $F^{-1}(\cdot)$ is the reverse of the distribution function, or the percent point function of $\pi$. For $\pi(\theta) = \theta^n$, $\theta = \pi^{\frac{1}{n}}$ and so $F(\pi) = \pi^{\frac{1}{n}}$. Take the derivative and we get the density function for $\pi$: $f(\pi) = \frac{1}{n} \pi^{\frac{1}{n} - 1}$. This way, the integral $\int \pi(\theta) d\theta$ can be computed.
Figure 4: The density function of $\pi$ when $\eta = 4$: $f(\pi) = \frac{1}{4} \pi^{-\frac{3}{2}}$.

ranges of central bank reserve levels. But the basic results, such as whether increases in central bank reserves would reduce the probability of bank runs, are insensitive to the values of $\eta$.

Table 2: Values of parameters

<table>
<thead>
<tr>
<th>A</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\bar{f}$</th>
<th>$d_t$</th>
<th>$\lambda$</th>
<th>$e_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.02</td>
<td>12.5</td>
<td>50</td>
<td>0.9</td>
<td>50</td>
</tr>
</tbody>
</table>

Other parameters are shown in Table 2. The value of $\alpha$ and $\delta$ are chosen according to the values usually used in the literature of business cycles. The meaning of $\bar{f} = 12.5$ is that the debt-equity ratio of banks is 12.5.\(^{16}\) We find that if the debt-equity ratio is very low, then bank runs may not happen in the equilibrium because banks will always be able to make the promised payment. And higher debt-equity ratio will result in higher probability of bank runs. We set the total deposit $d_t$ at 50, this value can be chosen freely. The bank equity is $e_b = d_t/\bar{f} = 4$. We then choose the level of $A$ such that the interest rate $R$ is neither too low or too high.\(^{17}\) We set $\lambda$ at 0.9. We find lower $\lambda$ would lead to higher probability of bank runs due to the higher liquidation costs. The specific values of the parameters do not affect the basic results of the example.

We set household endowment $e_h$ at 50. Please note that $e_h$ does not affect the choices of the commercial banks or the effects of central bank reserves on the probability of bank runs. $e_h$ only affects household’s utility. Higher $e_h$ means households are less affected by financial crises. The

\(^{16}\)The Basel accord requires that the risk-weighted loan-capital ratio of banks should not be higher than 12.5, and here we assume the debt-equity ratio of banks also takes a similar value.

\(^{17}\)We find that when we set $A = 1$, $A = 2$ and $A = 3$, the equilibrium lending rates are about 3%, 6% and 10% respectively. So we set $A = 2$
wage rate for households is 5.92 when banks allocate all deposits into the long-term projects. It is slightly lower when banks hold more reserves. We choose \( e_h = 50 \) so that if all long-term projects are liquidated and there is no wage income, the household income will be reduced by about 10%, which corresponds to a mild financial crisis in reality.

5.2 Basic effects of central bank reserves

Choice of commercial banks given a particular level of central bank reserve \( \bar{b} \)

First, we show how banks decide the optimal reserve ratio given a particular level of \( \bar{b} \). Since the maximum bank loan \( \bar{b} \) is proportional to the total central bank reserves, we can also use \( \bar{b} \) to denote the central bank reserve level.

The results are shown in Table 3 and Figure 5. In the example, we set \( \bar{b} = 2 \). The results are similar for other values of \( \bar{b} \).

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>Equilibrium Bank Runs</th>
<th>( R )</th>
<th>( \bar{r}^{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogenous information</td>
<td>0.1572</td>
<td>0.0105(1-( \theta ))</td>
<td>1.0603</td>
<td>1.0395</td>
</tr>
<tr>
<td>Heterogenous information</td>
<td>0.2175</td>
<td>0.0041(1-( \theta^{*} ))</td>
<td>1.0636</td>
<td>1.0419</td>
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</tbody>
</table>

In Figure 5, all banks \( j \neq i \) adopt the equilibrium reserve ratio and \( R \) is the equilibrium lending rate. The graphs show that when \( \gamma_i \) is at very low levels, increasing \( \gamma_i \) actually increases the deposit rate \( \bar{r}^{m} \). The reason is that higher \( \gamma_i \) reduces the probability of bank runs, and bank \( i \) can actually offer a higher deposit rate without violating the zero-expected-profit condition. The optimal \( \gamma_i \) is equal to the equilibrium reserve ratio.

Figure 6 shows how liquidity shocks affect the capital stock in \( t = 3 \). \( \theta_b \) in the figure is the level of \( \theta \) at which the bank has used up both its own reserve and the loan from the central bank and has to liquidate long-term assets. It is defined as

\[
\pi(\theta_b)\bar{r}_m d_t = \gamma d_t + \bar{b} \implies \pi(\theta_b) = \frac{\gamma d_t + \bar{b}}{\bar{r}_m d_t} \tag{32}
\]

Under homogenous information, bank runs happen when \( \theta > \theta_c \). If a bank run happens, the bank will liquidate all assets except \( \bar{b}/R \), which is the assets that will be used to repay the central

\[\text{The wage rate usually lies between 5 and 5.92.}\]
Figure 5: Bank $i$'s decision when $\overline{b} = 2$. The three graphs on the left are for homogenous information. The graphs on the right are for heterogenous information. Under heterogenous information, when $\gamma$ is very low, the probability of bank runs is very high and there is no equilibrium that can satisfy the zero-expected-profit condition of banks.

Figure 6: Liquidity shock $\theta$ and the level of unliquidated investments in $t = 3$. We set $\overline{b} = 2$
bank loan in \( t = 3 \). So in the figure, we can see that the level of remaining investments jumps down to \( \bar{b}/R \) at \( \theta_c \). Similarly, under heterogenous information, bank runs happen whenever \( \theta > \theta^* \), and so the remaining assets are \( \bar{b}/R \) for \( \theta > \theta^* \).

**Effects of different levels of central bank reserve \( \bar{b} \)**

Figure 7 shows the outcome when we change the level of central bank reserve exogenously. The figure shows that, although commercial banks will optimally choose to hold less reserves when the central bank reserve increases, overall, higher central bank reserves reduce the probability of bank runs.

Note that there will be no bank run when the central bank reserve is high enough. In that case, the results are the same under homogenous or heterogenous information because people only withdraw when they are sure that they are movers.

When there are bank runs, banks will choose to hold more reserves under heterogenous information, thus making less investments. Under both types of information, the equilibrium probability of bank runs chosen by banks are very low.\(^{19}\)

Figure (d) shows that under homogenous information, higher central bank reserves will reduce the value of \( 1 - \theta_b \) (the probability that liquidation of assets is needed). When \( \bar{b} \) is low, \( 1 - \theta_b \) is lower under heterogenous information because banks hold more reserves. Under heterogenous information, the increases in central bank reserves may actually cause \( 1 - \theta_b \) to increase. The reason is that increases in central bank reserves can cause rapid decreases in commercial bank reserves \( \gamma \) (see Figure a), which can cause \( 1 - \theta_b \) to increase.\(^{20}\)

Figure (e) shows that higher \( \bar{b} \) increases household’s expected utility. In this example, we assume that the central bank reserve is not consumed by households, so central bank reserves affect households’ expected utility only by affecting the wage income. Central bank reserves can be used to avoid liquidation of assets. Higher reserves also reduce the probability of bank runs and encourage banks to make more investments. All these lead to higher wage income. The expected utility is slightly lower under heterogenous information when the levels of central bank reserves is low. This is mainly because banks have to hold more reserves and make less

\(^{19}\)Depending on the values of the parameters we choose, the probability of banks runs under heterogenous information can either be higher or lower than that under homogenous information. But those two probabilities are always very close to each other and have low values.

\(^{20}\)Since \( \pi(\theta_b) = \frac{\gamma dt + \bar{b}}{R} \), if \( \gamma \) decreases quickly enough, then the nominator \( \gamma dt + \bar{b} \) can be decreasing even when \( \bar{b} \) is increasing, which means \( \theta_b \) can be decreasing in \( \bar{b} \).
Figure 7: The effects of central bank reserves. “Homo” means homogenous information, “Hete” means heterogenous information.
investments.

The main finding can be summarized as follows: First, higher central bank reserves reduces the probability of bank runs. It also gives households higher expected utility. Second, the possibility of self-fulfilling panics does not necessarily lead to higher probability of bank runs. If banks correctly take the risks into account and hold enough reserves, then the probability of bank runs will still be in low levels. We find these two results are robust to changes in the parameter values. The second result is intuitive since if banks correctly take into account the risks of self-fulfilling panics, then they will not deliberately choose to have high probability of bank runs since banks will lose all their capital if a bank run happens.

6 The central bank’s decision to accumulate reserves

This part use the simple two-generation model to analyze the central bank’s optimal choice of reserves. We assume that self-fulfilling panics are possible. The two generations are denoted as \(T = 1, 2\). The tax(transfer) for the first generation is \(\tau_1\). The central bank reserve level in \(T = 1, 2\) is \(b_1\) and \(b_2\), and \(b_2 = b_1 + \tau_1\).

The central bank reserve in \(T = 2\) has several effects: it will affect the optimal investments of banks, it can be lend to banks during liquidity shocks to reduce liquidations, it will also affect the probability of bank runs, and at the end of \(T = 2\), it will be consumed by the households.

The central bank maximizes the sum of households’ utility:

\[
 u_1 + Eu_2 = \frac{-1}{c_1} + E\frac{-1}{c_2}
\]  

where \(c_1\) and \(c_2\) are household consumption for the first and the second generation. \(c_1\) is equal to the initial income minus the tax(or plus the transfer), and

\[
 c_2 = e_h + (1 - \phi)w + b_2
\]

where \(e_h\) is household endowment, \((1 - \phi)w\) is the wage income, and \(b_2\) is central bank reserve.

In our example, we choose \([50, 55]\) as the range of the household income in \(T = 1\), and \([0, 4]\) as the range for the initial international reserves. Note that both the income and reserves in \(T = 1\) are exogenous given.

Figure 8 shows the central bank’s choice. First, Figure 8(a) and 8(d) show that with higher household income in \(T = 1\), the central bank will collect more taxes so as to allocate more
resources to the second generation. In addition, if the initial reserve is low, the central bank will collect more taxes to build up reserves for the second generation. Second, 8(b) and 8(e) show that the optimal reserve is increasing in the initial income and reserve level. Intuitively, with higher initial resources, the central bank can allocate more resources to the second generation. Third, 8(c) and 8(f) show that the social welfare is increasing in initial income and reserves.

The optimal reserve is slightly higher under heterogenous information than under homogenous information. This is not really caused by higher probability of bank runs. As we’ve seen from Figure 7, heterogenous information will lead to slightly lower household utility because banks hold more reserves and make lower investments. And so the central bank will have the
incentive to build up more reserves in order to encourage the banks to make more investments and also to allocate more consumption to the second generation.

7 When private banks hold less reserves

The previous analysis shows that if the commercial banks are very cautious and hold high reserves, then the incentive for the central bank to hold reserves may not be very high. For example, suppose originally people think there is no self-fulfilling panics. Later, people think self-fulfilling panics are possible. If the private sector takes the higher probability for bank runs into account and hold more reserves, then the probability of bank runs will be reduced to low levels and the central bank does not really need to hold a lot more reserves.

In this part, we show that if the private sector is less cautious than the central bank, then the central bank would hold more reserves. We will use a simple example. Suppose the private sector decides the reserve level according to the scenario that there is no self-fulfilling panics while the central bank is more cautious and decides its reserve level according to the scenario that self-fulfilling panics are possible, then the central bank will have a higher incentive to hold reserves.

In the case, according to the scenario used by the private sector, bank runs will happen when $\theta > \theta_e$. While according to the scenario used by the central bank, banks run happen with a higher probability when $\theta > \theta^*$. The result is shown in Figure 9. Since the private sector holds lower reserves, if self-fulfilling panics are possible, then the actual probability of financial crisis will be higher than the scenario used by the private sector, especially at low levels of central bank reserves. The central bank reserve also has a greater effect in reducing the probability of bank runs.

Because the central bank reserve has higher effect in reducing the probability of financial crisis, the central bank will have a higher incentive to build up reserves. The result when the initial reserve is zero is shown in Figure 10. We can see that central bank would want to hold more reserves.

An interesting finding in Figure 10 is that when the initial reserve is zero, the optimal tax is still positive even when the household income is 50. Recall that household endowment is 50. With positive $\tau_1$, household income in $T = 2$ must be higher than 50. Since the household utility function is concave, if the international reserve is purely used to smooth consumptions,
Figure 9: The probability of bank runs according the scenario used by the private sector and the central bank.

Figure 10: Central bank’s optimal reserve when the initial reserve $b_1 = 0$. “High private liquidity” means the private sector use the scenario with self-fulfilling panics and holds high reserve. “Low private liquidity” means the private sector ignores the possibility of panics and hold low liquidity.

then the tax should not be positive at this income level because positive tax will make the first period income lower than 50. But in this case, the benefit for international reserves is more than smoothing consumptions, it can also be used to reduce liquidations of assets and lower the probability of bank runs, which creates an extra incentive for the central bank to accumulate reserves.

8 Extension: external borrowing

In this part, we extend the model to allow for external borrowing. We show that an reevaluation of borrowing costs may encourage the central bank to accumulate more reserves. In addition, when the borrowing cost is high, the possibility for the central bank borrow external funds may cause the central bank to accumulate more reserves than if the central bank can not borrow at
all.

One explanation for higher reserve accumulation after the Asia financial crises is that countries find it is very costly to borrow from external sources such as the IMF during financial crises.\footnote{For example, countries may need to satisfy many conditions in order to qualify for the loan. Although those conditions may be necessary in order to make sure that the loan is repaid, countries may find it is very difficult to comply with those conditions.} In this part, we first show that if the central bank realizes that the external borrowing cost is higher than previously expected, then it will build up more reserves.

We make the following assumptions. After the central bank runs out of its own reserves, it can borrow up to $b_{ex}$ from external institutions. So the maximum central bank loan becomes

$$\bar{b} = b_c + b_{ex}$$

(35)

where $b_c$ is the reserve accumulated by the central bank. We assume that the central bank can not commit not to borrow from external sources and then lend to domestic banks. Domestic banks will take the additional central bank borrowing into account when they make their decisions. If the limit $\bar{b}$ is reached, the commercial banks must liquidate assets to meet the withdrawal needs. In order to be consistent with the previous analysis, we still assume that the lending rate by the central bank to commercial banks is normalized to zero. So there is no changes to the three-period bank run model except the level of $\bar{b}$.

We only analyze possible external borrowing in $T = 2$. We also assume that the loan in $T = 2$ must be repaid at the end of $T = 2$, so external loans are only used to make loans to commercial banks and they are not used to buy consumption goods for households. We assume the unit borrowing cost is $\psi$, this includes the interest cost, and also other costs due to the need to comply with the conditions of the loan. Denote the external loan as $b_{ex}$, then the cost is $\psi b_{ex}$. We assume that the value of these costs is equivalent to a reduction in household consumption by $\psi b_{ex}$.

The central bank still maximizes $-\frac{1}{c_1} + E \frac{1}{c_2}$, where $c_2$ becomes

$$c_2 = e_h + w(1 - \phi) + b_c - \psi b_{ex}$$

(36)

That is, $c_2$ is equal to the sum of household endowment, wage income and the central bank reserve, minus the cost of external borrowing.

**Numerical Example**
Figure 11: Higher external borrowing costs cause the central bank to hold more reserves, which in turn causes private banks to hold less reserves.

Figure 11 shows an example in which the initial reserve in $T = 1$ is zero. We show two cases: $\bar{b}_{ex} = 2$ and $\bar{b}_{ex} = 5$. We have two main results.

1. For the same external borrowing limit, the central bank will hold higher reserves when the borrowing cost is higher.

2. When the borrowing cost is low, the central bank will hold less reserves when it can borrow more. But if the borrowing cost is very high, the central bank may actually want hold more reserves when it can borrow more.

The first result is shown in Figure 11. For the same $\bar{b}_{ex}$, higher $\psi$ causes the central bank to hold more reserves, which will in turn cause the commercial banks to hold less reserves. An implication is that if the central bank suddenly realizes that the cost of external borrowing is higher than previous expected, then it will accumulate more reserves.
Figure 12: With very high borrowing cost, the central bank may hold more reserve when it can borrow more.

We can also see that increases in $\psi$ cause more changes in optimal central bank reserve under $\overline{b}_{ex} = 5$ than under $\overline{b}_{ex} = 2$. Intuitively, if there is no external borrowing (i.e., if $\overline{b}_{ex} = 0$), changes in $\psi$ will not affect the choice of the central bank. When $\overline{b}_{ex}$ is high, increases in $\psi$ will cause more increase in the borrowing cost.

The second result is shown in Figure 12. We can see that when $\psi = 0$, the reserve level is the highest when $\overline{b}_{ex} = 0$ and the lowest when $\overline{b}_{ex} = 5$, so the central bank would hold less reserve if it can borrow more. But if the cost is very high (for example, $\psi = 0.7$), the reserve for $\overline{b}_{ex} = 5$ is actually higher than the reserve for $\overline{b}_{ex} = 0$ and $\overline{b}_{ex} = 2$.

The reason is as follows. If private agents think that the central bank can borrow more from external sources, and if the central bank can not commit not to borrow those external loans and lend them to domestic banks at low costs, then domestic banks will hold less reserves, and the central bank need to be prepared to lend to private banks during liquidity shocks. The central bank will want to accumulate more reserves to reduce the probability of borrowing from external sources. It is less costly to accumulate more reserves to self-insure than being forced to borrowing from external loans at very high costs.

9 Summary and conclusion

In this paper, we applied the global game approach to build a model of bank runs caused by aggregate liquidity shocks, we then use the model to analyze the central bank’s motive to accumulate international reserves as self-insurance against financial crisis. The main findings can be summarized as follows:
1. We show that if liquidations of assets lead to social costs such as unemployment and low income of households, then the central bank will have incentive to hold reserves and lend them to banks during liquidity shocks in order to reduce the liquidations of assets.

2. We analyze the effects of central bank reserves on the probability of bank runs. We find that although higher central bank reserves will cause private banks to optimally choose to hold less reserves, overall, higher central bank reserves will reduce the probability of bank runs.

3. We also analyze the optimal choice of central bank reserves. We find that the central bank would want to hold more reserves when the private sector is less cautious and holds lower reserves.

We also analyze the case when the central bank can borrow from external sources, we find that an reevaluation of the borrowing cost will give the central bank higher incentive to hold reserves. We also find that if the government can not commit not to borrow from external sources, then domestic banks will hold less reserves. If the external borrowing cost is very high, then the government may find it is better to accumulate more reserves to self-insure than being forced to borrow at high costs.

In this paper, we only focus on liquidity risks. In reality, central banks may also want to build up reserves in order to maintain a stable exchange rate. We will explore these mechanisms in future works.

References


Appendix

A Collection of the main notations

e_h: endowment of the household;  
y = AK^\alpha L^{1-\alpha}: Production function  
k: capital-labour ratio  
d_t: bank deposit  
γ: bank reserve ratio  
w: wage rate  
\bar{r}_m: promised rate for period 3  
κ: minimum withdrawal in period 2  
\kappa_e: the \kappa above which r_n < \bar{r}_m  
\theta: fundamental factor  
\theta_i: private signal  
\theta_s: the \theta at which \pi(\theta) = \kappa_s  
\lambda: value for each unit of liquidated assets  
n: the number of withdrawers in stage 1 of the two-stage game  
n_f: the maximum share of depositors can be paid in period 2  
v(\cdot): the payoff differential function  
\Phi(\cdot): the expected payoff differential  
F(\pi): cumulative distribution function of \pi  
\tau_1: tax in T = 1  
\bar{b}_2: international reserve in T = 2  
\rho: coefficient for additional costs  
b_c: reserve of the domestic central bank  
ψ: costs for external loan  
e_b: endowment of the banker;  
A: Productivity factor  
\delta: depreciation Rate  
\bar{f}: bank’s debt-equity ratio  
R: gross return rate for capital goods  
\bar{r}_m: short-term deposit rate  
r^n: the actual rate paid in period 3  
\kappa_s: the \kappa above which r_n < \bar{r}_m  
\epsilon_i: noise in the private signal  
\pi(\theta): the share of movers  
\theta_b: liquidation is needed if \theta > \theta_b  
\theta_c: the \theta at which \pi(\theta) = \kappa_c  
\theta^*: equilibrium threshold level  
n_s(\theta): the \theta at which \kappa = \kappa_s  
n_e(\theta): the \theta at which \kappa = \kappa_c  
b: central bank loan  
\bar{b}: maximum central bank loan (also used to denote the central bank reserve level)  
f(\pi): density function of \pi  
\bar{b}_1: initial international reserve in T = 1  
T = 1, 2: the two generations  
b_{ex}: external borrowing  
\bar{b}_{ex}: maximum external borrowing
Table 4: Official reserves of countries with the largest holdings (Billions of U.S.
dollars, up until Oct. 2005)

<table>
<thead>
<tr>
<th></th>
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<td>Japan</td>
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<td>74.9</td>
<td>76.8</td>
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<td>41.1</td>
<td>48.7</td>
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</tr>
</tbody>
</table>

(1) Oct 2005. (2) Calculated according to the GDP and the market exchange rate at the end of 2004.

B Proofs

B.1 Proof of Proposition 1

The value of $\kappa_s$ can be computed as follows:

$$\kappa_s d_t \bar{r}_{mr} = \gamma d_t + \bar{b} + \lambda \phi_s [(1 - \gamma) d_t + e_b]$$

(37)

$$ (1 - \kappa_s) d_t \bar{r}_{mr} = (1 - \phi_s) [(1 - \gamma) d_t + e_b] R - \bar{b}$$

(38)

where $\phi_s$ is the liquidation at $\kappa_s$. The first equation means that at $\kappa_s$, the payment made in period 2 is $\kappa_s d_t \bar{r}_{mr}$. The second equation means that the bank’s resources in period 3 is just enough to give each of the remaining depositors the payment $d_t \bar{r}_{mr}$. The solution for $\kappa_s$ is shown in equation (12). Since $n_s(\theta) + (1 - n_s(\theta))\pi(\theta) = \kappa_s$, we have $n_s(\theta) = \frac{\kappa_s - \pi(\theta)}{1 - \pi(\theta)}$.

B.2 Proof of Proposition 2

The value of $r^n$ when $n > n_s$ (i.e., $\kappa > \kappa_s$) can be computed as follows:

$$\kappa d_t \bar{r}_{mr} = \gamma d_t + \bar{b} + \lambda \phi [(1 - \gamma) d_t + e_b]$$

(39)

$$ (1 - \kappa) d_t r^n = (1 - \phi) [(1 - \gamma) d_t + e_b] R - \bar{b}$$

(40)
The solution for \( r^n \) is \( r^n(\kappa) = \frac{R_{\infty}}{\lambda} + \frac{1}{1-\kappa}W \), where \( W \) is a constant and is defined in equation (14). The derivative of \( r^n(\kappa) \) with respect to \( \kappa \) is

\[
-\frac{1}{(1-\kappa)^2}W \tag{41}
\]

\(-\frac{1}{(1-\kappa)^2}\) is negative and \( W \) is a constant, which means the sign of the derivative is the same over \([\kappa_s, \kappa_e]\). Since we assume \( \kappa_e < 1 \), then by the definition of \( \kappa_e \), we have \( r^n = r_{\infty} \) at \( \kappa_e \). And since \( r^n = r_{\infty} > r_{\infty} \) at \( \kappa_s \), equation (41) must be negative, and so \( r^n \) is strictly decreasing over \([\kappa_s, \kappa_e]\), which means \( r^n \) is decreasing in \( n \) over \([n_s, n_e]\) because \( \kappa \) is strictly increasing in \( n \).

**B.3 Proof of proposition 3**

**B.3.1 Some intermediate results**

We first need to prove some intermediate results.

**Proposition 6.** When \( n_e(\theta) \) and \( n_s(\theta) \) are between \((0, 1)\), they are continuous and strictly decreasing in \( \theta \).

Proof: \( n_e = \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)} \), where \( \kappa_e \) is defined in equation (6) and is not a function of \( \theta \). The derivative of \( n_e \) with respect to \( \pi(\theta) \) is

\[
\frac{\kappa_e - 1}{(1 - \pi(\theta))^2} \tag{42}
\]

When \( n_e(\theta) \in (0, 1) \) (i.e., when \( \pi(\theta) < \kappa_e < 1 \)), equation (42) is negative and so \( n_e(\theta) \) is strictly decreasing in \( \pi(\theta) \). Since \( \pi(\theta) \) is strictly increasing in \( \theta \), \( n_e(\theta) \) is strictly decreasing in \( \theta \). Similarly, since \( n_s(\theta) = \frac{\kappa_s - \pi(\theta)}{1 - \pi(\theta)} \), when \( n_s(\theta) \) is between \([0, 1]\) (i.e., \( \pi(\theta) < \kappa_s < 1 \)), the derivative of \( n_s(\theta) \) with respect to \( \theta \) is \( \frac{\kappa_s - 1}{(1 - \pi(\theta))^2} \) and is negative, so \( n_s \) is decreasing in \( \theta \).

Since \( \pi(\theta) \) is continuous in \( \theta \), from the function form of \( n_e(\theta) \) and \( n_s(\theta) \), we can see that both \( n_e(\theta) \) and \( n_s(\theta) \) are continuous in \( \theta \).

**Proposition 7.** Suppose \( n_e(\theta) \in (0, 1) \). Then for \( n \in (n_e(\theta), 1) \), \( v(\theta) \) is positive and its value does not depend on \( \theta \). For \( 0 \leq n \leq n_e(\theta) \), \( v(\theta, n) \) is negative and strictly increasing in \( \theta \).

Proof: (The intuition are shown in Figure 13). Suppose \( \theta \) increases from \( \theta_1 \) to \( \theta_2 \). Since \( n_e(\theta) \) is strictly decreasing in \( \theta \), we have \( n_e(\theta_2) < n_e(\theta_1) \). For \( n > n_e(\theta_1) \), \( v(\theta_1, n) = v(\theta_2, n) \) (line \( CDE \)). \( n_f \) is not a function of \( \theta \). And from Table 1, we can see that for \( n > n_e(\theta) \), \( v(\theta, n) \) is
The single crossing property of $v(\theta, n)$ is line $ABCDE$, and $v(\theta_2, n)$ is line $A'B'C'D'E$. $v(\theta_1, n) = v(\theta_2, n)$ for $n_e(\theta_1) < n \leq 1$, and $v(\theta_2, n) > v(\theta_1, n)$ for $0 \leq n \leq n_e(\theta_1)$.

not a function of $\theta$. For $n \in (n_e(\theta_2), n_e(\theta_1)]$, we have $v(\theta_2, n) > 0 \geq v(\theta_1, n)$. For $n \in [0, n_e(\theta_2)]$, the value of $v$ is $v(\theta, n) = (1 - \pi(\theta))(r_{\text{m}} - rn)$, $1 - \pi(\theta)$ is strictly increasing in $\theta$, so in order to prove that $v(\theta, n)$ is strictly increasing in $\theta$, we need to show that $r^n(\theta)$ is weakly decreasing in $\theta$ over $n \in [0, n_e(\theta_2)]$, i.e., $r^n(\theta_2, n) \leq r^n(\theta_1, n)$.

First, if $n_e(\theta_2) \leq n_s(\theta_1)$, since $r^n(\theta_1, n) = \overline{r^n}$ for $n \leq n_s(\theta_1)$, we must have $r^n(\theta_1, n) \geq r^n(\theta_2, n)$ over $n \in [0, n_e(\theta_2)]$ since $r^n(\theta_2, n)$ can not be higher than $\overline{r^n}$.

Second, if $n_s(\theta_1) \leq n_e(\theta_2)$, then we can separate $[0, n_e(\theta_2)]$ into two parts $[0, n_s(\theta_1)]$ and $[n_s(\theta_1), n_e(\theta_2)]$. Over $[0, n_s(\theta_1)]$, we have $r^n(\theta_1, n) = \overline{r^n}$, since the highest $r^n$ is $\overline{r^n}$, we have $r^n(\theta_1, n) \geq r^n(\theta_2, n)$. Over $[n_s(\theta_1), n_e(\theta_2)]$, $r^n(\theta, n)$ is a function of $\theta$ and $n$ only because it is a function of $\kappa$. And the value of $r^n(\theta_1, n)$ is decreasing in $\kappa$ (equation 41). Since $\kappa = n + (1 - n)\pi(\theta)$, given $n$, $\kappa$ is increasing in $\theta$, which means $r^n(\theta_1, n)$ is decreasing in $\theta$, thus we have $r^n(\theta_1, n) \geq r^n(\theta_2, n)$.

**Proposition 8. The single crossing property of $v(\theta, n(\theta, \theta^*))$:** There exists only one level of $\theta$, denoted as $\theta_c$, such that $v(\theta, n(\theta, \theta^*)) > 0$ if $\theta > \theta_c$, and $v(\theta, n(\theta, \theta^*)) < 0$ if $\theta \leq \theta_c$.

The meaning is that $v(\theta, n(\theta, \theta^*))$ crosses the zero line from below once and only once.

Proof: We can derive this property of $v(\theta, n(\theta, \theta^*))$ from the properties of $v(\theta, n)$. Let $v(\theta_c, n)$ denote the value function $v(\theta, n)$ with $\theta = \theta_c$, and suppose $n_e(\theta_c) \in (0, 1)$. Then from Figure 13, we know that at $n = n_e(\theta_c)$, we have $v(\theta_c, n) = 0$, and $v(\theta_c, n) > 0$ for $n > n_e(\theta_c)$ and $v(\theta_c, n) < 0$ for $n < n_e(\theta_c)$.

Suppose the threshold of the equilibrium is $\theta^*$. In the equilibrium, $n(\theta, \theta^*)$ uniformly increases from 0 to 1 when $\theta$ increases from $\theta^* - \epsilon$ to $\theta^* + \epsilon$ (equation 16). On the other hand, we've shown in Proposition 6 that if $0 < n_e(\theta) < 1$, then $n_e(\theta)$ is continuous and decreasing in
\( \theta \). Then there should exist a fixed point \( \theta \in [\theta^* - \epsilon, \theta^* + \epsilon] \) at which \( n(\theta, \theta^*) = n_e(\theta) \). Denote this \( \theta \) as \( \theta_e \). And we have \( n(\theta_e, \theta^*) = n_e(\theta_e) \) and so \( v(\theta_e, n(\theta_e, \theta^*)) = v(\theta_e, n_e(\theta_e)) \). (We do not need all \( n_e(\theta) \) to be located between \( 0,1 \). But we know that there must exists some \( \theta \) at which \( n_e(\theta) \) is between \( 0,1 \). If all \( n_e(\theta) \) were equal to 0, then all \( v(\theta, n(\theta, \theta^*)) \) would be negative. And if all \( n_e(\theta) \) were equal to 1, then all \( v(\theta, n(\theta, \theta^*)) \) would be positive. But according to the definition of \( \theta^* \), the average value of \( v(\theta, n(\theta, \theta^*)) \) over \( [\theta^* - \epsilon, \theta^* + \epsilon] \) should be zero.)

We’ve shown in Proposition 7 that if \( \theta \) increases from \( \theta_1 \) to \( \theta_2 \), for \( n > n_e(\theta_1) \), the value of \( v(\theta, n) \) is positive and does not depend on \( \theta \), that is, \( v(\theta, n) = v(\theta_1, n) > 0 \) if \( n > n_e(\theta_1) \) and \( \theta > \theta_1 \). Let \( \theta_1 = \theta_e \), then for \( \theta > \theta_e \), we have \( n(\theta, \theta^*) > n_e(\theta_e) \), and so \( v(\theta, n(\theta, \theta^*)) = v(\theta_e, n(\theta_e, \theta^*)) > 0 \). That is, \( v(\theta, n(\theta, \theta^*)) > 0 \) for \( \theta > \theta_e \). We also show in Proposition 7 that if the value of \( \theta \) decreases from \( \theta_2 \) to \( \theta_1 \), for \( n < n_e(\theta_2) \), the value of \( v(\theta, n) \) is negative and increasing in \( \theta \), that is, \( v(\theta_1, n) < v(\theta_2, n) < 0 \). Let \( \theta_2 = \theta_e \), then for \( \theta < \theta_e \), we have \( n(\theta, \theta^*) < n_e(\theta_e) \), and so \( v(\theta, n(\theta, \theta^*)) < v(\theta_e, n(\theta_e, \theta^*)) < 0 \). As a result, \( v(\theta, n(\theta, \theta^*)) < 0 \) for \( \theta < \theta_e \). Thus, we proved that \( v(\theta, n(\theta, \theta^*)) \) changes from negative to positive values once and only once. (See Figure 14)

In addition, below \( \theta_e \), \( v(\theta, n(\theta, \theta^*)) \) is negative and strictly increasing in \( \theta \). This is because below \( \theta_e \), \( v(\theta, n(\theta, \theta^*)) \) is negative. We know that when \( v(\theta, n) \) is negative, given \( \theta \), the value of \( v(\theta, n) \) is weakly increasing in \( n \), and given \( n \), the value of \( v(\theta, n) \) is strictly increasing in \( \theta \). So for \( \theta < \theta_e \) we have

\[
\frac{dv(\theta, n(\theta, \theta^*))}{d\theta} = \frac{\partial v(\theta, n(\theta, \theta^*))}{\partial \theta} \frac{\partial n(\theta, \theta^*)}{\partial \theta} \geq 0
\]

(43)

\[
\frac{\partial n(\theta, \theta^*)}{\partial \theta} \geq 0 \text{ because for } \theta \leq \theta_e, n(\theta, \theta^*) \text{ is zero when } \theta \leq \theta^* - \epsilon, \text{ and is increasing in } \theta \text{ between } [\theta^* - \epsilon, \theta_e].
\]

**Dominance region**

Recall that \( n_e = \frac{s_e - \pi(\theta)}{1 - \pi(\theta)} \), and \( n_e \in (0,1) \) if \( \pi(\theta) < \kappa_e < 1 \). If \( \kappa_e < \pi(\theta) \), \( n_e \) is defined as 0, which means the bank would fail even if no one withdraw in stage 1. Let \( \bar{\theta} \) denote the level of \( \theta \) at which \( \pi(\theta) = \kappa_e \). Then for \( \theta > \bar{\theta}, n_e = 0 \), so \( v(\theta, n) > 0 \) for all values of \( n \) since \( v(\theta, n) \) is positive for all \( n > n_e(\theta) \). This means if \( \theta > \bar{\theta} \), it would be optimal to withdraw no matter what other people’s decisions are. Since the private signal \( \theta_i \) is uniformly distributed over \( [\theta - \epsilon, \theta + \epsilon] \), if people observe signal \( \theta_i > \bar{\theta} + \epsilon \), they would know that the true value of \( \theta \) is higher than \( \bar{\theta} \), and so they would choose to withdraw. We call the region \( [\bar{\theta} + \epsilon, 1] \) as the upper dominance region.
But as we’ve shown in the text, there is no lower dominance region when \( n_e(\theta) < 1(\kappa_e < 1) \). Because if a depositor thinks that all other depositors are going to run the bank, he will also run the bank. So we make Assumption 2 such that people will not run the bank for extremely good signals.\(^\text{22}\) Denote \( \epsilon \) as \( \theta \). If Assumption 2 holds, then given all other agents \( j \neq i \) do not run for \( \theta_j \in [0, \epsilon] \), agent \( i \) should also find that the expected payoff to run for \( \theta_i \leq \theta \) is negative.

We will use this result when we prove the unique equilibrium. In the limit, \( \theta = \epsilon \to 0 \). So as long as in the unique equilibrium, the threshold is above zero, then the equilibrium is consistent with Assumption 2. If we can not find such equilibrium above 0, then we simply assume that people will run the bank regardless of the signal.

We will prove Proposition 3 in two parts. In the first part, we show that there is a unique threshold equilibrium. In the second part, we show that there is no other non-threshold equilibrium.

### B.3.2 Part I: There is a unique threshold equilibrium

First, we need to show that the equilibrium condition only holds at one value of \( \theta^* \). Then, we need to show that the depositors do not want to deviate from the equilibrium strategy.

The equilibrium is defined by \( \Phi(\theta', \theta') = 0 \). We will show that there is one and only one value of \( \theta' \) which satisfies this condition.

First, assume that Assumption 2 holds, then it is strictly better not to withdraw when \( \theta_i \leq \theta \). So we have \( \Phi(\theta', \theta') < 0 \) when \( \theta' \leq \theta \). If the private signal is higher than \( \bar{\theta} + \epsilon \), we are in the upper dominance region and all \( v(\theta, n) \) in (17) is positive. So for \( \theta' > \bar{\theta} + \epsilon \) we have \( \Phi(\theta', \theta') > 0 \). This means \( \theta^* \) is between \( (\theta, \bar{\theta} + \epsilon) \). From the function form of \( \Phi(\theta', \theta') \), we can see that \( \Phi(\theta', \theta') \) is continuous because the boundary of the integral is continuous in \( \theta_i \), and the value of \( v(\theta, n) \) is bounded. So there exists at least one \( \theta' \) such that \( \Phi(\theta', \theta') = 0 \). In addition, we can prove that \( \Phi(\theta', \theta') \) is strictly increasing in \( \theta' \) between \( [\theta, \bar{\theta} + \epsilon] \). First, when the private signal (the first \( \theta' \) in \( \Phi(\theta', \theta') \)) and the threshold (the second \( \theta' \) in \( \Phi(\theta', \theta') \)) increase by the same amount, the expected distribution of \( n \) does not change, it is still uniformly distributed over \([0, 1]\) for \( \theta \in [\theta' - \epsilon, \theta' + \epsilon] \). Second, given the distribution of \( n \), we can show that the average value of \( v(\theta, n) \) is strictly increasing in \( \theta \). We’ve already shown that \( v(\theta, n) \) is weakly increasing in \( \theta \), so

\(^{22}\)Goldstein and Pauzner(2005) argue that we can make this kind of assumptions to help analyzing the equilibrium selection process when there is no genuine upper or lower dominance region.
we need to show that there are at least some $v(\theta, n)$ that are strictly increasing in $\theta$. Recall that in Proposition 7, we show that for $n < n_e(\theta)$, $v(\theta, n)$ is strictly increasing in $\theta$. This means we need to show that there exist some $n$ for which we have $n < n_e(\theta)$. When the private signal $\theta_i$ reaches the upper bound $\overline{\theta} + \epsilon$, the lower boundary of the integral in equation (17) is $\theta - \epsilon = \overline{\theta}$, so we need to show that when $\theta < \overline{\theta}$, there are at least some $n$ that satisfy $n < n_e(\theta)$. For $\theta < \overline{\theta}$, we have $n_e(\theta) > 0$. Since $n$ is uniform over $[0, 1]$, we must have some $n$ which is smaller than $n_e(\theta)$. As a result, there are at least some $n$ that satisfy $n < n_e(\theta)$, and so there are at least some $v(\theta, n)$ which are strictly increasing in $\theta$. Thus, $\Phi(\theta', \theta')$ is strictly increasing in $\theta'$ when $\theta' \in [\theta, \overline{\theta} + \epsilon]$, and there is only one level of $\theta' = \theta^*$ at which the equilibrium condition is satisfied.

The next step is to prove that depositors will not want to deviate from the equilibrium strategy. That is, not only $\Phi(\theta^*, \theta^*) = 0$, given the private signal $\theta_i$, $\Phi(\theta_i, \theta^*)$ is smaller than zero if $\theta_i < \theta^*$ and higher than zero if $\theta_i > \theta^*$. This result can be proved using the “single crossing property” proved in Proposition 8. As shown in Figure 14, $v(\theta, n(\theta, \theta^*))$ changes from negative to positive value once and only once, from $\theta^*$.

According to the definition of the equilibrium, we have

$$\Phi(\theta^*, \theta^*) = \frac{1}{2\epsilon} \int_{\theta^* - \epsilon}^{\theta^* + \epsilon} v(\theta, n(\theta, \theta^*))d\theta = 0$$

which means that the average value of $v(\theta, n(\theta, \theta^*))$ over $[\theta^* - \epsilon, \theta^* + \epsilon]$ is zero.

Now suppose the private signal received by a depositor is $\theta_i < \theta^*$. The value of $\Phi(\theta_i, \theta^*)$ is

$$\Phi(\theta_i, \theta^*) = \frac{1}{2\epsilon} \int_{\theta_i - \epsilon}^{\theta_i + \epsilon} v(\theta, n(\theta, \theta^*))d\theta$$

which is the average value of $v(\theta, n(\theta, \theta^*))$ over $[\theta_i - \epsilon, \theta_i + \epsilon]$. Since $v(\theta, n(\theta, \theta^*))$ changes from negative to positive values once and only once, from Figure 14, we can see that compared with

![Figure 14: Functions $n(\theta, \theta^*)$ and $v(\theta, n(\theta, \theta^*))$]


Φ(θ*, θ*), Φ(θi, θ*) replaces the positive values of v(θ, n(θ, θ*)) between [θi + ε, θ* + ε] with negative values between [θi − ε, θ* − ε], so Φ(θi, θ*) < 0 and the depositors would prefer to wait. Similarly, if θi > θ*, Φ(θi, θ*) > 0 and the depositors would prefer to withdraw. So in the equilibrium, the depositors will not deviate from the equilibrium strategy.

B.3.3 Part II: Any equilibrium must be a threshold equilibrium

The following part is largely taken from Goldstein and Pauzner(2005), we follow the exact steps in their proof. We modified the proof according to the different payoff function in our model. And we also add additional graphs and explanations to make the proof easier to understand.

First, we define some notations for the mixed strategy.

Agent i’s (mixed) strategy is a measurable function s_i: [0 − ε, 1 + ε] → [0, 1] that indicates the probability that the agent withdraw early given his private signal θi. A strategy profile is denoted by {s_i}_{i∈[0,1]}. A given strategy profile generates a random variable ˜n(θ) that represents the proportion of agents demanding early withdrawal at stage θ. ˜n(θ) is defined by the cumulative distribution function

\[ F_\theta(n) = \text{prob}[\tilde{n} \leq n] = \text{prob}_{\{i, i \in [0,1]\}} \left[ \int_{i=0}^{1} s_i(\theta + \epsilon_i) di \leq n \right] \]  

(46)

where s_i(θ + ε_i) is the probability for agent i to withdraw when the private signal is θ + ε_i. And we define the inverse CDF as

\[ n^x(\theta) = \inf\{n : F_\theta(n) \geq x\} \]  

(47)

Let Φ(θi, ˜n(·)) denote the expected difference in utility from withdrawing in stage 1 rather than in stage 2, when agent i observes signal θi and holds belief ˜n(·) about the proportion of agents who run at each stage θ. The posterior distribution of θ is uniform over [θi − ε, θi + ε]. And we have

\[ \Phi(\theta_i, \tilde{n}(\cdot)) = \frac{1}{2\epsilon} \int_{\theta=\theta_i-\epsilon}^{\theta_i+\epsilon} E_n[v(\theta, \tilde{n}(\theta))] d\theta = \frac{1}{2\epsilon} \int_{\theta=\theta_i-\epsilon}^{\theta_i+\epsilon} \left( \int_{n=0}^{1} v(\theta, n) dF_\theta(n) \right) d\theta \]  

(48)

If all agents have the same strategy, then the proportion of agents who run at each state θ is deterministic, and we can write ˜n(·) as n(θ), which is a deterministic number instead of a random variable. And equation (48) becomes

\[ \Phi(\theta_i, n(\cdot)) = \frac{1}{2\epsilon} \int_{\theta=\theta_i-\epsilon}^{\theta_i+\epsilon} v(\theta, n(\theta)) d\theta \]  

(49)
And if everyone has the same threshold $\theta'$, then we can also write $n(\theta)$ as $n(\theta, \theta')$.

$\Phi(\theta, \tilde{n}(\cdot))$ is continuous in $\theta_i$ because the boundary of the integral is continues in $\theta_i$ and the integrand is bounded.

**Proof of part II:** Suppose that $\{s_i\}_{i \in [0,1]}$ is an equilibrium and that $\tilde{n}(\cdot)$ is the corresponding distribution of the proportion of agents who withdraw as a function of the state $\theta$. Let $\theta_A$ be the lowest signal at which agents do not strictly prefer to wait:

$$
\theta_A = \inf \{ \theta_i : \Phi(\theta, \tilde{n}(\cdot)) \geq 0 \} \tag{50}
$$

By the continuity of $\Phi$, agents are indifferent at $\theta_A$ and we have $\Phi(\theta_A, \tilde{n}(\cdot)) = 0$. And according to Assumption 2, we must have $\theta_A > \epsilon$.

If we are not in a threshold equilibrium, then there are signals above $\theta_A$ at which $\Phi(\theta, \tilde{n}(\cdot)) \leq 0$. We define

$$
\theta_B = \inf \{ \theta_i > \theta_A : \Phi(\theta_i, \tilde{n}(\cdot)) \leq 0 \} \tag{51}
$$

And by continuity, agents are indifferent at $\theta_B$. So we must have

$$
\Phi(\theta_A, \tilde{n}(\cdot)) = \Phi(\theta_B, \tilde{n}(\cdot)) = 0 \tag{52}
$$

Figure 15: Function $\Phi$ in a (counterfactual nonthreshold equilibrium)

Figure 15 illustrates the (counterfactual) nonthreshold equilibrium. In the Figure, agents do not run when the signal is below $\theta_A$, run between $\theta_A$ and $\theta_B$, and may or may not run above $\theta_B$. We need to show that (52) cannot hold. We will show this in three steps.\(^{23}\)

**Step 1:** First note that $\theta_B$ is strictly above $\theta_A$, that is, there is a nontrivial interval of signals above $\theta_A$, where $\Phi(\theta_i, \tilde{n}(\cdot)) > 0$. The reason is that the derivative of $\Phi$ with respect to $\theta_i$ at $\theta_A$ is positive, so when $\theta_i$ is slightly higher than $\theta_A$, $\Phi$ is higher than zero. Given the definition of $\Phi$ in (48), the derivative is given by $E_n v(\theta_A + \epsilon, \tilde{n}(\theta_A + \epsilon)) - E_n v(\theta_A - \epsilon, \tilde{n}(\theta_A - \epsilon))$, where $\theta_A + \epsilon$ and $\theta_A - \epsilon$ is the upper and lower bound of $\theta$ when the private signal is $\theta_A$. When the state $\theta$\(^{24}\) We’ve drawn Figure 16 to help readers to understand the intuition of the proof. Also see the additional explanation at the end of the proof.
is \( \theta_A - \epsilon \), all agents have signal lower than \( \theta_A \), so no agent runs. Thus, \( \tilde{n}(\theta_A - \epsilon) \) is degenerate and is equal to 0. Now, for any \( n \geq 0 \), \( v(\theta_A + \epsilon, n) \) is higher than \( v(\theta_A - \epsilon, 0) \). The reason is as follows. First, \( v(\theta_A - \epsilon, 0) \) must be negative, otherwise it is impossible for \( \Phi(\theta, \tilde{n}(\cdot)) \) to be equal to zero at \( \theta_A \). This is because if \( v(\theta_A - \epsilon, 0) > 0 \), then we would have \( v(\theta, n) > 0 \) for all \( n > 0 \) and \( \theta > \theta_A - \epsilon \), and \( \Phi(\theta, \tilde{n}(\cdot)) \) would be positive. Second, if \( v(\theta_A + \epsilon, n) \) is positive, then of course \( v(\theta_A + \epsilon, n) > v(\theta_A - \epsilon, 0) \). If \( v(\theta_A + \epsilon, n) \) is negative, then \( v(\theta_A - \epsilon, 0) < v(\theta_A + \epsilon, 0) \leq v(\theta_A + \epsilon, n) \).

That is, given \( n = 0 \), the value of \( v(\theta, 0) \) is increasing in \( \theta \), and given \( \theta \), the value of \( v(\theta, n) \) is at least as high as \( v(\theta, 0) \) (See Figure 13). So \( v(\theta_A + \epsilon, n) \) is higher than \( v(\theta_A - \epsilon, 0) \) and the value of the derivative is positive.

We decompose the intervals over which the integrals \( \Phi(\theta_A, \tilde{n}(\cdot)) \) and \( \Phi(\theta_B, \tilde{n}(\cdot)) \) are computed into three parts. The first is a common part \( c = (\theta_A - \epsilon, \theta_A + \epsilon) \setminus (\theta_B - \epsilon, \theta_B + \epsilon) \). This common part may be empty if \( \theta_A + \epsilon < \theta_B - \epsilon \). And there are two disjoint parts: \( d^A = [\theta_A - \epsilon, \theta_A + \epsilon] \setminus c \) and \( d^B = [\theta_B - \epsilon, \theta_B + \epsilon] \setminus c \). Denote the range \( d^A \) as \( [\theta_1, \theta_2] \). And we also pair each point in \( d^A \) with a “mirror image” point \( \overline{\theta} \) in \( d^B \), such that as \( \theta \) moves from the upper end of \( d^A \) to its lower end, \( \overline{\theta} \) moves from the lower end of \( d^B \) to its upper end (See Figure 16).

\[
\overline{\theta} = \theta_B + (\theta_A - \theta) \tag{53}
\]

Using (48), we can rewrite (52) as

\[
\int_{\theta_1}^{\theta_2} E_n v(\theta, \tilde{n}(\theta)) = \int_{\theta_1}^{\theta_2} E_n v(\overline{\theta}, \tilde{n}(\overline{\theta})) \tag{54}
\]

This equation can be simplified because the proportion of agents who run at states \( \theta \in d^A = [\theta_1, \theta_2] \) is deterministic. The reason is that at \( \theta = \theta_1 (= \theta_A - \epsilon) \), no agent runs and thus \( n(\theta_1) = 0 \). And when we move from \( \theta_1 \) to higher stages in \( d^B \), we replace agents who get signals below \( \theta_A \) and do not run, with agents who get signals between \( \theta_A \) and \( \theta_B \) and run. As a result, \( n(\theta) \) increases at the fastest possible speed \( \frac{1}{2\epsilon} \). And \( n(\theta) \) is

\[
n(\theta) = (\theta - \theta_1)/2\epsilon \quad \text{for all } \theta \in [\theta_1, \theta_2] \tag{55}
\]

The RHS of (54) can be written as

\[
\int_{\theta_1}^{\theta_2} E_n v(\overline{\theta}, \tilde{n}(\overline{\theta})) = \int_{\theta_1}^{\theta_2} \int_{x=0}^{1} v(\overline{\theta}, n^x(\overline{\theta})) dx d\theta \tag{56}
\]

where \( n^x(\theta) \) is defined in equation (47) and is the inverse CDF of \( \tilde{n}(\theta) \). We will be able to show that (54) does not hold if we can show that for each \( x \),

\[
\int_{\theta_1}^{\theta_2} v(\theta, n(\theta)) d\theta < \int_{\theta_1}^{\theta_2} v(\overline{\theta}, n^x(\overline{\theta})) d\theta \tag{57}
\]
We will first derive some intermediate results.

**Step 2:** We first show that \( n(\theta) \) in the LHS of (57) changes faster than \( n^x(\bar{\theta}) \) in the RHS.

**Lemma 1.** \[ \left| \frac{\partial n(\bar{\theta})}{\partial \theta} \right| \leq \left| \frac{\partial n(\theta)}{\partial \theta} \right| = \frac{1}{2\epsilon} \]

This is based on the same idea that \( n(\theta) \) changes at the fastest feasible rate \( \frac{1}{2\epsilon} \) in \( d^A = [\theta_1, \theta_B] \).

We also need to collect some intermediate results:

**Claim 1:** For any \( \theta \in [\theta_1, \theta_2] \), \( \bar{\theta} > \theta_2 \).

This follows from the definition of \( d^A, d^B \) and \( \bar{\theta} \) in (53).

**Claim 2:** If \( c \) is nonempty, then for any \( \theta \in c = [\theta_2, \bar{\theta}_2] \), \( n^x(\theta) \geq n(\theta_2) \).

This follows from the fact that as we move up from \( \theta_2 \) to \( \bar{\theta}_2 \), we replace agents who get signals below \( \theta_A \) and do not run, with agents who get signals above \( \theta_A \) and may or may not run, which implies that the whole support of \( \tilde{n}(\theta) \) is above \( n(\theta_2) \).

**Claim 3:** The LHS of (57), \( \int_{\theta_1}^{\theta_2} v(\theta, n(\theta)) \), must be nonpositive.

When \( c \) is empty, this holds because \( \int_{\theta_1}^{\theta_2} v(\theta, n(\theta)) \) is equal to \( \Phi(\theta_A, n(\theta)) \), which equals to zero. When \( c \) is nonempty, if \( \int_{\theta=\theta_1}^{\theta_2} v(\theta, n(\theta)) \) were positive, then for some \( \theta \in [\theta_1, \theta_2] \) we would have \( v(\theta, n(\theta)) > 0 \). Since \( n(\theta) \) is increasing in the range \( [\theta_1, \theta_2] \), and since \( v(\theta, n) \) satisfies single crossing when both \( n \) and \( \theta \) are increasing, we must have \( v(\theta_2, n(\theta_2)) > 0 \). Applying claim 2 and using again the fact that \( v(\theta, n) \) satisfies single crossing when both \( \theta \) and \( n \) are increasing, we get that \( \int_{\theta \in c} E_n v(\theta, \tilde{n}(\theta)) \) is also positive. This contradicts the fact that the sum of \( \int_{\theta=\theta_1}^{\theta_2} v(\theta, n(\theta)) \) and \( \int_{\theta \in c} E_n v(\theta, \tilde{n}(\theta)) \), which is simply \( \Phi(\theta_A, n(\theta)) \), is equal to 0.

**Step 3:** In this step, we will show that (57) holds.

For any \( \theta \in [\theta_1, \theta_2] \), let \( n^x(\bar{\theta}) \) be a monotone version of \( n^x(\bar{\theta}) \), and \( \theta^x(n) \) be its “inverse” function.

\[
\begin{align*}
n^x(\bar{\theta}) &= \min(n^x(t) : \theta \leq t \leq \theta_2) \\
\theta^x(n) &= \max((\theta \in [\theta_1, \theta_2] : n^x(\bar{\theta}) \leq n) \cup \{\theta_1\})
\end{align*}
\]

\( n^x(\bar{\theta}) \) is weakly decreasing in \( \bar{\theta} \), and is the minimum \( n^x(\bar{\theta}) \) when we move \( \theta \) from \( \theta_2 \), the upper boundary of \( d^A \), to \( \theta_1 \), the lower boundary of \( d^A \). (i.e., when we move from \( \bar{\theta}_2 \), the lower
boundary of $d^B$, to $\theta_1$, the upper boundary of $d^B$.) $\theta^x(n)$ is the maximum $\theta$ (minimum $\theta$) at which $n^x(\theta) \leq n$. Note that if $n^x(\theta) > n$ for all $\theta \in [\theta_1, \theta_2]$, then $\theta^x(n)$ is defined as $\theta_1$, which is the lowest possible level for $\theta$ (When $\theta$ reaches the lowest value $\theta_1$, $\theta$ reaches the highest value $\theta_1$. See Figure 16 for an example.) Let $A(\theta)$ indicate whether $n^x(\theta)$ is strictly decreasing at $\theta$ (i.e., strictly increasing at $\theta$), if not, then there is a jump in $\theta^x(n^x(\theta))$:

$$A(\theta) = \begin{cases} 1 & \text{if } \frac{\partial n^x(\theta)}{\partial \theta} \text{ exists, and is strictly negative} \\ 0 & \text{otherwise} \end{cases}$$ (60)

Since $n^x(\theta_2) \geq n(\theta_2)$ (claim 2), we have $\theta^x(n(\theta_2)) \leq \theta_2$, and we can rewrite the RHS of (57) as

$$\int^{\theta_2}_{\theta^x(n(\theta_2))} v(\theta, n^x(\theta))d\theta + \int^{\theta^x(n(\theta_2))}_{\theta_1} v(\theta, n^x(\theta))(1 - A(\theta))d\theta + \int^{\theta^x(n(\theta_2))}_{\theta_1} v(\theta, n^x(\theta))A(\theta)d\theta$$

where the third summand equals $\int^{\theta^x(n(\theta_2))}_{\theta^x(\theta_1)} v(\theta^x(n), n)A(\theta^x(n))d\theta^x(n)$, and can be written as

$$\int^{\theta^x(n(\theta_2))}_{\theta^x(\theta_1)} v(\theta^x(n), n)A(\theta^x(n))d(\theta^x(n) - \theta(n)) + \int^{\theta^x(\theta_2)}_{\theta^x(\theta_1)} v(\theta^x(n), n)A(\theta^x(n))d\theta(n)$$ (61)

where $\theta(n)$ is the inverse function of $n(\theta)$ (equation (55)). Note that the second summand is equal to $\int^{\theta^x(\theta_2)}_{\theta^x(\theta_1)} v(\theta^x(n), n)d\theta(n)$. This is because $A(\theta^x(n))$ is 0 no more than countably many times (corresponding to the jumps in $\theta^x(n)$), and because $\theta(n)$ is differentiable.

We will consider two situations: $n^x(\theta_1) = n(\theta_1)$ and $n^x(\theta_1) > n(\theta_1)$.

55
First, suppose \( n^x(\overrightarrow{\theta}_1) = n(\theta_1) \). Since when we move from \( \theta_2 \) to \( \theta_1 \), \( n(\theta) \) decreases with the fastest rate, and because \( n^x(\overrightarrow{\theta}_2) \geq n(\theta_2) \) (This comes from claim 2 when \( c \) is nonempty. If \( c \) is empty, then \( \theta_2 = \theta_A + \epsilon < \overrightarrow{\theta}_2 = \theta_B - \epsilon \). And at \( \theta_2 \) or \( \overrightarrow{\theta}_2 \), all signals are between \( \theta_A \) and \( \theta_B \), and so all people will run and so \( n^x(\overrightarrow{\theta}_2) = n(\theta_2) = 1 \), and because \( n(\theta) \) changes with the fastest speed over \([\theta_1, \theta_2]\), it implies that \( n^x(\overrightarrow{\theta}_2) = n(\theta_2) \) and \( n^x(\overrightarrow{\theta}) = n(\theta) \) for \( \theta \in [\theta_1, \theta_2] \). So each \( v(\theta, n(\theta)) \) in \( d^A \) is paired with \( v(\overrightarrow{\theta}, n(\theta)) \) in \( d^B \) with the same \( n \) but with \( \overrightarrow{\theta} > \theta \). Recall that given \( n \), \( v(\theta, n) \) is weakly increasing in \( \theta \), and if \( v(\theta, n) < 0 \), then it is strictly increasing in \( \theta \).

Since some \( v(\theta, n) \) in \( d^A \) must be negative, so the sum of \( v(\theta, n) \) over \( d^B \) must be higher than the sum of \( v \) over \( d^A \), so (57) will hold.

Now consider the situation in which \( n^x(\overrightarrow{\theta}_1) > n(\theta_1) \).

The LHS of (57) can be written as

\[
\int_{\theta_1}^{\theta_2} v(\theta, n(\theta))d\theta = \int_{\theta_1}^{n(\theta_2)} v(\theta, n(\theta))d\theta + \int_{n(\theta_2)}^{n(\theta_1)} v(\theta(n), n)d\theta(n)
\]

(62)

And we know that

\[
\int_{n(\theta_2)}^{n(\theta_1)} v(\theta(n), n)d\theta(n) \leq \int_{n(\theta_1)}^{n(\theta_2)} v(\overrightarrow{\theta}(n), n)d\theta(n)
\]

(63)

This is because \( \overrightarrow{\theta}(n) > \theta(n) \), and we know that given \( n \), \( v(\theta, n) \) is weakly increasing in \( \theta \).

So the final step is to show that

\[
\int_{\theta_1}^{n(\theta_2)} v(\theta, n(\theta))d\theta < \int_{n(\theta_2)}^{\theta_2} v(\overrightarrow{\theta}, n^x(\overrightarrow{\theta}))d\theta + \int_{\theta_2}^{n(\theta_2)} v(\theta, n^x(\overrightarrow{\theta}))(1 - A(\theta))d\theta + \int_{n(\theta_1)}^{n(\theta_2)} v(\overrightarrow{\theta}(n), n)A(\theta^x(n))d(\theta^x(n) - \theta(n))
\]

(64)

First, if taken together, the integrals in the RHS of (64) have the same length as the integral in the LHS of (64) (the length is measured with respect to \( \theta \)). The reason is that the Length of the LHS(RHS) of (64) is the length of the LHS(RHS) of (57), which is \( \theta_2 - \theta_1 \), minus the length of the LHS(RHS) of (63), which is \( \int_{n(\theta_1)}^{n(\theta_2)} 1d\theta(n) \). Second, the weights on the values of \( v(\theta, n) \) in the RHS of (64) are always positive because from Lemma 1, we know that \( \theta^x(n) \) changes faster than \( \theta(n) \), which means \( \theta^x(n) - \theta(n) \) is weakly increasing.

The LHS of (64) is the sum of values of \( v(\theta, n) \) with \( \theta \) in \( d^A \) and \( n \) below \( n^x(\overrightarrow{\theta}_1) \), and the RHS of (64) is the sum of values of \( v(\theta, n) \) with \( \theta \) in \( d^B \) and \( n \) above \( n^x(\overrightarrow{\theta}_1) \). We consider two cases.

(1) If \( v(\theta(n), n^x(\overrightarrow{\theta}_1)), n^x(\overrightarrow{\theta}_1)) > 0 \), then since when \( \theta \) and \( n \) are both increasing, \( v(\theta, n) \) satisfies the
single crossing property, so all the values of \( v \) in the RHS of (57) are positive because they have higher \( n \) and \( \theta \) than \( v(\theta(\bar{n}(\bar{\theta}_1)), \bar{n}(\bar{\theta}_1)) \). Since the LHS of (57) is non-positive, so (57) should hold. (2) If \( v(\theta(\bar{n}(\bar{\theta}_1)), \bar{n}(\bar{\theta}_1)) < 0 \), then since when \( v(\theta, n) \) is negative, it is strictly increasing when both \( \theta \) and \( n \) are increasing (see the result from Proposition 8), and also because \( v(\theta, n) \) satisfies the signal crossing property, all values of \( v \) in the LHS of (64) are lower than those in the RHS, and since the integrals on both sides have the same length, (64) should hold, as a result, (57) should hold. Q.E.D.

The basic intuition of the above proof can be shown with the imagined example in Figure 16. Denote the total length of \([\theta_1, \theta_2]\) as \( k \). Then the LHS of (63) is the sum of \( v \) over \([\theta(\bar{n}(\bar{\theta}_1)), \theta_2]\), the length of which is \( k_1 \). The RHS of (63) is the sum of \( v \) over \([\bar{\theta}^n(n(\theta_2)), \bar{\theta}_1]\), with the density scaled by \( \frac{k_1}{k_2} \), so the length measured in \( \theta \) is \( k_1 \). The LHS of (64) is the part between \([\theta_1, \theta(\bar{n}(\bar{\theta}_1)))\), the length of which is \( 1 - k_1 \). The RHS of (64) is composed of two parts. The first part is the \( \theta \) between \([\bar{\theta}_2, \bar{\theta}^n(n(\theta_2))]\), the length of which is \( 1 - k_2 \). The second part is the \( \theta \) between \([\bar{\theta}^n(n(\theta_2)), \bar{\theta}_1]\), with the density scaled by \( 1 - \frac{k_1}{k_2} \), so the total length on the RHS is \((1 - k_2) + k_2(1 - \frac{k_1}{k_2}) = 1 - k_1\).

**B.4 Proof of Proposition 4**

If \( n_s(\theta^*) > 0 \), we have

\[
\int_0^1 v(\theta^*, n) dn = \int_0^{n_s(\theta^*)} (1 - \pi(\theta^*)) (\bar{r}^m - \bar{r}^n) dn + \int_{n_s(\theta^*)}^{n_f(\theta^*)} (1 - \pi(\theta^*)) (\bar{r}^m - r^n(\theta^*, n)) dn + \int_{n_f(\theta^*)}^{n_f} \frac{1 - n_f}{1 - n} r^n dn + \int_{n_f}^{n_f} \frac{1}{n_f} \bar{r}^m dn = n_e(\theta^*)(1 - \pi(\theta^*)) \bar{r}^m - n_s(\theta^*) \bar{r}^m - (1 - \pi(\theta^*)) \int_{n_s(\theta^*)}^{n_f(\theta^*)} r^n(\theta^*, n) dn + (1 - n_f) \bar{r}^m \ln(1 - n_e(\theta^*)) - \ln(1 - n_f) - n_f \bar{r}^m \ln n_f
\]

(65)

We can then replace \( n_e(\theta^*) \) with \( \frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)} \) and \( n_s(\theta^*) \) with \( \frac{\kappa_s - \pi(\theta^*)}{1 - \pi(\theta^*)} \). And when \( n \in [n_s(\theta^*), n_e(\theta^*)] \), \( r^n(\theta^*, n) \) can be computed with equation (13), and we have

\[
(1 - \pi(\theta^*)) \int_{n_s(\theta^*)}^{n_e(\theta^*)} \frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)} r^n(\theta^*, n) dn = (1 - \pi(\theta^*)) \int_{n_s(\theta^*)}^{n_e(\theta^*)} \frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)} \left( \frac{R}{\lambda} \bar{r}^m + \frac{1}{1 - \pi(\theta^*)} W \right) dn
= \frac{R}{\lambda} \bar{r}^m (\kappa_e - \kappa_s) + W \ln\left(\frac{1 - \kappa_s}{1 - \kappa_e}\right)
\]

Arranging terms and we get the result in Proposition 4. The case when \( n_s(\theta^*) = 0 \) is similar.
B.5 The results of $Eu^d$ and $E\Pi$

This part shows the result of depositors’ expected payoff and banker’s expected profit when we assume $\pi(\theta) = \theta^n$.

B.5.1 Homogenous information

**Depositor’s expected payoff:** Let $\theta_s$ denote the $\theta$ at which $\pi(\theta) = \kappa_s$. When $\theta \in [0, \theta_s]$, the bank will be able to pay $r^n = \bar{r}$. When $\theta \in (\theta_s, \theta_e]$, the value of $r^n$ can be computed from the following two equations:

$$
\pi(\theta)d_t\bar{r}m = \gamma d_t + \bar{b} + \lambda \eta[(1 - \gamma)d_t + e_b] 
$$

(66)

$$
(1 - \pi(\theta))d_tr^n = (1 - \eta)[(1 - \gamma)d_t + e_b]R - \bar{b} 
$$

(67)

Equation (66) means the withdrawal in $t = 2$ by movers is equal to bank’s own reserve, plus the central bank loan and the proceeds from liquidating $\eta$ of the capital. Equation (67) means the payment available to depositors in period $t = 3$ is equal to the proceeds from the unliquidated investment minus the loan payment.

The solution for $r^n$ is $\frac{R}{\lambda} \bar{r}m + \frac{1}{1-\pi(\theta)}W$. Substituting it into (25), we have

$$
\frac{Eu^d}{d_t} = (1 - \theta_e)n_f\bar{r}m + \int_{\theta_s}^{\theta_e} [\pi(\theta)\bar{r}m + (1 - \pi(\theta))\bar{r}m]d\theta
$$

$$
+ \int_{\theta_s}^{\theta_e} [\pi(\theta)\bar{r}m + (1 - \pi(\theta))\frac{R}{\lambda} \bar{r}m + W]d\theta 
$$

(68)

Given $\pi(\theta) = \theta^n$, (68) becomes

$$
\frac{Eu^d}{d_t} = (1 - \theta_e)n_f\bar{r}m + \int_{\theta_s}^{\theta_e} [\theta^n \bar{r}m + (1 - \theta^n)\bar{r}m]d\theta + \int_{\theta_s}^{\theta_e} [\theta^n \bar{r}m + (1 - \theta^n)\frac{R}{\lambda} \bar{r}m + W]d\theta
$$

$$
= (1 - \theta_e)n_f\bar{r}m + \frac{\theta_e^{n+1}}{\eta + 1} + \bar{r}m(\theta_s - \frac{\theta_s^{n+1}}{\eta + 1}) + \left(\frac{R}{\lambda} \bar{r}m + W\right)(\theta_e - \theta_s)
$$

$$
- \frac{R}{\lambda} \bar{r}m \frac{\theta_e^{n+1} - \theta_s^{n+1}}{\eta + 1} 
$$

(69)

where $\theta_e = \kappa_e^n$ and $\theta_s = \kappa_s^n$ and $\kappa_e$ and $\kappa_s$ are defined in (6) and (12).

**Bank’s expected profit:** Equation (28) can be written as

$$
E\Pi = \int_{0}^{\theta_s} (\gamma d_t - \pi(\theta)d_t\bar{r}m) + [(1 - \gamma)d_t + e_b]Rd\theta + \int_{\theta_s}^{\theta_e} -b + [(1 - \gamma)d_t + e_b]Rd\theta
$$

$$
+ \int_{\theta_e}^{\theta_b} -\bar{b} + (1 - \phi(\theta))(1 - \gamma)d_t + e_b]Rd\theta - \int_{0}^{\theta_b} (1 - \pi(\theta))\bar{r}m d\theta - e_b R 
$$

(70)
Over $\theta \in [0, \theta_a]$, the withdrawal can be met with bank’s own reserve, we have $b = 0$ and $\phi = 0$. When $\theta \in [\theta_a, \theta_b]$, the withdrawal can be met by the loan from the central bank, we have $b \in [0, \bar{b}]$ and $\phi = 0$. Above $\theta_b$, the bank needs to liquidate assets and $\phi > 0$. $\theta_a$ and $\theta_b$ can be computed from

$$\pi(\theta_a) d_t \bar{r}^m = \gamma d_t \Rightarrow \theta_a = \left( \frac{\gamma}{\bar{r}^m} \right)^{\frac{1}{\eta}} \quad (71)$$

$$\pi(\theta_b) d_t \bar{r}^m = \gamma d_t + \bar{b} \Rightarrow \theta_b = \left( \frac{\gamma d_t + \bar{b}}{d_t \bar{r}^m} \right)^{\frac{1}{\eta}} \quad (72)$$

We set $\theta_a$ or $\theta_b$ to 1 if (71) or (72) is higher than 1.

The first integral in (70) is equal to

$$(\gamma d_t + [(1 - \gamma) d_t + e_b] R(\theta_a - \theta_a) - d_t \bar{r}^m \frac{\theta_a^{\eta+1}}{\eta + 1}$$

The second integral is equal to

$$[(1 - \gamma) d_t + e_b] R(\theta_b - \theta_a) - \frac{\theta_b^{\eta+1} - \theta_a^{\eta+1}}{\eta + 1} d_t \bar{r}^m + \gamma d_t (\theta_b - \theta_a)$$

where we use $b = \pi(\theta)d_t \bar{r}^m - \gamma d_t$ when $b \in [0, \bar{b}]$. The third integral is equal to

$$(\bar{b} + [(1 - \gamma) d_t + e_b] R(\theta_s - \theta_b) - \frac{R}{\lambda} d_t \bar{r}^m \frac{\theta_b^{\eta+1} - \theta_a^{\eta+1}}{\eta + 1} + \frac{R}{\lambda} (\gamma d_t + \bar{b})(\theta_s - \theta_b)$$

where we use $\phi(\theta) = \frac{\pi(\theta)d_t \bar{r}^m - \gamma d_t - \bar{b}}{\lambda((1 - \gamma) d_t + e_b)}$, which is solved from (26) and (27). And the forth integral is equal to

$$- \left[ \bar{r}^m d_t \theta_s - \frac{\theta_s^{\eta+1}}{\eta + 1} \bar{r}^m d_t \right]$$

**B.5.2 Heterogenous information**

**Depositor's expected payoff:** If $\theta^* \geq \theta_s$, then over $[0, \theta^*]$, there is no bank runs and the payoff is the same as in the same as in the homogenous information case. And we only need to change $\theta_e$ in equation (69) into $\theta^*$. If $\theta^* < \theta_s$, then $r^n = \bar{r}^m$ over $[0, \theta^*]$, and we have

$$\frac{E u^d}{d_t} = (1 - \theta^*) n f \bar{r}^m + \int_0^{\theta^*} \left[ \pi(\theta) \bar{r}^m + (1 - \pi(\theta)) \bar{r}^m \right] d\theta$$

$$= (1 - \theta^*) n f \bar{r}^m + \bar{r}^m \theta^* + \frac{(\theta^*)^{\eta+1}}{\eta + 1} \left( \frac{\bar{r}^m}{\bar{r}^m - \bar{r}^m} \right)$$

(73)
Bank’s expected profit: If \( \theta^* \geq \theta^* \), then the expected profit can be computed by equation (70). If \( \theta^* < \theta^* \), then the method is still same except that the upper bound is changed into \( \theta^* \). There are three possibilities: \( 0 \leq \theta^* \leq \theta_a, \theta_a < \theta^* \leq \theta_b \) and \( \theta_b < \theta^* < \theta_s \). We only need to change equation (70) accordingly. The details are omitted.

C Simulation steps

We will only explain the key steps of the simulation. What follows are the steps to compute the equilibrium when the information is heterogenous. Recall that given the reserve ratio \( \gamma_j \) chosen by other all banks \( j \neq i \), a representative bank \( i \) chooses the reserve ratio and the lending rate \( r^m \) to maximize depositors’ expected utility, in the equilibrium, the optimal \( \gamma_i \) is equal to \( \gamma_j \).

1. Suppose all banks \( j \neq i \) adopt the same reserve ratio \( \gamma_j \). We need to test whether it is optimal for a representative bank \( i \) to adopt the same reserve ratio. We can test this by computing the derivative of \( Eu^d \) with respect to \( \gamma_i \). In order to compute the derivative, we need to compute the value of \( Eu^d \) when \( \gamma_i \) is slightly above and below \( \gamma_j \). Denote the value of \( \gamma_j \) as \( \gamma \). And define \( \gamma = \gamma + \epsilon, \gamma = \gamma - \epsilon \), where \( \epsilon \) is a small number.

2. Given \( \gamma_j = \gamma \), compute \( R \) according to equation (2).

3. Compute \( Eu^d \) by assuming that bank \( i \) sets \( \gamma_i = \gamma \)

3.1 Given \( \gamma_i = \gamma \), we need to find out the highest \( r^m \) that bank \( i \) can offer subject to the zero expected profit condition.

a. Choose a level of \( r^m \), and then compute \( \theta^* \) according to equation (20).

b. Compute the expected profit of the bank according to equation (28) (if \( \theta^* \geq \theta_s \)) or equation (30) (if \( \theta^* < \theta_s \)).

c. Repeat 2.1 and 2.2 until we find the highest \( r^m \) at which the zero expected profit condition of the bank is satisfied.

3.2 Given \( \gamma, r^m \) and \( \theta^* \), we can compute the expected utility of depositors according to (29).

4. Repeat the previous step for \( \gamma_i = \gamma \), and compute the derivative.

\[
\frac{Eu^d(\gamma_i = \gamma) - Eu^d(\gamma_i = \gamma)}{2\epsilon}
\]  

(74)
If the derivative is higher than zero, it implies that bank $i$ would like to adopt $\gamma_i > \gamma_j$, and we need to increase the value of $\dot{\gamma}$. If the derivative is lower than zero, we need to reduce the value of $\dot{\gamma}$.

5. Repeat all the above steps until the derivative is zero. And the final value of $\gamma_j$ is the equilibrium $\gamma$. Given the equilibrium $\gamma$, we then compute the equilibrium $r^\Pi$ according to step 3.1.

The steps for homogenous information are the same except that in step 3.1 bank run happens when $\theta > \theta_e$ instead of $\theta > \theta^*$. 