Contagion of Fire Sale on Security-Trader Network

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May 2012

Abstract

This paper studies the contagion of fire sale across asset markets through the price effect. When a distressed trader liquidates her stock holding quickly, the fire sale will depress the asset price and other shareholders will incur mark-to-market losses. If the losses are large enough, these shareholders may also have to liquidate other assets they hold, and the liquidation can spread out and become contagious over the financial markets. We show that contagion happens only when the network is neither too sparse nor too dense. The simulation results confirmed this, and also suggest that the asset market networks exhibit a robust-yet-fragile tendency: when the financial network is relatively dense and hence the probability of contagion becomes small, the consequence can be disastrous (close to 100% bankruptcy rate) if it happens. I also examine the effects of front running on contagion and show that front running aggravates both probability of contagion and its consequence. The complex asset markets are depicted by random graph-based bipartite networks, which allow for multiple assets and multiple agents with arbitrary portfolios and distinguishes this paper from the the current literature on contagion of default (counterparty risk). By using the techniques of generating functions, the probabilities of widespread contagions and their extent can be analytically derived.

Keywords: Financial stability, contagion, fire sale, network

1 Introduction

When financial institutions are in distress, forced sales of assets, either prescribed by regulatory constraints or forced by margin calls, may further depress the prices and adversely affect other shareholders. This adverse effect can induce further round of forced sales in other asset markets, and the rapid price declines can be contagious and spread extensively across markets. In modern financial systems, institutions are interlinked by portfolios across various assets, and face a

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system risk of fire sale contagion when the initial shock is large enough, as demonstrated by the recent crises.

Financial contagions generally spread through two channels - direct credit exposures and indirect linkages through changes in asset prices. The former having been extensively studied, and this paper concentrates on the latter. Most recent crises exhibit large price declines and highlight the important role of liquidity risks in spreading the distress.

In the US subprime crisis of 2008, the Dow Jones Industrial Average declined by 18% as of October 10, the largest weekly decline ever. In the same week, The FTSE100 declined by 20%, again the worst ever. This crisis transmitted so fast to Europe, because various US stocks and derivatives were purchased by European investors, and the price effects are felt almost instantly by all market participants around the world. The real estate properties also serve as a contagion channel because they are themselves - when housing price declines by 15% or more from its peak in 2007, direct property investors, banks and other institutions financing mortgages, collateralized debt obligations (CDOs) buyers are all likely to incur loses, which may induce further liquidations, e.g. Bear Stearns. The 1987 stock markets crash bear similar features. The crash began in Hong Kong on 19 October, spread west through international time zones to Europe, hitting the United States later that day. The program trades are blamed to exacerbate the asset price declines in the contagion. The LTCM crisis in 1998 can also be seen as an instance where this knock-on effect propagating market distress. The liquidation of Salomon Brothers bond arbitrage group reduced the market liquidity significantly, where LTCM is also a large player. This initial shock, escalated with Russian default in the same year, led to the failure of LTCM.

This paper aims to model the contagion of forced liquidation, or fire sale, spreading from one asset market to another, and predicts the probability of a wide spread contagion on complex random graph-based financial networks. The random graph/network highlights its topology complexity, compared with most financial networks models, and bears some key features of the real world financial networks, which makes it a good substitute in system risk research. Also, multiple assets in my model are explicitly represented by nodes distinct from agents (Figure 1 shows an example), in contrast to most financial network models where only agents/banks with counterparty risks (and at most one common asset) are modeled. With multiple assets
distinguished from agents, prices of different assets can be individually identified, and the contagion process from one market to another becomes self-evident. This network structures can also accommodate predatory behaviors (front running) and allows study of their effects on contagion. To highlight the price effects, I abstract away the direct exposures and study a network of pure asset portfolio linkages.

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\text{Fig. 1: Representation of a securities-traders network}
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Allen and Gale (2000) is the first paper that study the financial contagion via direct linkage over a network. With a simple network of four banks, they show that spread of contagion depends crucially on the pattern of interconnectedness between banks - when banks only have exposures to a few others, the counterparty risk is not well diversified and the system is more vulnerable; when every bank has exposures to all other banks, the risk is diversified across more counterparties, and default may be absorbed, so contagion is less likely.

Some researchers also study the financial systems from a perspective of endogenous network formation, yet the strong assumptions imposed in their models lead to stark predictions and restrict their applicability to the real financial world. Besides, the financial systems in practice has been evolved over hundreds of years from its very primitive and simplest form to a highly complicated and intricate network where history, trust and local interaction play important role, a one-shot game in which agents simultaneously decide to how to form the network may not be able to provide convincing evidence in optimal design of the financial systems.

Although the insights from simple and rigid network structures are seminal, its generality to the real world financial systems is doubtful. As indicated by Cifuentes et al. (2005), in a more complicated network, there is a non-monotonic relationship between connectedness and
financial stability, which incorporates Allen and Gale (2000)’s finding, but also identifies new network dynamics. Cifuentes et al. (2005) study a regular network of 10 banks with direct credit exposure to counterparties and sharing a common illiquid asset. Under idiosyncratic shock, the disposal of asset drives down the price and may induce defaults on debts. Their simulation results show a non-monotonic relationship: the contagion is small either when there are few links or when there are many, but is large when the network are moderately connected.

Gai and Kapadia (2010) use the same setting, but with random graph-based network structures, which can accommodate arbitrary and complex networks. A random graph/network is a graph generated by some random process. In particular, how many links does each node has is determined by a given probability distribution. On top of that, who is connected to who is also determined by a random process that implement this distribution. They introduce the generating function techniques from the literature on complex networks (Newman et al. (2001)) and derive an elegant analytical solution of the probability of system wide contagion. Besides the non-monotonicity, they also find a robust-yet-fragile tendency of the financial networks: the probability of contagion may be low, the effects can be extremely widespread when initial problems occur.

While both Cifuentes et al. (2005) and Gai and Kapadia (2010) incorporate the asset price effects, there is only one generic illiquid asset in their models, and contagion still spreads through credit channel per se - without default, there will be no contagion. The last three decades have seen a transition from bank-based financial system to a market-based financial system. As demonstrated earlier, contagion across assets markets via price effects plays an important role in market-based systemic events. In contrast, my model investigates the contagion of forced liquidations across multiple distinct assets markets on a random graph network and evaluates the systemic risks. Though not addressing the issue of network formation, random graph-based networks accommodate all networks. So the results of this paper are compatible to and apply to all possible networks, including those endogenously formed or optimally designed networks.

Following Gai and Kapadia (2010), I use the generating function techniques to derive the condition under which the contagion of fire sale will be widespread. The analytical results suggest a non-monotonic relationship between risks and connectedness similar to that in Gai

\footnote{A network where every node has exactly the same number of links}
I will also discuss the effects of front running, or predatory trading, on the contagion of fire sale. The simulation results suggest that with front running, both the probability of contagion and the extent of the contagion are larger than the benchmark model.

The rest of the paper are organized as follows: in Section 2, the setting of the model is given, and I discuss the contagion process, and give analytical results of when and how far is the contagion. Section 5 uses numerical simulations in both models and give the likelihood and extent of contagion. In Section 6, I discuss the effect of the front running. A final section concludes.

2 Setting

2.1 Network Representation

We use nodes and links to represent the relationship among multiple traders and multiple assets (see Figure 1). Nodes are divided into two groups - traders and assets (or securities, I use them interchangeably). A link between a trader and an asset means that this trader holds some share of this asset. As shown in the figure, one trader may hold shares of some assets but not others, and one asset may be held by some traders but not every trader.

We assume that there are in total $S$ assets, $T$ traders in the network. In network literature, the number of links of a node is called its degree. Let $d_s$ denote security $s$’s degree and $d_t$ trader $t$’s degree.

2.2 Asset and Pricing

The total asset of a trader is her portfolio represented in the network. Assets are restricted to illiquid assets. Government bonds and treasury bills are sometimes considered perfectly liquid assets so they may not apply to this model. Each trader is assumed to be a large player, so that her behavior will affect the prices.

Let $V_s$ denote mark-to-market initial value of security $s$ held by traders in the network. “initial” means before any trader is hit by any shock and forced to liquidate. We impose that $V_s = V$, $\forall s$, i.e. the initial value of each security $s$ held in the network is identical$^2$. We also

\footnote{See Footnote 6.}
require that $V_s$ is evenly distributed among shareholders of security $s$, and the value of a link of security $s$ is $\frac{V_s}{d_s}P_s$, where $P_s$ is the price of security $s^3$.

It is assumed that there are also investors who are outside the network. When people trade, traders in the network do not trade with each other, instead they trade with only people outside the network: as traders in the networks buy the security, the outside market has a limited supply and the price is pushed up; and as the traders sell the security, the outside market has a limited capacity to absorb and the price is depressed. Price $P_s$ is thus a strictly increasing function of $x_s$:

$$P_s = \rho(x_s)$$

(1)

where $x_s$ is the fraction of security $s$ held in the network. By this pricing formula, assets are assumed to be illiquid, and investors outside the network are assumed to be passive to the short term price fluctuations. These investors can be regarded as long term investors who are willing to take the assets being liquidated by traders in the networks because the prices are depressed low enough. Now we impose that, before any contagion happens, $x_s = x, \forall s$.

Short selling is not allowed.

### 2.3 Contagion

Time is discrete. At time= 0, a randomly chosen trader is hit by a shock and is forced to liquidate all her portfolio, which depresses the prices of those assets and shareholders may be in distress. When the fire sale spreads and other traders are forced to liquidate, they must liquidate all their portfolios as well. For a shareholder of any affected asset, if the mark-to-market loss is larger than a constant capital buffer $k$, this shareholder will be forced to liquidate all the assets she holds as well in the next period$^4$. This is sometimes referred to as zero recovery assumption. These liquidations may induce yet further round of liquidations of other traders. In the real world a constrained trader forced to to reduce assets off her balance sheet does not always end up with total liquidation, but large share sale often depress the price and may make other shareholders constrained. When others are forced to sell as well, price will be depressed.

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$^3$Therefore prices may differ from security to security, and shares may differ from link to link

$^4$A capital buffer that is proportional to a trader's total asset may seem to be more natural, however this assumption leads to a diverge condition too complicated to get an analytically intuitive explanation.
further and this triggers a downward spiral in price. This process is similar to a system in a high but unstable equilibrium disturbed and evolving to a low but stable equilibrium. Cifuentes et al. (2005) discuss this procedure in details and have a similar result. When crisis is fermented and investors are highly uncertain, this type of processes are more likely than often to occur. It is also assumed that $V > k$.

A trader does not respond to price fluctuations until she finds that her loss is already larger than $k$ and then she liquidates in the next period.

The contagion continues until a period no trader is liquidating, and then the contagion ends.

### 2.4 Network Structure

The network is a random graph with securities' degree distribution $p_j$ and traders' degree distribution $q_k$ exogenously given ($j$ and $k = 0, 1, 2, ...$). The distributions prescribe that, of the $S$ securities, a fraction of $p_j$ has exactly degree of $j$; and of the $T$ traders, a fraction of $q_k$ has exactly degree of $k$. But exactly who is connected to who is determined by a stochastic process that complies with the degree distributions. Let $\mu = \sum_j j p_j$ and $\nu = \sum_k k q_k$, so the average degree of security is $\mu$ and the average degree of trader is $\nu$, and we have $S\mu = T\nu = \text{total number of links in the network}$.

Assume there is no loop in the network\(^5\), any trader will suffer from this price impact at most once. No trader is to be subject to multiple impacts (either simultaneously or consecutively). If the trader does not fall in a round of liquidation, she stands till the contagion ends. If loops are allowed, traders will suffer multiple impacts and this only makes the contagion more likely and more extensively. The no-loop restriction will be relaxed in simulations.

### 3 Vulnerability of Security

In Figure 2, when trader $F$ is forced to liquidate all her positions of security 2 and 3, $x_2$ (the fraction) decreases from $x$ to $x - \frac{\rho(1)}{\rho(2)}$, and price $P_2$ decreases from $\rho(x)$ to $\rho(x - \frac{\rho(1)}{\rho(2)})$. Trader $D$’s holding of security 2 is initially worth of $\frac{V d_2}{\rho(2)}$. This price decline leads to a loss of $L(d_2) \equiv \frac{V d_2 \rho(x) - \rho(x - \frac{\rho(1)}{\rho(2)})}{\rho(x)}$. If $L(d_2) > k$, trader $D$ will go bankrupt and be forced to liquidate all

\(^5\)This is required by the generating function techniques. This assumption is not as strong as it looks. It turns out that that most loops are allowed. See Footnote 8 for explanation.
her positions of securities 1 and 2, otherwise $D$ survives. Since $\rho$ is a strictly increasing function, $L(d_2)$ is strictly decreasing in $d_2$, and the equation $L(d_2) = k$ has exactly one solution, and denote it $d^*$. If any asset’s degree $d < d^*$, we have $L(d) > k$, otherwise, $L(d) < k$.

Since $V$, $x$ and $k$ are constant parameters, it turns out that if a security’s degree is less than $d^*$, one shareholder’s liquidation will induce the bankruptcy of all other shareholders and force them to liquidate all their holdings of other securities; otherwise, other shareholders will survive. In short, if a security is held by only a few traders, it will be vulnerable. We thus define the vulnerability of a security as follows:

**Definition 3.1** A security $s$ is vulnerable if

$$d_s < d^*$$

(2)

Define an indicator function $v(d_s)$,

$$v(d_s) = \begin{cases} 
1 & \text{if } d_s < d^*; \\
0 & \text{if } d_s \geq d^*.
\end{cases}$$

A particularly simple example is when $\rho(x) = \gamma x$, where $\gamma$ is a constant, i.e. the price is linear in $x$. Again in Figure 2, when trader $F$ is forced to liquidate all her positions of security 2 and 3, $x_2$ decreases by $\frac{1}{d_2}$, and since price is linear in $x_2$, $P_2$ also decreases by $\frac{1}{d_2}$. Trader $D$’s holding of security 2 is initially worth of $V_2$. This price decline leads to a loss of $\frac{V_2}{d_2}$ for trader $D$. If $\frac{V_2}{d_2^2} > k$, i.e. $d_2 < \sqrt{\frac{V}{k}}$, trader $D$ will be forced to liquidate all her positions.

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$^6$Alternatively, we can make both $V$ and $k$ random variables, so that the rigid assumptions on constant $v$ and $k$ are gone and assets and traders becomes heterogenous, and this indicator function will become a probability. But we will lose the tractability of the model and the analytical result will not be available.
of securities 1 and 2, otherwise $D$ survives. This shows that, the smaller the ratio $\frac{V}{k}$, the less likely a security is vulnerable to contagion, which implies that higher capital buffer may reduce the chance of contagion.

4 Extent of Contagion

Now we use the generating functions to calculate the size of contagion. If properly derived, the generating function can give us the probability distribution of the size of contagion. Appendix A lists some basic properties of the generating functions. In a infinite network (infinite number of nodes), if the average size of contagion diverges, that means there is a non-trivial fraction of the traders are infected, and we consider it a crisis.

4.1 Probability Generating Functions

- The probability that a randomly chosen security has degree $j$ is $p_j$, and the probability that a randomly chosen security is vulnerable and has degree $j$ is $p_jv(j)$, and the latter probability distribution can be represented by a generating function $f_0(x) = \sum_j p_jv(j)x^j$;

- The probability that a randomly chosen trader has degree $j$ is $q_j$, and this probability distribution can be represented by a generating function $g_0(x) = \sum_j q_jx^j$;

- If a security is vulnerable, we say that all its links are contagious, otherwise, all its links are not contagious. Randomly choose a link and follow it to a security $s$, the probability that $s$ has $j - 1$ contagious outgoing links (excluding the incoming link by which we followed to $s$. Or equivalently $s$ is vulnerable and has degree $j$) is $\frac{jp_jv(j)}{\sum_j jp_j} = \frac{jp_jv(j)}{\mu}$, in an infinite network, and its generating function is

$$f_1(x) = \frac{\sum_j jp_jv(j)x^{j-1}}{\mu} = \frac{f_0'(x)}{\mu}$$

The reason why $f_1(x)$ is different from $f_0(x)$ is that, if we randomly choose a link and follow it to a node, the probability that we arrive at a node with particular degree is proportional to the degree of that node. This is why we multiply $p_jv(j)$ by $j$. This

\footnote{The use of generating functions requires an infinite network. Watts (2002) shows that the infinite network can be well approximated by a network with 10,000 nodes, and Gai and Kapadia (2010) shows that the results from networks with 1,000 nodes and those from networks with 10,000 nodes agree quite well.}
probability also needs to be normalized, that is why we divide it by $\mu$. See Newman et al. (2001), Newman (2003) and Feld (1991) for more explanations.

- Randomly choose a link and follow it to a security $s$, the probability that $s$ is not vulnerable is $1 - \sum_j \frac{j p_j v(j)}{\mu} = 1 - f_1(1)$. The generating function of the probability distribution that $s$ has $j$ contagious outgoing links is

$$f_2(x) = 1 - f_1(1) + f_1(x) = 1 - f_1(1) + \frac{1}{\mu}[p_1 v(1) + p_2 v(2)x + p_3 v(3)x^2 + ...]$$

- Randomly choose a link and follow it to a trader $t$, the probability that $t$ has $j-1$ outgoing links (excluding the incoming link, in an infinite network) can be generated by

$$g_1(x) = \frac{\sum_j j q_j x^{j-1}}{\nu} = \frac{g'_0(x)}{\nu}$$

- Randomly choose a link and follow it to trader $t$, the probability distribution of number of traders that share vulnerable securities with $t$ (excluding $t$ itself) can be generated by

$$G_1(x) = g_1(f_2(x)) = \frac{1}{\nu}[q_1 + q_2(f_2(x)) + q_3(f_2(x))^2 + ...]$$

(3)

- Randomly choose a link and follow it to a trader $t$, the probability distribution of the total number of traders that can be infected, directly and indirectly, through vulnerable securities starting from $t$, $H_1(x)$, must satisfy

$$H_1(x) = x G_1(H_1(x))$$

(4)

Randomly choose a link and follow it to a trader $t$, the tree-like vulnerable cluster we can reach from there can take many different forms, given by distribution $H_1(x)$, and this is represented in Figure 3 by the LHS (a square). $t$ may or may not have links directly to vulnerable securities. $t$ and these possible vulnerable securities are all encapsulated in

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8The use of probability generating functions requires a tree-like network, i.e. there cannot be any loop in the network. The generating function $H_1(x)$ aims to give (the distribution of) the total number traders that can be infected. If there are loops, nodes on the loop may be repeatedly counted. However we only need that there can not be loops between low degree (vulnerable) assets. If high degree assets have loops, nodes on the loop are unlikely to be counted at all, because the high degree nodes (safe assets) obstruct the infection. Notice that the higher the degree, the more likely that the node is in a loop, because it has more links. So most loops have high degree nodes involved, but we do not need to worry about them because they are insulated. The only thing we should worry is the loops consisted of purely low degree assets, though there are not many of them. This whole idea is reflected by the presence of term $v(j)$ in the generating functions.
the circle on RHS in Figure 3. There can be all together 0, 1, 2, 3, ... outgoing links (represented by $G_1(x)$) from this single circle (from these possible vulnerable securities) to other traders, and these possibilities correspond to the subfigures on the RHS\(^9\). All these possibilities “sum” to the vulnerable cluster on LHS. Note that each of the vulnerable clusters (the squares) on the RHS has the same size distribution as the original vulnerable cluster on the LHS, and we have

$$H_1(x) = x r_0 + x r_1 H_1(x) + x r_2 [H_1(x)]^2 + ...$$

$r_i$ is just the coefficients of $x^i$ in the generating function $G_1(x)$ (Equation 3). Therefore, we have Equation 4. See Newman et al. (2001) for details.

The leading $x$ in Equation 4 on RHS accounts for the first trader following the initial chosen link. In this recursive form of $H_1$, $x$ actually accounts for all the traders infected on route. Without this $x$, we are merely calculating the infected traders “on the farmost border”. The reason is that $G_1$ does not include the original trader.

- Randomly choose a trader $t$, the generating function of the distribution of the total number of traders that can be infected through vulnerable securities starting from $t$, can be generated by

$$H_0(x) = x g_0(f_2(H_1(x))) \quad (5)$$

The leading $x$ on RHS accounts for the very initially randomly chosen trader.

### 4.2 Expected Size of Contagion

Unfortunately, a closed form solution for $H_0(x)$ does not exist for Equation 5 in general. However we do not need the whole distribution $H_0(x)$, the mean is enough for us to infer the system

\(^9\)The 0 link case is where trader $t$ has no outgoing link, or $t$ has outgoing links but all to safe securities.
stability. We can calculate the expected size of contagion $\langle s \rangle$ by

$$\langle s \rangle = H'_0|_{x=1} = g_0(f_2(H_1(1))) + g'_0(f_2(H_1(1)))f'_2(H_1(1))H'_1(1)$$

Since $H_1(x)$ is a standard generating function and it includes all possibilities so that it sums to 1, $H_1(1) = 1$. By the same reason, $f_2(1) = 1$ and $g_0(1) = 1$. Also $g'_0(1) = \nu$. So we have

$$\langle s \rangle = 1 + \nu f'_2(1)H'_1(1)$$

(6)

Also $f'_2(1) = f'_1(1) = \frac{f''_0(1)}{\mu}$, and from Equation 4 we have $H'_1(1) = G_1(H_1(1)) + G'_1(H_1(1))H'_1(1) = G_1(1) + G'_1(1)H'_1(1) \Rightarrow H'_1(1) = \frac{G_1(1)}{1-G'_1(1)}$, substitute into Equation 6, we have

$$\langle s \rangle = 1 + \frac{\nu}{\mu} f''_0(1) \frac{G_1(1)}{1-G'_1(1)}$$

From $v(j) = 0$ if $j \geq d^*$ we know $f''_0(1) = \sum_j j(j-1)p_jv(j)$ must has a finite value, and $G_1(1) = 1$. So if $1 - G'_1(1) = 1$, then the expected size of contagion diverges.

$$G'_1(1) = g'_1(f_2(1))f'_2(1) = g'_1(1)f'_2(1) = \frac{g''_0(1)}{\nu} \frac{f''_0(1)}{\mu} = 1 \Rightarrow$$

$$\sum_j \sum_k jk(j-1)(k-1)p_jq_kv(j) = \nu \mu$$

(7)

Therefore the expected size of contagion $\langle s \rangle$ diverges when the above condition holds. From the definitions of $\mu$ and $\nu$, we can rewrite Equation 7 to

$$\sum_j \sum_k jk(j-1)(k-1)p_jq_kv(j) = \sum_j \sum_k jkp_jq_k$$

(8)

For typical degree distributions (such as binomial distribution or Poisson distribution\(^\text{10}\), which will be used in the simulations.), as $p$ increases, the probability mass moves to $p_j$ and $q_k$ with higher $j$ and $k$, which can be see in Figure 4. As $p$ increases, the LHS of Equation 8 first increases and then decreases, whereas the RHS increases monotonically (see Figure 5a for example). The two curves have two intersections (The left intersection can be seen in Figure 5b, which is an expansion of Figure 5a in the area close to 0), where Equation 7 or 8 holds. The intuition is that:

\(^\text{10}\)If assuming degree distributions are binomial, then security’s degree distribution, $p_j$, is $\binom{T}{j}p^j(1-p)^{T-j}$ (and trader’s degree distribution, $q_k$, is $\binom{S}{k}q^k(1-q)^{S-k}$). The binomial distribution can be well approximated by Poisson distribution $\frac{e^{-T}T^j}{j!}$, so this type of networks are often called Poisson networks. Note that the mean of binomial distribution is $Tp$. In bipartite networks, to keep $S\mu = T\nu$ (total links of securities equal total links of traders), it is required that $p = q$. 

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Figure 4: Binomial Distributions. $n = 100$, $p = 0, 0.005, 0.010, 0.015, ...$

Figure 5: LHS and RHS of the diverge condition Equation 7. $S = T = 100$, $p = 0, 0.005, 0.010, 0.015, ..., 0.150$

1. When $p$ is small and most of the mass of $p_j$ and $q_k$ is on $j, k = 0, 1, 2$, adding $(j - 1)(k - 1)$ on the LHS actually decreases the sum. This is because, when $j$ or $k = 0$, adding $(j - 1)(k - 1)$ has no effect; when $j$ or $k = 1$, either $j - 1$ or $k - 1 = 0$, so some significant terms are removed from the sum; when $j = k = 2$, $(j - 1)(k - 1) = 1$, again there is no effect. Only when both $j$ and $k > 2$, adding $(j - 1)(k - 1)$ increases the terms, but since $p_j$ and $q_k$ are small for $j$ and $k > 2$, their contributions are negligible, and the overall effect of adding $(j - 1)(k - 1)$ is negative. This is why LHS < RHS in Figure 5b when $p$ is small.

2. When $p$ increases and most of the probability mass moves out of $p_j$ and $q_k$ with $j, k = 0, 1, 2$, and the $j$ of $p_j$ with large probability mass is not too large so that $v(j) = 1$, adding $(j - 1)(k - 1)$ on the LHS increases the sum, as expected, and this makes LHS > RHS for
median $p$.

3. When $p$ increases even further, the large mass of $p_j$ and $q_k$ move way too right and falls in the range where $v(j) = 0$. For small $j$ that $v(j) = 1$, the LHS decreases with $p$ (see Appendix B) and eventually LHS < RHS again for large $p$.

In between the two intersections, when LHS > RHS, we have $G'_1(1) > 1$. From Equation 3, we know that $G_1(x)$ is the distribution of number of traders that share securities with a trader (this trader is reached by randomly choose a link and follows to), and $G'_1(1)$ is its expected value. Intuitively, if on average, such a trader can reach more than one neighboring traders (excluding the incoming one) through vulnerable securities, it implies that the number of traders we can ultimately reach through vulnerable securities diverges in an infinite network. This is why the crisis frequency curve has a hump in the middle and converges to 0 on the left and right hand side.

5 Numerical Simulations

5.1 Methodology

In simulations we relax the no-loop restriction on the network structures and allow multiple impacts occur to a trader consecutively and simultaneously. We consider networks with 1000 securities and 2000 traders. We assume a random graph in which each possible link between a trader and a security is present with independent and identical probability $p$ (binomial distribution). The binomial distribution is chosen for simplicity. The distribution is implemented by Configuration Model\(^{11}\).

I draw 500 realizations of network for each $p\(^{12}\) and in each of these draws, I randomly choose a trader and force her to liquidate all her portfolio. This whole procedure is repeated 10 times. Any trader whose accumulated loss on all affected securities is larger than $k$ must liquidate all their assets as well.

I assume the price is linear in the fraction of the asset held in network. By varying the ratio $V$ to $k$, we examine the effect of capital buffer on the contagions. By varying the value of $p$, we

\(^{11}\)See Jackson (2008), Section 4.1.4

\(^{12}\)Traders and securities have independent degree distributions.
have networks with different degree distributions and thus different average degrees of traders and securities. The average degree of a network is an indicator of how well or poor a network is connected. We would like to see when system-wide contagion is more likely and when it is rare. When more than 5% of the total traders are infected, we consider it a system-wide contagion or crisis. Conditional on there is a system-wide contagion, we also examine the extent of that contagion.

5.2 Simulation Results

The simulation results of the model is shown in Figure 6. We can see that when the average degree is either very low or very high, system-wide contagion is not likely to happen. Whereas within a certain window where the average degree is moderate, an extensive contagion is more likely, but is non-monotonic in average degree. This confirms the results in Section 4.2. The extent of contagion, conditional on that it has infected more that 5% of the trader population, is approximately the same as the frequency of contagion at the range of lower average degrees. But at the higher end of average degree, although system-wide contagion is rare, but once it happens, it will be a severe one. So this model has the same robust-yet-fragile feature as Gai and Kapadia (2010).

![Figure 6: Crisis frequency and conditional extent (k/V = 0.01)](image)

Figure 7 shows how changes in the capital buffer and average degree jointly affect the frequency of extensive contagion. When the capital buffer is high, system risks are low in all
average degrees. When the capital buffer declines, probability of contagion reach its peak first within a small window at the low average degree range. As capital buffer drops further, this windows expands to high average degree range. This tells that increasing capital buffer can effectively decrease the probability of widespread contagion; if there is no restriction on the leverage ratios, even very well connected network faces huge system risks.

6 Discussion: Front Running and Contagion

In this section we examine the effect of predation on contagion by allowing someone front running the distressed trader.

The front running, or predatory trading, is in line with Brunnermeier and Pedersen (2005): when some traders are in distress and forced to liquidate illiquid assets, other shareholder may take the advantage by selling before the distressed and then later buy back the asset to make a profit. Front running, though generally illegal, has long been suspected on Wall Street. Investigations and convictions appear in newspaper from time to time\textsuperscript{13}. There are evidences suggesting that, during the 1998 LTCM collapse, several market participants front run LTCM’s liquidation\textsuperscript{14}.

As shown in Figure 8a, when some distressed agents are forced to liquidate their positions,

\textsuperscript{13}See Khan and Lu (2009)
\textsuperscript{14}See Cai (2003)
without predation the price will decline permanently after the forced liquidation. In Figure 8b, if another shareholder (predator) knows about the oncoming liquidation, the predator will try selling before the liquidation, then buy back after the liquidation. By selling at a higher price at the beginning and buy back after at a lower price, the predator can make a profit. Notice that because of the front running, there is excessive price decline, called price overshoot,

![Price dynamics in liquidation](image)

(a) Without front running  
(b) With front running

Figure 8: Price dynamics in liquidation

where the price is lower than that without predation. This price overshooting is the reason why we are concerned about predatory behavior in the networks, because it further writes down the asset price during the liquidation, aggravates the distress on other shareholders, and may induce further rounds of distressed liquidation.

The model we discuss here has everything else the same as the benchmark model, including the network and the asset pricing, except that there is front running. We only consider a very simple situation, where the predator can actually front run the distressed trader: the predator can sell all her position before the forced liquidation starts, and buys back after the forced liquidation finishes. We also assume that, whenever there is a distressed trader forced to liquidate, there is one and exactly one non-distressed trader who knows it and preys on the distressed. Other traders do not know the liquidation and remain inactive. This is consistent with the fact that front running is illegal so the predator will want to keep the secret to herself. Besides, the more predators, the less profit for each predator can be made out of the predation.\textsuperscript{15}

\textsuperscript{15}See BP2005 for details.
Because there is no loop in the network, there is at most one distressed present in any asset market.

The predators are myopic in the sense that they do not forecast whether or not they themselves will be made distressed because of the excessive price declines - when they do find themselves in trouble, they will liquidate in the next period. The myopic behavior can be justified by predators’ incomplete information about the market participants and others’ positions, etc, in which case predators might originally be optimistic about the predation, only find out later that it is not as what they have expected. In real world people do not always prey on the distressed, while in the model I assume an extreme case in which people always do so. Alternatively we can assume people prey with some probability, but this will only change the results quantitatively, not qualitatively.

Assume that predators must return to their original positions after the predation. Since all traders have the same positions at beginning, and both the predator and distressed are to sell all their positions, the extra price decline, or the price overshooting, will double the permanent price decline, exactly as shown in Figure 8b. Recall Figure 2, where trader \( F \) is forced to liquidate all her positions of security 2, and a predator is also to sell the same position, then \( X_2 \) decreases by \( \frac{2}{d_2} \). According to Equation 1 (the pricing formula), \( P_2 \) also decreases by \( \frac{2}{d_2} \). Trader \( D \)’s holding of security 2 is initially worth of \( \frac{V_2}{d_2} \). This price decline leads to a loss of \( \frac{2V_2}{d_2} = \frac{2V}{d_2} \) for trader \( D \). If \( \frac{2V}{d_2} > k \), i.e. \( d_2 < \sqrt{\frac{2V}{k}} \), trader \( D \) will be forced to liquidate all her positions of securities 1 and 2, otherwise \( D \) survives.

So the vulnerability of a security changes to:

\[
d_s < \sqrt{\frac{2V}{k}}
\]  

Clearly, this condition expands the vulnerable degree range by \( \sqrt{2} \), and more assets become vulnerable. As a result, contagion tends to spread further than in the benchmark model.

At high degree assets, the price overshooting is small, and the prices converge to the case where there is no predatory behavior. So it is expected that when the network are better connected, the effect of predatory behavior on contagion will be small.

I will not derive a analytical solution in this case, but use simulations to compare the two models instead, which is shown in Figure 9. We see that at the lower end of the average degree,
Figure 9: Contagion with front running vs. without front running \((k/V = 0.01)\)

the two models do not differ much. When average degree is beyond 3, the front running makes a difference and outweighs the benchmark model. With front running, both the contagion frequency and the conditional extent of contagion stay near 100% much longer than without. This reminds us that, when predatory behavior is common, the security markets are exposed to a much larger risk. The robust-yet-fragile feature is even more evident: crisis may be rare, but if it comes, it will be a real disaster.

7 Conclusion

In this paper I investigate the contagion of fire sale in complex bipartite networks of multiple assets and traders. The initial idiosyncratic shock that forces an individual to liquidate can potentially transmit across asset markets and spread to a large population, which causes a huge market crash just like the crash in 1987. The results of the model suggest a robust-yet-fragile tendency similar to Gai and Kapadia (2010). This model can also incorporate some behavioral elements such as front running so that we can examine their effects on the contagion.

This model applies to a variety of systems in which agents holding arbitrary financial portfolios in illiquid and volatile assets, where common elements in the portfolios link two agents together. While diversified portfolios reduce the probability of contagion on an individual basis, a high connectivity increase the probability of contagion when problems do occur.
This model provides a preliminary method to evaluate the stability of the asset markets as a whole against the contagion of fire sale. When the true structures of the markets are unknown, it is reasonable to use random networks which are consistent with the true markets in some key features. It would be useful to extend the model by relaxing the zero recovery assumption so that distressed traders do not necessarily liquidate all their portfolios, but only part of them. It may also be interesting to allow short selling so that the predation would be more fierce, and this would allow a new link (temporarily) created in the network and thus introduce more flexibility on the network structures, and may shed some light on the network formation or evolvement issue. By tweaking some parameters in the extended model, traders in the predatory process may face a coordination problem discussed in BP2005, and this equilibrium selection problem may have some interesting implication on the contagion. In the real world, it is usually not a default on a bilateral loan (as discussed in Gai and Kapadia (2010)) or the knock-on effect of fire sale alone that causes the trouble, but the combination of the two. It might be possible to combine the two models to analyze the contagion on both channels, but it remains to see whether it is technically feasible. I will leave these for future works.

References


Feld, Scott L. (1991) ‘Why your friends have more friends than you do.’ *American Journal of Sociology* 96(6), 1464–77


Appendix A: Generating Functions

Let $D$ be a discrete random variable taking values 0, 1, 2, ..., and let $p_j = \text{Prob}[D = j]$ for $j=0, 1, 2, ...$

The probability generating function of the distribution, $p_j$, of the random variable $D$ is

$$f(x) = E(x^D) = \sum_{j=0}^{\infty} P(D = j)x^j = \sum_{j=0}^{\infty} p_jx^j$$

Note that

$$f(1) = \sum_{j=0}^{\infty} p_j = 1$$

The probability distribution $p_j$ can be uniquely determined by the generating function $f(x)$ in the following sense:

$$p_j = \frac{1}{j!} \left. \frac{d^j f(x)}{dx^j} \right|_{x=0} = \frac{1}{j!} f^{(j)}(0)$$

**Moments.** The average over the probability distribution is given by

$$\mu = \langle D \rangle = \sum_{j=0}^{\infty} jp_j = f'(1)$$

and higher moments are given by

$$\langle D^n \rangle = \sum_{j=0}^{\infty} [(x \frac{d}{dx})^n f(x)]|_{x=1}$$

**Distribution of sum.** If $D_1$, $D_2$, ..., $D_n$ are independent discrete random variables with generating functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, then the generating function of $D_1 + D_2 + ... + D_n$ is $f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)$. For example, if $D_1$ and $D_2$ are i.i.d. random variables from distribution $p_j$, then the distribution of $D_1 + D_2$ is generated by

$$[f(x)]^2 = \left[ \sum_i p_ix^i \right]^2$$

$$= \sum_j \sum_k p_jp_kx^{j+k}$$

$$= p_0p_0x^0 + (p_0p_1 + p_1p_0)x^1 + (p_0p_2 + p_1p_1 + p_2p_0)x^2$$

$$+ (p_0p_3 + p_1p_2 + p_2p_1 + p_3p_0)x^3 + ...$$
Appendix B: LHS Decreases with \( p \) When \( p \) is Large

the LHS of Equation 8 is

\[
\sum_j \sum_k j(k-1)(k-1)p_jq_kv(j) = \sum_j j(j-1)v(j)p_j[\sum_k k(k-1)q_k]
\]

The mean of binomial distributions are \( Tp \) and \( Sq \), and variances are \( Tp(1-p) \) and \( sq(1-q) \). From Footnote 10 we know \( p = q \). We have

\[
\sum_k k(k-1)q = \sum_k k^2q_k + \sum_k kq_k = \text{var}_{\text{trader}} - \nu + \nu^2 = S^2(q^2 - q^4) - Sq
\]

Let \( A_j = p_j[\sum_k k(k-1)q_k] \), so LHS = \( \sum_j j(j-1)v(j)A_j \). \( dA_j/dp = \left( \begin{array}{c} T \\ j \end{array} \right) p^{j-1}(1-p)^{T-j-1}Sp[(j-pT)(S(p-p^3) - 1) + (1-p)[S(p-p^3) - 1] + (1-p)[S(p-3p^3)]] \). Since \( p \) has increased further and not too small, \( S \) and \( T \) are very large, \( j \) is small so that \( v(j) = 1 \), we have \( j - pT >> 1 - p \). So \( (j-pT)(S(p-p^3) - 1) >> (1-p)[S(p-p^3) - 1] \) and \( (j-pT)(S(p-p^3) - 1) >> (1-p)[S(p-3p^3)] \). Therefore \( dA_j/dp \) must be negative, so all \( A_j \) with small \( j \) and large \( p \) decrease with \( p \), and eventually LHS< RHS.