Booms, bubbles, and crashes
(Job Market Paper)

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Abstract

In this paper we attempt to answer the fundamental question of why bubbles exist and design a tentative tax/subsidy scheme that reduces bubbles. We first generate bubbles from a simplified version of Abreu and Brunnermeier (2003) model, where rational traders optimally ride the bubble for sufficient rise in price. We then show that the model is equivalent to a reverse discriminatory price (first-price) common value auction. We further reveal that bubbles exist because in these auctions prices fail to reflect asset values, and two (reverse) bid shading incentives together give rise to the bubbles. A surprising implication is that a tax on the capital gain during the boom, which is designed to suppress the bubble, may actually inflate the bubble. A simple tax/subsidy schedule is devised which can reduce the size of bubbles by half.

In an extended model, we analyze how traders decide to first purchase and then sell. When the price is so high that no one wants to buy any more, not even the trader with the highest signal, the bubble bursts. In a unique equilibrium, traders collectively create and optimally ride the bubble for sufficient long. Bubble size is decreasing in the transaction cost, and traders maintain a substantial markup between their purchase and sale prices.

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1 Introduction

The asset bubbles that are relevant in this paper are broadly defined as a relatively large upward price deviations from their fundamental values, which can last for extended periods. Such phenomena have long been intriguing to economists because they not only affect the financial sector, but also have huge impact on the real economy. Historical examples of bubbles include the Dutch tulip mania of the 1630s, the South Sea bubble of 1720 in England, the Mississippi bubble in France, the Great Recession of 1929 in the United States, DotCom bubble in the late 1990s, and the most recent housing bubble crash since 2008.

However, bubbles have been difficult to explain and generate theoretically. One major hurdle is the “no trade theorems”. One version of the theorem states that, if, the initial allocation is efficient, there is common knowledge that all traders are rational, and agents have common priors about the distribution of asset values and private information, then agents do not have incentives to trade\(^1\). In a dynamic setting, in particular, the standard neoclassical theory precludes the existence of bubbles by backward induction.

Facing these impossibilities, economists elicit trade by relaxing different aspects of the assumptions in the theorems. For surveys, refer to Brunnermeier (2009), Brunnermeier (2001), Brunnermeier and Oehmke (2012) and Scherbina (2013).

One strand of the literature allows heterogeneous priors and agents to “agree to disagree”. Harrison and Kreps (1978) show that when agents disagree about the probability distributions of dividend streams, a trader may buy an asset at a price that exceeds her own valuation because she believes that in the future she will find a buyer with more optimistic belief. In a similar model, Scheinkman and Xiong (2003) justify this reasoning by studying overconfidence as a source of disagreement.

In a related strand of literature, Allen et al. (1993) and Conlon (2004) allow agents to have heterogeneous priors and hold worthless assets in hopes of selling it to greater fools. Instead of relying on “agreeing to disagree”, it is the lack of common knowledge about the distribution of belief or a higher order belief about asset value that generate the bubbles. One advantage of these models is that all agents are rational.

Both of the above strands help explain in theory why trade exists in the market, but they are hard to apply to the bubbles in the real world because, to some extent, they lack a clear price path with steadily growth and then a sudden collapse. Moreover, the complexity of these models implies that agents need to be hyper-rational to deduce the posterior beliefs\(^2\).

\(^1\)See Grossman and Stiglitz (1980) and Milgrom and Stokey (1982).
so that bubbles emerge. These restrictions have prompted many economists to be inclined to believe that behavioral traders must be active in the market.

Recent empirical and experimental studies and psychological studies\(^2\) have provided convincing evidence that agent’s behavior is far from rational in daily trade. Hence, the third strand of literature studies bubbles that arise from the interaction between rational traders and behavioral traders. In De Long et al. (1990), rational traders buy the asset and induce positive-feedback traders to follow and buy the asset from (behavioral) passive traders. After the price is further pushed up, the rational traders can profitably unload. Although simple and intuitive, the bubble in their model arises exactly because behavioral traders are buying. If we remove the positive-feedback traders and leave only the passive traders, there will be no bubble. In addition, the bubble exists also because time is discrete in their model. If we transform the model into a continuous time one and assume that not all traders can receive the full pre-crash price if they all sell together, then the bubble will not survive because rational traders will then have incentive to preempt each other, which suppresses the size of the bubble until it drops to zero.

The non-robustness in the above model actually confirms the efficient markets hypothesis, which states that prices are consistent with the fundamental values, and that well-informed sophisticated investors will undo the price impact of behavioral traders so that any bubble cannot persist for an extended period.

Herding is an intuitive explanation of the rise of bubbles. Literature on rational herding challenges the efficient markets hypothesis and attempts to simulate the emergence of bubbles and subsequent crashes\(^3\). Avery and Zemsky (1998) introduce a sequential trade model, where a trader is informed with probability \(\mu\) and a liquidity trader with probability \(1 - \mu\). To produce an informational cascade as well as a crash, three dimensions of uncertainty are needed: the signal about the asset value, the quality of the signals, and the uncertainty about the \(\mu\). The market maker adjusts the price after observing each round of transaction. If many of the poorly informed traders are herding and trading with the market maker, the price will keep increasing above its liquidation value, and then, after many liquidity trades, the bubble bursts. Lee (1998) studies an information avalanche, where a trader must pay a transaction cost but can decide when to trade. The market maker can adjusts the price only ex post every round. All traders are informed but differ in the precision of their information. A partial informational cascade can occur if many use the “wait and see” strategy but be shattered by an extreme signal that triggers all

\(^3\)See Brunnermeier (2001) and Chamley (2004) for surveys on this topic.
previous inactive traders to sell. However, both models suffer from the critique that the bubble followed by a crash happens with very small probabilities. In addition, in almost all rational herding models, traders have essentially only one opportunity to trade.

Abreu and Brunnermeier (2003) (henceforth AB2003) also challenge the efficient markets perspective. In their model, the price keeps growing as a result of behavioral traders’ actions, and, at some random moment, the growth rate of fundamental value falls behind that of the price, hence a bubble emerges. Rational traders become aware of this sequentially. The bubble will burst when a certain fraction of traders have sold. A trader’s problem is to decide when to sell the asset. Ideally one wants to sell just prior the crash. But this is difficult. A trader understands that, by selling early, she can make a small profit; by selling late she might be able capture a large price appreciation but also have the risk of getting caught in the crash. She needs to balance between selling at a higher price (riding the bubble) and avoiding getting caught in the crash. A key ingredient is that traders are uncertain of their position in this awareness queue, i.e., a trader does not know how many others became aware ahead of her. Therefore, every trader has a different posterior belief about the time at which the bubble first emerges and the bubble size. This dispersion of belief induces dispersion of exit strategies, which is what allow the bubble to arise and grow.

AB2003 captures the “greater fool” dynamic well. More importantly, they show that it is optimal for rational traders to ride the bubble, which is shown to be consistent with recent empirical studies of stock market data. Temin and Voth (2004) show that a major investor in the South Sea bubble knew that a bubble was in progress and nonetheless invested in the stock and hence was riding the bubble profitably. Brunnermeier and Nagel (2004) and Griffin et al. (2011) both study the Tech bubble in the late 1990s. They show that, instead of correcting the price bubble, hedge funds turned out to be the most aggressive investors. They profited in the upturn, and unloaded their positions before the downturn.

Doblas-Madrid (2012) (henceforth DM2012) constructs a discrete-time version of AB2003 and addresses certain issues that were under criticism in AB2003. In AB2003, the exogenous price increase because of behavioral agents are buying, and only the arbitrageur’s liquidation decision is explicitly modeled. In addition, the price does not respond to rational traders’ sales until a certain threshold is crossed. DM2012 removes the behavioral agents from the model and allows rational traders to buy and sell every period, hence price is (partially) determined in equilibrium and becomes responsive to sales. Two complications arise out of this change. First, when the price is low no one wants to sell so there is no trade. To initiate sales at low prices, traders are randomly drawn and hit by liquidity
shocks and must sell all their shares. Second, to maintain an increasing price path, it is necessary that the money available for traders to purchase the asset increases over time, since the expected aggregate forced liquidation is constant over time. We will further discuss these complications in the extended model. Observing the price, agents update their belief and, when there is a large enough price drop, they liquidate. Hence, the price path dynamic is more realistic.

AB2003, together with DM2012, provide a framework of bubbles that is intuitive and justifiable, both theoretically and empirically. However, why bubbles exist is still not clear. AB2003 explains that the lack of common knowledge prevents traders from perfectly coordinating, and hence the backward induction has no bite and a bubble can exist. This, however, is only a necessary condition for bubbles to exist. Brunnermeier and Morgan (2010), who study a discrete trader version of AB2003, show that such a game can be recast as an auction. But they do not pursue this direction. In particular, they are silent on how the emergence of bubbles is related to auction theory.

In this paper, we will attempt to answer the fundamental question: why bubbles arise? We first show that a simplified version of AB2003 is equivalent to a reverse common value auction. As a result, why bubbles arise is because price fails to reflect the true value of the object in this auction. Then we identify two bid shading incentives in traders’ strategies that generate bubbles. Next, we design a simple tax/subsidy scheme which can reduce the size of bubble. Specifically, we simplify AB2003 by removing all non-essential features, and transform the uncertainty on the dimension of time in AB2003 to the dimension of value/price. That is, in addition to a common prior belief about the asset’s fundamental value, each trader receives a private signal, and their strategies are functions of price only. This transformation abstracts time away from the model and removes the stark assumption of sequential awareness. As a consequence, the intuition of AB2003 becomes clearer and, with price being the only concern for the agents, it is not difficult to show that the model is equivalent to a reverse (procurement) auction with a common value outside option. The auction is one where traders simultaneously submit bids, and the lowest bids win and each receives their individual bid amount, while the rest receive a unknown common value outside option, which corresponds to the fundamental value of the asset.

The results of the benchmark model, as well as models in the recent auction literature, show that the prices\(^4\) do not converge to the true value in these common value auctions\(^5\). It thus becomes clear that why bubbles arise is for the same reason why prices deviate

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\(^4\) The price here is the marginal bid, i.e., the highest winning bid or lowest losing bid.

\(^5\) A detailed review of this literature is postponed to Section 2.2.2.
from the true values in common value auctions. As a trading mechanism, common value auctions fail to aggregate information dispersed in the bidders and bidders overly revise their bids. Equivalently, market often overreacts to shocks because of the inability to reach a consensus among the privately informed investors, and, as a result, each investor optimally rides the bubble. This equivalence might be interpreted as “bad news” for efficient market hypotheses, because it re-confirms that the bubbles and crashes arise from the information asymmetry. Nevertheless, it prompts further studies in this issue from a mechanism design perspective to develop policies that could possibly minimize bubbles and economic fluctuations. It is worth mentioning that three frictions are at work allowing the bubble to arise and grow: 1. limited short selling, 2. lack of common knowledge of the population’s belief, and 3. unobservability of individual’s trade.

We further show that, in this discriminatory price reverse common value auction, two (reverse) bid shading incentives contribute to the rise of bubbles: 1. traders’ efforts to avoid both (reverse) winner’s and loser’s curse and 2. traders’ price-setting incentives (because they get what they bid upon winning) in first-price auctions compared to price-taking behavior in second-price auctions. The result is that the bubble size equals price setting incentive. Compared with previous auction literature, the two incentives in my model can be explicitly separated in the bidding strategies, which may also help explain why price deviate from true value in the discriminatory price common value auctions. In addition, they have opposite response to the change of ratio between “winning slots” and bidders, which can explain different results from several models.

Based on above interpretation, several tax schemes are tested and a surprising result is that that taxing only on pre-crash sales and not otherwise actually makes the bubble size larger. This result puts an alert to the policies that some governments use to subdue the housing prices. With the above lessons learned, we design a simple tax/subsidy scheme that could reduce the size of the bubble by half. Targeting at the uniform price common value auction where there is no bubble, if we subsidize the pre-crash sales and let them all have receive the marginal bid, there will be no bubble. However, the budget is unbalanced. A revised scheme which subsidize early sellers and tax on late sellers, and which results in identical payoff for pre-crash sellers and a balanced budget, reduces the size bubble by half.

In the extended model, we analyze how traders decide to first purchase and then sell, with the hope of riding the bubble and selling before the crash. The price keeps increasing exactly because there are rational traders buying. Traders buy at a low price and wish to

6Bid shading is the practice of a bidder placing a bid below what she believes a good is worth.
sell at a high price before the crash. With the transaction cost, they will no longer seek to buy if the current price is too close to their scheduled sale price, otherwise they will not be able to profit. In the unique equilibrium, a bubble exists when transaction cost is small compared to the dispersion of belief about fundamental value, and bubble size decreases in transaction cost. In addition, a small transaction cost induces a large gap between the the purchase and sale prices. In contrast to AB2003, where an exogenous burst threshold is imposed, the bubble bursts in our model because the price is so high that no one, not even the trader with the highest signal, wants to buy; hence, the price stops increasing, and it becomes common knowledge that the asset is indeed overpriced. When the transaction cost is large compared to the dispersion of beliefs about the fundamental value, there is no bubble. A trader stops buying when the price approaches her estimate of fundamental and sells after uncertainty is resolved. There is no bubble riding.

Both AB2003 and DM2012 require that the price increases exponentially with time so that simple solution forms are possible. Our extended model allows the price to increase in an arbitrary way, because time is not essential in our model. Instead, what matters is the price, and traders' decisions only depend on the price.

The remainder of the paper is organized as follows. Section 2 investigates the origin of bubbles by examining the benchmark model. Section 2.1 introduces the benchmark model and characterizes the equilibrium. Section 2.2.1 shows its equivalence to the common value auction. Section 2.2.2 shows that prices generally do not converge to true value in a common value auctions. Section 2.2.3 clarifies the two reverse bid shading incentives which induce the bubble, Section 2.2.4 devises a simple tax/subsidy scheme to reduce the bubble size, and Section 2.2.5 shows the two incentives respond differently to $\kappa$. In Section 3, we study the extended model. Section 3.1 shows how traders’ strategy spaces are reduced. In Section 3.2, we solve an individual trader’s problem backwards, by first solving a trader’s optimal selling price and then her stop-buy price. The equilibrium is characterized in Section 3.3. Section 3.4 briefly discusses a recession, which is the reverse process of the bubble. Section 4 concludes.

2 A benchmark model: the origin of bubble

2.1 The benchmark model

In this section we present a simple model that is very close to AB2003 and then illustrate its equivalence to a discriminatory price common value auction, which shows that bubbles
occur for the same reason as in the common value auctions, where information aggregation fails and price deviates from true value in the limit. To focus on the intuition the origin of bubbles, I will make the benchmark model as simple as possible. For a detailed explanation and justification of the assumptions, please see AB2003.

Consider a model that parallels AB2003. Specifically, in an environment with continuous time and only one asset, the asset’s fundamental value is $\theta$, which is a random variable and unobservable. There is a unit mass of risk neutral rational traders (henceforth traders), each holding 1 unit of asset at the beginning. Without loss of generality, each trader’s asset position is normalized to $[0, 1]$, so that only limited short selling is allowed. A trader can sell her shares at any time. The price can be publicly observed and is denoted $p(t)$. When $t > 0$, the price starts to increase continuously and deterministically. At any time, when the price rises above $\theta$, we say there is a bubble.

As in AB2003, the backdrop is that the asset price keeps increasing continuously and deterministically, which can be interpreted as behavioral agents buying the asset. These behavioral agents believe that a positive technology shock has permanently raised the productivity and growth rate of this industry, and they simply keep buying this asset, which pushes up the price, as was the case of the tech bubble in the 1990s. The belief of behavioral agents is also confirmed by the fact that rational traders are all holding the asset. However, as will be introduced shortly, each rational trader has a private belief on $\theta$, and they will start to sell one by one when they gradually believe that the price is too high. Initially this sale is disguised by the noise in the price, but when more and more rational traders have sold, its impact on the price will be observed by all behavioral traders, which ultimately subverts their belief, and all of them start to sell. Although there is no price noise in our model, the above story is summarized as a threshold: we assume that when a fraction of $\kappa$ traders has sold, the price stops increasing and jumps instantly to its fundamental value and stays there thereafter, i.e., the bubble bursts. In line with AB2003, we call this an endogenous crash. Note that behavioral agents are not modeled here, and with the threshold assumption, we essentially remove behavioral traders from our model and a rational trader is only concerned about how many other rational traders have sold. An example of the price path is depicted in Figure 1. The bubble size is the gap between the price $p(t)$ and $\theta$ when $p(t) > \theta$. There is an upper bound $\overline{B}$ for the bubble size, and the bubble will also burst when the bubble size is larger than $\overline{B}$. We call this exogenous crash. This is to rule out the possibility that all traders hold the asset forever and never

\[^7\text{In contrast to AB2003, we no longer require that price increases exponentially and no discounting is necessary.}\]
sell. After all, when an asset’s value is more than the GDP of the whole economy, no one, not even behavioral traders, will still believe that the price is a fair reflection of its fundamental value. We are only interested in the endogenous crash, so we assume that $\mathcal{B}$ is large enough so that the bubble will always burst due to threshold $\kappa$.

$\theta$ has density $\phi(\theta)$ distribution over $[0, \infty)$. We restrict to two alternative distributions: an improper uniform distribution on $[0, \infty)$\(^8\) and an exponential distribution with density $\phi(\theta) = \lambda e^{-\lambda \theta}$\(^9\). They both give simple solutions, but we will focus on the uniform case in most of our analysis because of its simplicity. At $t = 0$, each trader receives one, and only one, private signal $s$, which is uniformly distributed on $[\theta - \eta, \theta + \eta]$\(^10\). $s$ can be regarded as a trader’s type. Since $\theta$ is random, a trader is not sure about her signal’s position in $[\theta - \eta, \theta + \eta]$, i.e., a trader does not know how many others’ signals are lower than hers, and how many are higher than hers. This is an important element in the model because this lack of common knowledge prevents agents from perfectly coordinating with each other. In contrast, in the standard finance literature, perfect coordination leads to backward induction, which rules out the possibility of a bubble. Note that in the extended model, we will allow traders to buy first then sell, and the price increases precisely because

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\(^8\)The improper uniform distribution on $[0, \infty)$ has well defined posterior belief when we specify how signals are distributed.

\(^9\)The uniform prior case is adapted from Li and Milne (2012) and the exponential prior case is adapted from Abreu and Brunnermeier (2003)

\(^10\)When $\theta$ is close to zero, some traders will receive negative signals, which has no consequence since traders know that $\theta$ cannot be negative. The implication that some low signal traders’ posterior has smaller support will be addressed shortly.
of rational traders’ purchases.

Let \( \Phi(\theta|s) \) be the posterior CDF about \( \theta \) of a trader with signal \( s \), and \( \phi(\theta|s) \) be the corresponding PDF. In the exponential prior case, \( \Phi(\theta|s) = \frac{e^{\lambda s} - e^{\lambda(s + \frac{\eta}{2} - \theta)}}{e^{\lambda \eta} - 1} \), and in the uniform prior case, \( \Phi(\theta|s) = \frac{\theta - (s - \frac{\eta}{2})}{\eta} \). Given signal \( s \), the support of the posterior is \([s - \frac{\eta}{2}, s + \frac{\eta}{2}]^{11} \). Figure 2 depicts the posterior belief about \( \theta \) for trader \( s \), \( s' \) and \( s'' \). These specifications make all agents’ posterior belief have exactly the same shape, except

![Diagram](image)

Figure 2: Posterior beliefs

a horizontal shift. Given one’s belief, a trader will sell at the price where her marginal benefit and cost of selling at a slightly higher price are equal. Since no one knows how many others are below or above herself, everyone behaves in a similar way. If the chance of being a low type trader is large enough, then, when observing that the price is increasing, it is optimal for everyone to ride the bubble sufficiently high before selling.

We assume that a trader’s purchase and sale cannot be observed by other traders. There is no discounting.

**Corollary 2.1.** A trader will not hold the asset forever.

To simplify the problem and focus on the results and interpretations, we assume the following:

**Assumption 2.1.** A trader uses a trigger strategy: she sells only once, whereby she sells all her shares and will never buy back.

\(^{11}\)When \( s \) is below \( \frac{\eta}{2} \), the support of posterior of trader \( s \) is truncated at 0 because \( \theta \) cannot be below zero. This cause traders in the lower boundary \([\frac{-\eta}{2}, \frac{-\eta}{2} + \eta \kappa]\) behave differently from those above. This will be clarified in Proposition 2.1 where we characterize the equilibrium. The rest of the analysis will ignore this special case.
In this reduced strategy space, a trader need only consider at which price to sell. AB2003 starts with weaker assumptions and derives this trigger strategy as a result. In this benchmark model, we simply assume the trigger strategy. In the extended model, we will start with weak assumptions and derive similar results instead of assuming them.

Assume all other agents use strategy $P^*(s)$, where $s$ is signal. To further simplify the analysis, we have the following assumption.

**Assumption 2.2.** $P^*(\cdot)$ is continuous, strictly increasing and differentiable on $\left(-\frac{\eta}{2} + \eta \kappa, \infty\right)$.

This guarantees that agents with higher signals must sell at higher prices. It also implies that $P^{*-1}(\cdot)$ is well defined.

**Lemma 2.1.** Any equilibrium strategy $P^*(\cdot)$ must be that $P^*(s) - s$ is bounded, $\forall s$.

See Appendix A.1 for proof.

Let burst price be $p_T$, hence $p_T = P^*(\theta - \frac{\eta}{2} + \eta \kappa)$ and $\theta = P^{*-1}(p_T) + \frac{\eta}{2} - \eta \kappa$. Since $\theta$ is a random variable, so is $p_T$. Suppose an agent $i$ decides to sell at price $p$, then if $p < p_T$, she will be able to flee the market before the crash; otherwise, she will get caught. By inverting this relationship, we know that she will get caught in the crash if $\theta \in [s - \frac{\eta}{2}, P^{*-1}(p) + \frac{\eta}{2} - \eta \kappa]$, and she will flee the market successfully before the crash if $\theta \in [P^{*-1}(p) + \frac{\eta}{2} - \eta \kappa, s + \frac{\eta}{2}]$. Therefore, given that all others use strategy $P^*(s)$, the expected payoff for a particular trader $s$ is

$$E[R|s] = \max_p \int_{\theta=s-\frac{\eta}{2}}^{P^{*-1}(p)+\frac{\eta}{2} - \eta \kappa} \theta \phi(\theta|s)d\theta + \int_{P^{*-1}(p)+\frac{\eta}{2} - \eta \kappa}^{s+\frac{\eta}{2}} p \phi(\theta|s)d\theta$$ (1)

The above formulation is a static game. We first solve for the static equilibrium strategy $P^*(s)$. Given that all other use this strategy, it is not difficult to verify that a trader’s problem is well defined (SOC< 0). Then we show that the strategies derived from the static formulation are the same as those from the original dynamic game, hence traders do not change their initial plan even when they update beliefs as price increases. Lastly, we claim, in Proposition 2.1, that this is a unique equilibrium and clarify traders’ strategies at the lower boundary (when $s$ is close to $-\frac{\eta}{2}$).

Differentiating $E[R|s]$ w.r.t $p$, imposing $P^*(s) = p$, and set $\frac{dE[R|s]}{dp} = 0$, we have

$$1 = \left[P^* - (s + \frac{\eta}{2} - \eta \kappa)\right] \frac{1}{P^*} \frac{\phi(s + \frac{\eta}{2} - \eta \kappa|s)}{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa|s)}$$ (2)

See footnote 11
The FOC (Equation (2)) can be interpreted in terms of marginal benefit (MB) and marginal cost (MC). For a trader who evaluates selling at \( p \) vs. \( p + \Delta \), the benefit of selling at \( p + \Delta \) instead of \( p \) is \( \Delta \). The cost is that she could get caught in the crash if that happens in between \( p \) and \( p + \Delta \). This equals loss \( p - \theta \) (due to bubble burst) multiplied by \( \frac{\Phi(P^*-1(p + \Delta) + \frac{\eta}{2} - \eta\kappa|s) - \Phi(P^*-1(p) + \frac{\eta}{2} - \eta\kappa|s)}{1 - \Phi(P^*-1(p) + \frac{\eta}{2} - \eta\kappa|s)} \) (the probability of bursting between \( p \) and \( p + \Delta \)). Dividing both sides by \( \Delta \) and letting \( \Delta \to 0 \), we have \( \text{MB}=1 \) and \( \text{MC}=(p-\theta)\frac{1}{\phi(s + \frac{\eta}{2} - \eta\kappa|s)} \), which is the LHS and RHS of the FOC.

**Proposition 2.1.** There is a unique equilibrium, in which a trader sells at price

\[
\begin{cases}
B, & \text{if her signal } s < -\frac{\eta}{2} + \eta\kappa \\
P^*_s(s) = s + \frac{\eta}{2} - \eta\kappa + B, & \text{if her signal } s \geq -\frac{\eta}{2} + \eta\kappa
\end{cases}
\]

This gives rise to a bubble of the size \( B \), where \( B = \begin{cases} \eta\kappa \text{ in the uniform prior case} \\
\frac{1-e^{-\lambda\eta\kappa}}{\lambda} \text{ in the exponential prior case}
\end{cases} \)

See Appendix A.2 for proof\(^\text{13}\). The equilibrium strategy is depicted in Figure 3.

![Figure 3: Equilibrium strategy](image)

Then we show that the strategies derived from the static formulation are the same as those from the original dynamic game even when they update beliefs as price increases. In

\(^{13}\)The proof of uniqueness requires an additional technical assumption: for those traders in the lower boundary \([-\frac{\eta}{2}, -\frac{\eta}{2} + \eta\kappa] \), when any positive mass of traders sell at the same price and the bubble bursts right at that price, only some of them (random draw) can sell at pre-crash price.
Equation (1) the trader’s belief is $\Phi(\theta|s)$ with support $[s - \frac{\eta}{2}, s + \frac{\eta}{2}]$, which is her initial belief. See Figure 4 panel (a) (where an arbitrary density $\phi(\theta|s)$ is depicted). However, as the price increases, this belief can change. Consider that a trader re-examines her situation at any price $p_e > 0$. Given that all others use equilibrium strategy $P^*(s)$, the bubble will burst at $p_T = P^*(\theta - \frac{\eta}{2} + \eta\kappa)$ and hence $\theta = P^{-1}(p_T) + \frac{\eta}{2} - \eta\kappa$. When the current price $p_e$ is such that $P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa < s - \frac{\eta}{2}$, where $s - \frac{\eta}{2}$ is the lowest possible $\theta$, bubble bursting is impossible for trader $s$, i.e., when $p_e < s - \eta + \eta\kappa$ the bubble will certainly not burst. When price $p_e$ increases such that $p_e > s - \eta + \eta\kappa$, from trader $s$’s point of view, the bubble can burst at any moment. The fact that the bubble has not burst below $p_e$ shrinks the support of $\theta$ from below and hence changes the belief of trader $s$. Specifically, that the bubble has not burst below $p_e$ implies that $\theta$ cannot be below $\theta = P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa$ (i.e., the bubble bursts right at price $p_e$). Hence, the support of $\theta$ is now $[P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa, s + \frac{\eta}{2}]$, and the updated belief about $\theta$ is $\Phi(\theta|s, p_e) = \frac{\Phi(\theta|s) - \Phi(P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa|s)}{1 - \Phi(P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa|s)}$, with corresponding density $\phi(\theta|s, p_e) = \frac{\phi(\theta|s) - \Phi(P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa|s)}{1 - \Phi(P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa|s)}$. See Figure 4 panel (b).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Update of belief}
\end{figure}

The trader $s$’s problem now is

$$E[R|s, p_e] = \max_p \int_{P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa}^{P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa} \theta \phi(\theta|s, p_e) d\theta + \int_{P^{-1}(p_e) + \frac{\eta}{2} - \eta\kappa}^{s + \frac{\eta}{2}} p \phi(\theta|s, p_e) d\theta$$

(3)
Notice that $\phi(\theta|s, p_e)$ differs from $\phi(\theta|s)$ only on its denominator, which is constant (does not include $\theta$). Therefore problem (3) and problem (1) are equivalent and their solutions are identical, and this result does not depend on particular forms of prior distributions of $\theta$ or signal distributions.

This equivalence is similar to the equivalence between first-price sealed bid auction and Dutch auction. In the Dutch auction, a bidder has a initial posterior belief about the highest signal among others. When price is decreasing and no one has claimed the object yet, the upper bound of this belief also decreases. However their biding price does not change under the new belief and, as has been showed many times in the literature, Dutch auction is strategically equivalent to the first-price sealed bid auction. This is because, in Dutch auction, the new belief differs from old one only by a constant denominator, which does not change the optimal choice of bidding price. The difference is that, the new belief in our benchmark model takes the form of $\frac{\phi(\cdot|s)}{1 - \Phi(constant|s)}$, while the new belief in Dutch auction takes the form of $\frac{\phi(\cdot|s)}{\Phi(constant|s)}$. If we further consider the equivalence between a reverse first-price sealed bid auction and reverse Dutch auction, the new belief in the reverse Dutch auction is exactly of the form $\frac{\phi(\cdot|s)}{1 - \Phi(constant|s)}$, and the lower bound of posterior belief is increasing but the hazard rate at the planned selling point does not change.

### 2.2 Why do bubbles exist

As mentioned in the introduction, why bubbles exist is not clear in AB2003. AB2003 explains that the lack of common knowledge prevents traders from perfectly coordinating, and hence the backward induction has no bite and a bubble can exist. However, this is only a necessary condition. Brunnermeier and Morgan (2010), who study a discrete trader version of AB2003, show that such a game can be recast as an auction. But they do not pursue this direction. In particular, they are silent on how the emergence of bubbles is related to auction theory.

In the rest of this section, we will show that the benchmark model is equivalent to a reverse common value auction, and bubbles exist because in this auction the price fails to reflect asset value. We then show that the equilibrium selling price $P^*(s)$ can be decomposed as: The marginal bidder incentive is traders’ effort to avoid both winner’s and loser’s curse; the price setting incentive is traders’ effort to set the price in first-price auc-

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14Reverse auctions will be introduced in Section 2.2.1. Simply put, Reverse auctions are procurement auction, where the auctioneer wants to buy and bidders want to sell. Winners are those who bid the lowest.

15reverse Dutch auction can be thought as an English auction, but with only the increasing price observable. Bidders’ drop outs are unobservable.
\[ P^*(s) = s + \text{marginal bidder incentive} + \text{price setting incentive} \]
\[ = s + \frac{\eta}{2} - \eta \kappa + \frac{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa | s)}{\phi(s + \frac{\eta}{2} - \eta \kappa | s)} \]

tions, compared to the price-taking behavior in second-price auctions. Also, the size of the bubble \( B \) is
\[ B = \text{price setting incentive} = \frac{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa | s)}{\phi(s + \frac{\eta}{2} - \eta \kappa | s)} \]

Therefore, it is the two incentive that generate the bubble. This result sheds light on how to reduce bubble size, which we discussed briefly. In addition, the fact that the two effects have opposite response to a change of \( \kappa \) can explain different results among different models.

### 2.2.1 Relationship to an auction

The benchmark model in Equation (1) is actually a pure common value auction (mineral rights model), except that bidders are continuous instead of discrete. Specifically, this is a reverse discriminatory price (first-price) sealed-bid multi-unit auction with a single unit demand and common value outside option. We will explain the terms step by step. All traders are selling/bidding in this one single auction. See Figure 5. The solid line is the price path. Winners in the auction are those who sell before the crash, whose selling/bidding prices are represented by the thick solid line. The losers are those who are caught in the crash, and the thick dashed line represents their planned selling prices, though never happened since bubble bursts before they have a chance to sell. All traders/bidders are...
involved in this single auction. The fact that there is a fraction $\kappa$ of winners means there is the same mass of continuous identical contracts are up for tender, and each trader/bidder bids to procure only one contract. We say contracts because this is a reverse auction (also called procurement auction), where the auctioneer wants to buy, say, labor or service via a mass of $\kappa$ of contracts, and traders/bidders are selling their labor or service to the auctioneer. Bidders who bid lowest (selling at lowest prices) win, and each winner is awarded a contract and receives contract value equal to their bidding price (selling price). In contrast, in the normal auctions, winner are those who bid highest. It is called a discriminatory price auction because winner pays/gets her own bidding price, which corresponds to the first-price in the single object auction. The winners need not exert any effort in the contract. The losers receive a common value outside option $\theta$. An agent’s beliefs about $\theta$ is the same as what was introduced in the benchmark model. The expected payoff with a signal $s$ is exactly the same as in Equation (1).

To win, a trader necessarily bids low enough to be in the low range. But the lower the bid, the lower the payoff upon winning. Since an agent does exert any effort and either receives $\theta$ or $p$ (her bidding price), so she compares the two alternatives and her bidding strategy only depends on her belief about the common value outside option $\theta$: the lower her belief about $\theta$, the lower she bids. This is why we say it is a common value auction.

The idea is even clearer when we further alter the above auction so that there is no outside option. Instead, winners need to exert an uncertain, common effort $\theta$ to fulfill the contract. Then the expected payoff of trader $s$ changes to

$$E[R|s] = \max_p \int_{p^{-1}(p) + \frac{\eta}{2} - \eta \kappa}^{s + \frac{\eta}{2}} (p - \theta) \phi(\theta|s) d\theta$$

This expected payoff is different from the auction with outside option in Equation (1), but the first order condition turns out to be the same as in Equation (2), so the equilibrium strategy is the same.

AB2003 demonstrates that the lack of common knowledge prevents traders from perfectly coordinating, and hence backward induction has no bite and a bubble can exist. That, however, is only a necessary condition for bubbles to survive, but does not explain why bubbles arise. Still, it is not clear as to why rational traders are willing to wait and sell at prices above fundamental, causing a bubble to arise. AB2003 also indicates that the model is not a global game and strategic complementarity is not at work. Brunnermeier and Morgan (2010) show that such a game can be recast as an auction. However, they did not pursue this direction further. In particular, they are silent on how the emergence of bubbles is related to auction theory. In what follows, we will show that as a trading mech-
anism the above auction, as well as some other common value auctions, fails to aggregate information and prices deviate from the asset values.

2.2.2 A review of common value auction literature: price convergence

Whether this a bubble is equivalent to whether, in the reverse auction, the marginal/pivotal bidding price is higher than or equal to (or lower than) $\theta$ (See Figure 5). The marginal/pivotal bid is the highest winner’s bid, which is also the lowest loser’s bid. We have seen that, in the benchmark model, the marginal bidding price is higher than the fundamental value. This subsection shows that, this result holds in more general settings as demonstrated in the recent auction literature.

That price converges to true value of object in the discreet bidder/objects common value auction is called information aggregation. Theories in common value auction have been used to demonstrate the price convergence and market efficiency. Wilson (1977) and Milgrom (1979) showed that, in a first-price common value single unit auction, the winning bid converges in probability to the value of the object as the number of bidders $n$ becomes large. Milgrom (1981) showed that, in a uniform-price (second-price) common value auction with $k$ identical objects for sale and each bidder only desiring one item, where all winning bidders pay the (same) $k + 1$th bid (which corresponds to the second price in single object auction), if we fix $k$ and let the total number of bidders $n \rightarrow \infty$, then the price (the highest loser’s bid, which is also the $k + 1$th highest bid) also converges to true value $v$. These results demonstrate that, while no one knows for sure the exact value, the auction, a price formation process, aggregates diffuse information through the economy and the price tends to reveal the value. All above results not only require the monotone likelihood ratio property (MLRP), but also require that the likelihood ratio approaches to zero. Let $f(s|\theta)$ be the density distribution of a bidder’s estimate when true value is $\theta$. MLRP requires that, if $\theta_1 < \theta_2$, $\frac{f(s|\theta_1)}{f(s|\theta_2)}$ decreases in $s$, which most distributions satisfy. But the price convergence also requires that $\frac{f(s|\theta_1)}{f(s|\theta_2)} \rightarrow 0$ as $s \rightarrow \bar{s}$, where $\bar{s}$ is the upper bound of the support of $s$.

However, as recent literature has shown, the requirement that the likelihood ratio $\rightarrow 0$ turns out to be too strong, and when it does not hold, the price generally fails to converge to true value in the common value auction. Kremer (2002) shows that both first- and second-price single object common value auctions fail to aggregate information. Jackson and Kremer (2007) show that, in the discriminatory price common value auction with $k$ identical objects for sale and each bidder desiring only one item, the information aggregation also fails and the price does not converge to asset value even when both $k \rightarrow \infty$ and $n \rightarrow \infty$. 17
In particular, they show that the marginal/pivotal bid in the auction is lower than the true value of the asset. The only situation where the price does converge to true value is Pesendorfer and Swinkels (1997), who show that when both $k \to \infty$ and $n \to \infty$, the price converges to true value in the uniform price common value auctions.

To compare the strategies in our benchmark model and standard auctions and gain further insight, it is convenient if we expand and alter the benchmark model (which is a discriminatory price common value auction) to uniform-price and to private value auctions, under the framework of reverse auction with continuous bidders and contracts. Under uniform-price, all winners receive the marginal bidder’s bidding price; in the private value auctions, a winner needs to exert an effort equal her signal $s$, instead of an common value $\theta$. We will use some of the results from these variant models.

These price convergence in normal auctions with both $k \to \infty$ and $n \to \infty$, as well as the benchmark models, are summarized in Table 1.

<table>
<thead>
<tr>
<th>Normal auctions, $k \to \infty$, $n \to \infty$</th>
<th>Benchmark model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform price</td>
<td>Discriminatory price</td>
</tr>
<tr>
<td>Common value</td>
<td>$P \to V$</td>
</tr>
</tbody>
</table>

Table 1: Price convergence

In Jackson and Kremer (2007), the marginal bidding price is lower than true value. Since the benchmark model is a reverse auction, it is natural that the marginal price is higher than the true value in this reverse auction, which is exactly the case in the benchmark model. This conformity shows that bubbles arise from the benchmark model is not due to some peculiar assumptions. It holds in more general cases. Note that in uniform price cases, there is no bubble, which prompts potential opportunities to reduce the bubble.

The equivalence of our benchmark model to the discriminatory price common value auction, and the fact that the marginal bid in this auction does not converge to true value, show that why bubbles arise is for the same reason of why marginal bid does not equal object true value. As a result, to generate bubbles, we do not need such fancy information structures as heterogenous prior beliefs, or explicit higher order beliefs. One as simple as common value auction is enough.

In addition, when private information is dispersed in market participants and no one is sure about the asset fundamental value, the market fails to magically aggregate these
information, and price fails to reflect average belief. Riding the bubble becomes optimal to each investor. Compared to previous literature, this interpretation makes the intuition especially clear that the bubbles and crashes arise from the information asymmetry, which is an intrinsic characteristic of the market. This is a bad news for market efficiency hypothesis, and implies that every time a news arrives, there could be a coordination failure and hence a bubble arises.

In what follows, we first clarify two incentives that shape the bidding strategies, then show how they affect the marginal bid so that it deviates from true value, and hence bubbles arise.

### 2.2.3 Two bid shading incentives

We have seen that the benchmark model, as well as recent auction literature show that the marginal price/bid deviate upward (downward) from the true value in the reverse (normal) discriminatory price common value auctions. One may ask, why is it necessarily a upward deviation in the reverse auctions. Why can’t it be a downward one in the reverse auctions. That is, why is there necessarily a crash in the benchmark model, instead of a jump-up at the threshold $\kappa$? After all, when we defined the “endogenous crash”, we indicated that the price jumps, but the direction was not specified. A more fundamental question is that, when no one is sure about the fundamental value, why is a bubble and a crash created in equilibrium? Why it is impossible for them to sell such that there is no bubble or there is even a negative “bubble” where price jumps up at threshold $\kappa$? There does not seem to be any particular reason in AB2003, or DM2012, or our model that forces the traders to sell high enough that a bubble emerges. The fact that the behavioral traders drive price up is actually not critical to inducing a bubble. There is no clear answer in AB2003 or DM2012 to this question; we will try to answer this question in this subsection by showing that traders have two (reverse) price shading incentives, which can explain why bubble arise and also help explain why marginal bid deviates from true value in the discriminatory price common value auctions.

1. **Marginal bidder incentive**: bidders to avoid both winner’s and loser’s curse, so that the marginal bidder bids exactly at true value. It is defined as bidding by conditioning on being the marginal bidder. This incentive exists in all common value auctions and

---

16To summarize, three frictions are at work allowing the bubble to arise and grow: 1. limited short selling, 2. lack of common knowledge of the population’s belief, and 3. unobservability of individual’s trade.
in common value auctions only.

This incentive is best illustrated in uniform-price common value (normal) auction, where there is no interference of the other incentive. The winner’s curse is that, if a bidder turns out to be the winner, it means that her signal $s$ is one of the highest among all bidders and hence biased upward. Hence, if she have bid naively, it is very likely that she overestimated and overbid. This is more striking when there are $n$ bidders but only one object for auction. Hence sophisticated bidders uniform-price common value auction bid conditioning on winning. The loser’s curse is that, upon losing, a bidder finds that her signal is among the lowest (biased downward) and hence she may have underestimated and underbid. This is more striking when there are $n$ bidders and $n - 1$ objects for auction, in which case, sophisticated bidders would bid conditioning on losing. In general cases where there are $n$ bidders and $k < n$ objects, the bidding strategy in uniform-price common value auction is to bid conditioning on being the marginal bidder. In fact, in both single object and $n - 1$ objects cases, we have always been conditioning on being the marginal bidder.

Bidding conditioning on being the marginal bidder is actually the only symmetric equilibrium bidding strategy in uniform-price common value auctions. It is because, if the marginal bidder’s bid is higher than true value, then all winners will be paying a price higher than true value due to the uniform-price setting. Then all bidder will try not to win, which is no longer an equilibrium. If the marginal bid is lower than true value, then a bidder has incentive to raise her bid to have higher probability of winning, which essentially has no impact on the price she will pay upon winning. Hence this not an equilibrium either. Ex ante, no one knows whether she will be the marginal bidder. Hence in equilibrium, everyone behaves as if she is the marginal bidder. This is why in the normal discrete bidder uniform-price common value auctions, the equilibrium strategy is $E[v|X_1 = s, Y_k = s]$, where $X_1$ is my signal, and $Y_k$ is the $k$th highest signal among all other bidders.

In our uniform-price variant of benchmark model, the bidding/selling strategy is $s + \frac{n}{2} - \eta \kappa$, because conditional on she is the marginal bidder, i.e. her signal is $\theta - \frac{n}{2} + \eta \kappa$, her bid will be exactly the true value $\theta$. Therefore, $\frac{n}{2} - \eta \kappa$ is bidders’ effort to avoid both winner’s and loser’s curse and to bid exactly at $\theta$ if being marginal bidder.

2. **Price setting incentive**: traders to set the price to seize extra surplus in discriminatory price auctions, compared to price-taking behavior in uniform price auctions.
This incentive exists in all discriminatory price auctions and in discriminatory price auctions only.

In uniform price (second-price) auctions, a bidder does not pay what she bids and her bid essentially has no impact on her payment. Hence she behaves like a price-taker. In contrast, in discriminatory price (first-price) auctions, a bidder pays exactly what she bids upon winning. Hence, in normal auctions she has incentive to her bid. It is well-known that, in the first-price private value auctions, if a bidder bids exactly her signal, she always has zero surplus. To extract positive surplus, she shades her bid. In first-price common value auctions, this incentive still exists. But when she lowers her bid, she also lowers her winning probability. So the equilibrium strategy would be to balance these two forces.

In the benchmark model, where the lowest bids win the contracts, this incentive is reversed: upon winning, a bidder in discriminatory price gets paid what she bids, so she has incentive to increase her bid. As demonstrated previously, the price setting incentive only appears in the discriminatory price auctions. In Table 2, the competition effect in the uniform prior case is (positive) $\eta \kappa$, and in the exponential prior case $\frac{1}{\lambda} - e^{-\frac{\lambda}{\eta \kappa}}$. These terms are actually the inverse of the hazard rates in the FOCs (recall that the hazard rates are $\frac{1}{\eta \kappa}$ and $\frac{\lambda}{1 - e^{-\frac{\lambda}{\eta \kappa}}}$ in the uniform prior and exponential prior cases, respectively). If we re-write Equation (2), we have

$$P^* = (s + \eta \kappa) - \eta \kappa + \frac{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa | s)}{\phi(s + \frac{\eta}{2} - \eta \kappa | s)}$$

where we assume $P^* = 1$ and ignore it. Note that the FOCs in the uniform price cases do not have the term $\frac{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa | s)}{\phi(s + \frac{\eta}{2} - \eta \kappa | s)}$. In uniform price reverse auctions, since a bidder does not pay what she bids, she would bid infinitely high if she is guaranteed to win. When she has to consider the possibility of losing, then the situation where she is the marginal winner matters. In this case, $\frac{1 - \Phi(s + \frac{\eta}{2} - \eta \kappa | s)}{\phi(s + \frac{\eta}{2} - \eta \kappa | s)}$ is exactly what she can add to her bid to balance between seizing extra value and not forgoing too much the opportunity of winning. Finally, notice that both $\eta \kappa$ and $\frac{1}{\lambda} - e^{-\frac{\lambda}{\eta \kappa}}$ increase in $\kappa$, which means that in these reverse auctions, higher $\kappa$ relieves competition and leads to higher bids (selling prices).

These two incentives are well-known in auction literature, maybe under different names. But in previous literature they are entangled in the bidding strategies and the definitions are somewhat ambiguous. The benchmark model provides a unique opportunity where we can see them explicitly defined and separated, due to its information structures and
distribution specifications\textsuperscript{17}. The selling/bidding strategies of the four variant benchmark models are listed in Table 2.

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<th>Uniform prior</th>
<th>Exponential prior</th>
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<td></td>
<td>Uniform price</td>
<td>Discriminatory price</td>
</tr>
<tr>
<td>Private value</td>
<td>$s$</td>
<td>$s + \eta \kappa$</td>
</tr>
<tr>
<td>Common value</td>
<td>$s + \frac{\theta}{2} - \eta \kappa$</td>
<td>$s + \frac{\theta}{2} - \eta \kappa + \eta \kappa$</td>
</tr>
</tbody>
</table>

Table 2: Bidding strategies in continuous bidder/object reverse auctions

Table 3 shows the bursting price in the four variant benchmark models. One can observe that the marginal bidder incentive exists in all common value auctions and in common value auctions only, and the price setting incentive exists in all discriminatory price auctions and in discriminatory price auctions only. In particular, in the original

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<tr>
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<td>Discriminatory price</td>
</tr>
<tr>
<td>Common value</td>
<td>$\theta$</td>
<td>$\theta + \eta \kappa$</td>
</tr>
</tbody>
</table>

Table 3: Marginal bids (“bursting prices”) in continuous bidder/object reverse auctions

\textsuperscript{17}The reason why our benchmark model has a simple solution form and always has bubbles is due to the lack of common knowledge adopted from AB2003. Specifically, except at the lower boundary, there is no common knowledge about the lower or upper bound for current population’s signals. In contrast, Jackson and Kremer (2007) assume that signals are always distributed within $[0, 1]$, irrespective of the realization of object value. The object value only affects the shape of the signal distributions. This specification introduces an extra complication: for those whose signals are high enough, they know for sure that they are among the highest types and they will win, hence they have incentive to further shade their bids; symmetrically, for those whose signals are low enough, they have incentive to bid even higher. As a result, bids are somewhat concentrated in the middle. When the realized object value is high, the average transaction price is below the true value; when the object value is very low, the average transaction price is higher than the true value. However, on average, the expected price is lower than the expected value, and in particular, the marginal bid is always lower than the true value.
benchmark model (discriminatory price common value case), the bubble bursts at

\[ p_T = P^*(\theta - \frac{\eta}{2} + \eta \kappa) = (\theta - \frac{\eta}{2} + \eta \kappa) + \text{marginal bidding incentive} + \text{price setting incentive} \]

\[ = \theta - \frac{\eta}{2} + \eta \kappa + \frac{\eta}{2} - \eta \kappa + \frac{1 - \Phi}{\phi} = \theta + \frac{1 - \Phi}{\phi} \]

By definition, the marginal bidding incentive is to ensure the marginal bidder to bid true value \( \theta \), hence the marginal bidding incentive is canceled out by the gap between marginal bidder’s signal and true value \( \theta \). Therefore, the size of the bubble is determined by the price setting incentive alone.

Since price setting incentive is increasing in \( \kappa \) (introduced in next subsection), in both uniform and exponential prior cases, bubble size is increasing in \( \kappa \).

### 2.2.4 Reduce the size of bubble: tax and subsidy

We have already seen that the size of the bubble equals the price setting incentive, which is induced by the fact that everyone gets what she bids upon winning. If somehow we can adjust a trader’s payoff such that she does not get what she bid/sells at, then the bubble size can be altered.

Consider the benchmark model in the uniform prior case, but write the expected payoff for a particular trader \( s \) in a general form:

\[ E[R|s] = \max_p \int_{\theta=s-\frac{\eta}{2}}^{P^*+\frac{\eta}{2}-\eta \kappa} B\phi(\theta|s)d\theta + \int_{P^*+\frac{\eta}{2}-\eta \kappa}^{s+\frac{\eta}{2}} A\phi(\theta|s)d\theta \]

(4)

where \( A \) is the pre-crash payoff, and \( B \) is the post-crash payoff. In the benchmark model \( A = p \) and \( B = \theta \) and the equilibrium strategy is \( P^*(s) = s + \frac{\eta}{2} \), the bubble bursts at \( p_T = \theta + \eta \kappa \) and the bubble size is \( \eta \kappa \).

To gain some insight, we first consider three experiments, which are 1. taxing on all traders; 2. taxing only on winners; 3. taxing only on losers.

1. If we levy a tax of rate \( t \) on the sale revenue of all traders, irrespective of their signals, i.e., irrespective of whether they sell before or after the crash. Then \( A = (1-t)p \) and \( B = (1-t)\theta \). Solving Equation (4), we find that this tax does not change traders’ strategies, because the tax is imposed on everyone and becomes irrelevant, just like a sunk cost. Hence the bubble size is unchanged. This may partially explain why the effect of Tobin tax is controversial and empirical evidences tend to show that Tobin tax does not reduce price volatilities.
2. If we levy the same tax only on pre-crash sales, then $A = (1 - t)p$ and $B = \theta$. Solving the problem we find that a trader’s equilibrium strategy becomes

$$P^*(s) = \frac{s + \frac{\eta}{2}}{1 - t}$$

and the bubble size becomes $\frac{t\theta + \eta \kappa}{1 - t}$, which is larger than the original bubble size $\eta \kappa$. The intuition is that taxing only on pre-crash sales makes winning less attractive and losing less hideous, compared to without the tax. This drives every trader to bid higher or sell at even higher price to avoid this punishment. Hence the bubble becomes larger. This is surprising because usually we impose a tax to reduce the bubble size, but the result from this model shows that, when traders know that there will be no tax or a dramatic tax cut after the crash, a tax on pre-crash sale revenue actually makes the bubble larger.

This counterintuitive result has imperative policy relevance. In some of the major cities in UK, China and Canada, it is suspected that housing bubbles are heating up, and governments are ready to or already intervened. Chancellor George Osborne of British parliament said that from April 2015 he would introduce a capital gains tax on future gains made by non-residents who sell a residential property in the United Kingdom. In 2013 Chinese government introduced a new 20% tax on capital gain from selling residential properties if a family owns multiple properties. In Canada, there are more and more clamour for capital gain tax to suppress the housing price are put forward. The above result warns that policy makers need to have a second thought before implementing such taxes.

3. If we levy the same tax only on post-crash sales, then $A = p$ and $B = (1 - t)\theta$. Solving Equation (4) we find that a trader’s equilibrium strategy becomes

$$P^*(s) = (1 - t)(s + \frac{\eta}{2})$$

and the bubble size becomes $(1 - t)\eta \kappa - t\theta$, which is smaller than the original bubble size $\eta \kappa$. If, instead, we tax on post-crash sales and subsidize the pre-crash sales my further reduce the bubble size. Although the bubble becomes smaller, unless the bubble size is close to zero, taxing on those who are get caught in the crash and are financially distressed may further aggravate the situation and induce more bankruptcies and, since financial institutions are interconnected, more widespread crises. In addition, traders get caught in the crash may not choose to sell the asset right away, so the effect of the tax is somewhat uncertain.
Recall that in the uniform price common value auctions, the price equals the fundamental value in the continuous models. So if we can somehow make that happen, then there will be no bubble. See Figure 6.

![Figure 6: No bubble scheme with unbalanced budget](image)

As shown in Table 2, the equilibrium bidding strategy in this case is $s + \frac{n}{2} - \eta \kappa$. Hence the lowest type sells at $P^*(\theta - \frac{n}{2}) = \theta - \eta \kappa$, and the marginal/pivotal type sells at $\theta$. If we can subsidize these traders who successfully flee the market before the crash so that in the end they all get the marginal bidding price, then in equilibrium the traders will adjust their selling strategies so that their strategies change to the bidding strategies in the uniform price common value auction and there is no bubble. Notice that we are guaranteeing that everyone gets the marginal bidding price, not $\theta$.

However, without tax revenue, this subsidy scheme is not feasible. There is not much we can do about the fact that $\theta$ has the same value to every trader, but it is possible that we can alter the payoff structure so that every trader who sells before the crash has the same payoff. Consider a plan where we subsidize the lower half of these successful traders and tax the higher half, and maintain a balanced budget. Since these successful traders’s signals are uniformly distributed in $[\theta - \frac{n}{2}, \theta - \frac{n}{2} + \eta \kappa]$, we assume that after the tax/subsidy, everyone gets $P^*(\theta - \frac{n}{2} + \frac{\eta \kappa}{2})$. Then $A = P^*(\theta - \frac{n}{2} + \frac{\eta \kappa}{2})$ and $B = \theta$. Solving Equation (4) we have

$$ P^*(s) = s + \frac{n}{2} - \frac{\eta \kappa}{2} $$

and the bubble size is $\frac{\eta \kappa}{2}$, which is half of the original bubble. There is still a bubble because some pre-crash trades are taxed. The bubble becomes smaller because some pre-crash trades are subsidized. Successful traders’ selling prices are uniformly distributed in $[\theta - \frac{\eta \kappa}{2}, \theta + \frac{\eta \kappa}{2}]$. Since eventually everyone gets $\theta$, taxing a trader $s \in [\theta - \frac{\eta \kappa}{2}, \theta - \frac{\eta \kappa}{2} + \eta \kappa]$ a lump sum amount $s + \frac{n}{2} - \frac{\eta \kappa}{2} - \theta$ will balance the budget (a negative amount means subsidizing instead of taxing).
However, the social planner/government does not know $\theta$, but she may estimate a value, implement the tax based on the estimate, and, after the crash and $\theta$ revealed, tax more if $\theta$ is overestimated. Or, before the boom and crash, the government announces that, after the crash with $\theta$ already revealed, it will retrieve transaction records and subsidize early sellers and tax late sellers. In either case, allocate the tax revenue to early sellers so in the end every successful trader receives the same amount. This result justifies the assumption that everyone gets $P^*(\theta - \frac{\eta}{2} + \frac{\kappa}{2})$. See Figure 7. As has shown, the bubble size is halved in this scheme.

The above tax/subsidy scheme is of course very primitive, and there are many practical issues needed to be solved before it can be implemented. But it provides a new perspective to approach the problem.

2.2.5 Opposite response of the two incentives to $\kappa$

Now we show that the two incentives have opposite responses to a change of $\kappa$, which can explain different results from several models. In the benchmark model (a reverse auction),

- Marginal bidder incentive: $\kappa \nearrow \implies$ bid lower
  $\kappa \nearrow \implies$ winner’s curse becomes smaller (or equivalently, loser’s curse becomes larger), because winners’ signals are less biased compared to the population. This prompts the sophisticated bidders not to bid very high to counteract the winner’s curse, and hence bidding prices become lower.

- Price setting incentive: $\kappa \nearrow \implies$ bid higher
  When winning slots are scarce, the competition is more intense, therefore, bidders bid
aggressively (bid lower) and this incentive is thwarted. When \( \kappa \uparrow \Rightarrow \) competition becomes less intense, they can bid higher without losing much winning opportunities.

The above intuition is expanded and summarized in Table 4.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>Reverse auction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \uparrow )</td>
<td>( \Rightarrow ) bid lower</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \Rightarrow ) bid higher</td>
</tr>
</tbody>
</table>

Table 4: Opposite response to change of \( \kappa \)

Depending on the model specification, when \( \kappa \) changes, the response of one incentive may dominate the response of the other, which can explain why several models have different results. In the exponential prior case of our benchmark model, strategy \( P^*_s(s) \) decreases in \( \kappa \). That is, when more traders are allowed to sell before the crash, a trader becomes more cautious and sells earlier. This seemingly counterintuitive result emerges because the marginal bidder incentive dominates the price setting incentive in this case, hence the strategy’s overall response to \( \kappa \) is negative. In the uniform prior case, the strategy does not depend on \( \kappa \), which suggests that the two incentives perfectly offset each other. In Brunnermeier and Morgan (2010), which is a discrete agents version of AB2003, strategies are increasing in \( \kappa \). This is because the payoff in that model is an exponential function of the bidding/selling price, while the effort to fulfill the contract is not. So when \( \kappa \) changes, the price setting incentive has larger coefficient multiplying the inverse hazard rate, hence dominating the marginal bidder incentive.

3 The extended model

Consider the benchmark model with uniform prior belief about \( \theta \), but with the following changes. At \( t = 0 \), the asset’s fundamental value jumps up from \( p_0 \geq 0 \) to an unobservable value \( \theta \) and does not change henceforth. Without loss of generality, we assume \( p_0 = 0 \), which will simplify the notation. There are two types of agents: rational traders (henceforth traders) and a large passive (behavioral) agent. Each trader receives a private signal \( s \), which has the same distribution as in the benchmark model. The measure of traders is
no longer fixed and finite; instead, we assume that there is an infinite mass of traders for each type $s$, and $s$ is continuously distributed on $[\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2}]^{18}$. We will see that, in the extended model, the signal distribution become irrelevant. A trader’s belief about $\theta$ is deduced from the uniform prior belief and and the fact that she has been granted the loan, which will be introduced shortly. At $t = 0$, all of the asset is held by the passive agent while traders have no share. Since initial price $p(0) = 0$, when the fundamental value jumps above 0, every trader wants to buy at this moment. These traders however will not be able to enter the market all at once, because they are financially constrained and are waiting for loans, which will be explained shortly. Assume the asset has a positive supply that is large enough so that when traders want to buy, they are always able to do so. With the fundamental $\theta$ unknown at this moment, the behavioral passive agent has an inverse supply function:

$$p(t) = \alpha(D_r(t))$$

where $\alpha(\cdot)$ is a continuous, strictly increasing function with $\lim_{x \to -\infty} \alpha(x) = \infty$ and $D_r(t)$ is the total shares sold by the passive agent, which is also the traders’ aggregate position at $t$. This supply function indicates that when traders buy, the price is pushed up and the behavioral passive agent is willing to continue to supply the asset at a higher price; conversely when traders sell, the selling pressure depresses the price and the passive agent is willing to buy back at a lower price.

This setting, with some unconventional assumptions (an infinite mass of traders, infinite asset supply, etc.), serves the sole purpose of ensuring that, as traders are buying, the asset price keeps increasing and it is always possible that the price could exceed the fundamental $\theta$. Since the fundamental $\theta$ can be arbitrarily high, a finite asset supply will make a trader with a high enough signal (which is generated by a high enough realization of $\theta$) know for sure that, after traders have bought all of the asset from the passive agent, the price is still lower than $\theta$. Foreseeing this, this trader will never sell the asset, which, on the one hand, is counterfactual in a certain some sense because in the real world, the higher the signal a trader has, the more jittery she is. On the other hand, this will complicate the equilibrium strategies a lot, and we will not have a simple solution. The infinite mass of traders also serves to avoid this, since a fixed measure of traders implies that there will be situations where we know that the asset fundamental is very high, but the price fails to catch up with the fundamental because we are running out of buyers! Again, the assumption $\lim_{x \to -\infty} \alpha(x) = \infty$ has the same goal.

---

18It can be even weakened to allow up to countably infinite point discontinuities on the support $[\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2}]$. But for simplicity, we assume a continuous support.
Similar behavioral asset supplies with explicit function forms have been used in the literature. This behavior may be due to risk aversion or due to institutional frictions that make the risky asset less attractive for the passive agent. The same type of passive agents appears in De Long et al. (1990), where they supply the asset at increasing price when others are buying. In Brunnermeier and Pedersen (2005) and Carlin et al. (2007), speculators try to predate distressed traders when the price falls under liquidation pressure and rises when the pressure is relieved. A similar pricing relationship is also obtained in Pritsker (2005) for the price impact of large traders when institutional investors transact in the market.

If one is not satisfied with the presence of behavioral traders, then, alternatively, we can assume that there is no behavioral agent in this model but only the rational traders, as Doblas-Madrid (2012) does. In this case, the asset is initially held by certain rational traders. To engender initial selling (no one wants to sell at $t = 0$, since the price is zero) and hence ensure that the price increases steadily, assume that all shareholders are subject to random liquidity shocks, under which a trader must sell all her shares. Hence, the price is determined by equilibrium in real time, instead of by an exogenous supply function. However, to make the price increase over time, the money available to buy the asset at each instant must also increase over time, which might be difficult to justify empirically.

The model with behavioral asset supply assumptions exhibits some price inertia, which captures the belief that previous prices do reflect asset fundamentals and should not be overwhelmed by current (temporary or noisy) price fluctuations, so that as long as there are subsequent purchases, the price will keep increasing. The model with an equilibrium price, despite being more appealing in theory, is very sensitive to supply and demand; hence, to support an increasing price path, ever-increasing money injections are required. During a boom, the real-world price dynamics are somewhere in between the two extremes. Demand and supply must play an important role. However, herding, agency problems for fund managers, irrational behavior, etc. are all at work. Hence, the two alternative assumptions can be viewed as simplified mechanisms, such that, as long as there are purchases, the bubble will not burst.

To make traders enter the market sequentially, we assume that purchases are subject to the availability of the loan. That is, traders finance their purchases entirely by borrowing, and loans are granted only over time. As such, the price increases steadily and in a realistic manner. This assumption is, however, not essential in our model, since, without it, traders enter the market all at once, pushing the price up to the peak of the bubble, at which point it plummets back instantly. Though the price path looks unrealistic, the bubble still exists.
In waiting for their loan, at each instant, traders who wish to buy are uniformly randomly chosen and granted the amount of loan they need, irrespective of their types (signals). Notice that this uniformly random choice implies that, given a trader’s signal and the fact that she gets the loan, her posterior belief about $\theta$ must be a uniform distribution. We will clarify this in Section 3.2. Since each trader has a limited capacity and is infinitesimal, we assume that, once chosen, each trader gets just enough of a loan to buy her desired amount of shares. Hence, if a trader wishes to buy, she must wait until she is chosen. If she wants to sell, she can do so right away. Also, all her purchases (if the loan is granted) or sales, i.e., any position change within $[0, 1]$, are done instantly. The stream of loans available is long enough so that everyone who keeps requesting a loan will eventually get one. Denote $v(t)$ the rate of influx of loan at each instant $t$, with $v(t) > 0$, $\forall t \geq 0$, which is exogenous and is public knowledge. The interest rate is zero.

A trader’s purchases and sales cannot be observed by other traders. Since the price is publicly observable, to avoid complex belief updating, we assume that all sales are instantly absorbed by other traders’ purchases and cannot be observed, as long as there are traders who wish to buy. Hence, strategic sales are not reflected in the price path. Specifically, if some traders sell the asset to the passive agent and repay the loan, this repayment instantly becomes a loan available to others, who can buy from the passive trader. This internal loan transfer is not subject to loan control and thus is not part of $v(t)$. These procedures take place instantly, so that these transactions are not reflected in the price. Hence, $v(t)$ is the net injection of loans into the trader population at instant $t$.

In the alternative setting where there is no (behavioral) passive agent and the price is determined in equilibrium in real time, to conceal the very first strategic sale from the price, one must assume that there is an extra amount of money injected to balance the unanticipated extra sale at that moment. This glitch dissuades us from fully embracing this alternative setting.

Since all transactions are executed at the current price prescribed in Equation (5), the price dynamics are then determined by $v(t)$. Since $v(t)$ is known, the entire price path is fully anticipated, until the bubble suddenly bursts. At instant $t$, traders’ aggregate position increment is $\frac{v(t)}{p(t)}$, and the aggregate position is $D_r(t) = \int_{\tau=0}^{t} \frac{v(\tau)}{p(\tau)} d\tau$. With this equation, combined with $p(t) = \alpha(D_r(t))$ and the boundary condition $p(0) = 0$, we can solve for $p(t)$. For example, when $p(t) = \rho D_r(t)$, where $\rho$ is a constant, $p(t) = \sqrt{\frac{2}{\rho}} \int_{\tau=0}^{t} v(\tau) d\tau$. Therefore, price will keep increasing in the predetermined manner, and no one is sure whether a sale has happened until the bubble bursts.

---

\[19\] The assumption is that trading volume is not observable.
There is a fixed transaction cost $c$ each time a trader changes her position, no matter how much is traded. The transaction cost plays a more material role in this model than in AB2003. In AB2003, this cost provides only technical convenience to show that in equilibrium a trader only takes extremal (long and short) positions, but no positions in the middle. In our model, the transaction cost induces a gap between the stop-buy price and the selling price, which turns out to be much larger than the cost itself. This will be explained in detail shortly.

Since sales do not affect price, the bubble is revealed not by accumulated sales, but by the fact that no one wants to buy and the price stops rising. At $t = 0$, $p(0) = 0$ and the fundamental value is above 0. At least some traders want to buy, and the purchase pushes up the price. The price will keep increasing as long as there are traders seeking to buy. If at a certain point, the price becomes so high that no one wants to buy any more, the price stops increasing. Denote this random time $T$ and the random price $p_T$. Given the strategy profile of all types of traders, it turns out that $\theta$ can be perfectly inferred from $p_T$ (this will be explained after Restriction 3.1). If $p_T > \theta$, we say there is a bubble and, since $\theta$ is revealed, the bubble bursts at $p_T$. After that, we assume that the passive agent is willing to buy any shares only at price $p = \theta$, i.e., the price is fixed at $\theta$ after $T$. All traders who still hold the asset have to liquidate at price $\theta$ to passive agent. In the end, no trader holds the asset and all shares go back to the passive agent. Similar to the benchmark model, to rule out the possibility that some types of traders never stop buying so that the bubble grows forever, we assume that there is an upper bound $\bar{B}$ for the bubble size. Once $p(t) - \theta > \bar{B}$, the bubble bursts exogenously. However, we are only interested in the endogenous burst; therefore, we assume that $\bar{B}$ is large enough and we will show that in equilibrium the upper bound is never binding.

Figure 8 shows a simple example of the dynamics. In each panel, traders’ signals are continuously distributed between $\theta - \frac{\eta}{2}$ and $\theta + \frac{\eta}{2}$ along the horizontal axis, given $\theta$, while the vertical axis measures the number of traders (at each signal $s$). The shaded area is the measure of traders who hold the asset. If everyone who has bought holds the same number of shares, the shaded area is proportional to $D_r(t)$. In panel (a) at $t = 0$, everyone wants to buy but none of them have bought any shares. In panel (b), traders are chosen and become shareholders, and there are still traders of all types want to buy. In panel (c), the price is high enough such that some low type traders no longer want to buy but simply hold (the shaded area of “hold” does not rise any more), while the rest are still buying. In panel (d), the price is even higher such that not only does “no-buy” spreads to higher

\[20\] This will be clarified in section 31.
types of traders, but low types start to sell. However, there are still types who wish to buy. Recall that each type has an infinite number of traders waiting to enter the market. In panel (f), the “no-buy” finally reaches the highest type, $\theta + \frac{n}{2}$, who stops buying. As a result, no one buys any more and the price stops rising, which makes the existence of bubble a common knowledge and the bubble bursts. Note that in Figure 8, we assumed a simple strategy, which is that never re-enter the market once one exits. This turns out to be the equilibrium strategy illustrated later.

number of traders

![Diagram of trader behavior](image)

Figure 8: Dynamics of the mass of shareholders

A trader’s profit from trading depends on her purchase prices and sale prices, minus the transaction costs. Upon the burst, no one is buying. However, there maybe sales at the time of the crash. We assume that if there is zero measure of traders selling exactly at the instant when the bubble bursts, they all get the pre-crash price. When a strictly positive measure of traders sell right upon the crash, then only the first randomly chosen orders will be executed at the pre-crash price while the remaining orders are only executed
at the post-crash price.

With the transaction cost incurred each time she changes her position, a trader will buy or sell only a finite number of times. A general form of payoff function in this case, which parallels that in AB2003, is complex and is relegated to Appendix B.2.

3.1 Preliminary analysis

In this section, we define the equilibrium, impose a restriction on traders’ strategies and show that the dynamic game can be simplified to a static-like game by reducing traders’ strategy space.

We use the following notion of equilibrium.

Definition 3.1. A trading equilibrium is defined as a Perfect Bayesian Nash equilibrium in which traders hold the following two (correct) beliefs:

1. Whenever a trader of type $s$ decides not to buy (temporarily or permanently), she (correctly) believes that all traders with signal $< s$ do not want to buy.

2. Each trader of type $s$ with a position less than 1 (correctly) believes that the position of all traders with signal $< s$ are less than 1.

The definition imposes two natural assumptions on traders’ beliefs. Without these assumptions, it is difficult to characterize any equilibrium.

As in AB2003, we show that, at any price, it is optimal for traders to either hold the maximum long position or to hold no asset at all. No trader will hold a position in $(0, 1)$.

Proposition 3.1. A trader either hold the maximum long position or does not hold any shares at all.

The payoff function in Appendix B.2 shows that at any price and any position, a trader’s asset value is linear in her position. In the presence of the transaction cost, if increasing the position is profitable, then it is optimal to increase to the maximum long position; conversely, if selling is profitable, it is optimal to sell all shares. Thus, Proposition 3.1 reduces traders’ position space to $\{0, 1\}$.

At any price, a trader has three optional actions: sell all her shares, hold and not change her current position, or attempt to buy to the maximum long position and wait for the loan. Let $A(p, s)$ denote the strategy of a trader $s$ at price $p$, $A : [0, \infty) \times [0, \infty) \to \{\text{buy, hold, sell}\}$. To clarify, if $A(p, s) = \text{sell}$ for some $p$ and $s$, it means all traders of
type $s$ should sell all shares at price $p$, though some of them do not hold any shares and thus have nothing to sell. If $A(p, s) = \text{buy}$ for some $p$ and $s$, it means all traders of type $s$ should try buying to the maximum long position at price $p$, though some of them already hold the maximum position and thus cannot buy any more and will not be granted any more loans. In short, traders with the same signal use the same strategy, irrespective of their current positions. Now we can formally define bursting price $p_T$:

$$p_T = \inf\{p | A(s, p) \neq \text{buy}, \forall s \in [\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2}]\}$$

where $p$ is price. That is, the bubble bursts when no one wants to buy.

Definition 3.1 immediately implies Corollaries 3.1 and 3.2. Corollary 3.1 states that when trader $s$ decides not to buy, all types higher than her have already decided, or will at that moment decide, not to buy; Corollary 3.2 states that when she is buying, all types lower than her are still buying. When trader $s$ is selling, all types lower than her are selling or have already done so; when she is not selling, all types higher than her are not.

**Corollary 3.1.** $A(p, s) = \text{buy} \implies A(p, s_j) = \text{buy}, \forall s_j \geq s$ and $A(p, s) \neq \text{buy} \implies A(p, s_j) \neq \text{buy}, \forall s_j \leq s$.

**Corollary 3.2.** $A(p, s) \neq \text{sell} \implies A(p, s_j) \neq \text{sell}, \forall s_j \geq s$ and $A(p, s) = \text{sell} \implies A(p, s_j) = \text{sell}, \forall s_j \leq s$.

Let $P^*_b(s)$ denote the lowest price at which trader $s$’s action is to not buy (temporarily or permanently) for the first time. Let $P^*_s(s)$ denote the price at which trader $s$ sells any of her shares for the first time. Because traders have no shares at $t = 0$, if they want to sell, they must first buy. That is, $P^*_b(s) \leq P^*_s(s), \forall s$.

Corollary 3.1 implies that, when the highest type $\theta + \frac{\eta}{2}$ decides to stop buying for the first time, i.e. $p(t) = P^*_b(\theta + \frac{\eta}{2})$, all other types have already done so. Therefore, no one is buying at this moment, which triggers the burst. Let $p^*_T(\theta)$ denote the bursting price of the bubble for a given realization of $\theta$. Then we have $p_T = p^*_T(\theta) = P^*_b(\theta + \frac{\eta}{2})$.

To derive an equilibrium, we impose the following assumption:

**Assumption 3.1.** $P^*_b(s)$ and $P^*_s(s)$ are continuous in $s$.

**Lemma 3.1.** $P^*_b(s)$ is strictly increasing in $s$.

Then the inverse function $P^*_b^{-1}(\cdot)$ is well defined, and $\theta(p_T) = P^*_b^{-1}(p_T) - \frac{\eta}{2}$. This is how everyone can perfectly infer $\theta$ from the bursting price $p_T$. It follows immediately that $p^*_T(\theta)$ is strictly increasing and continuous in $\theta$. 

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Proposition 3.2. (*Trigger-strategy*): In equilibrium, trader $s$ tries to buy the asset to her maximum long position until the price rises to $P^*_b(s)$, then she stops buying. She sells out all her holding, if she has any, when the price reaches $P^*_s(s)$ and never returns to the market.

Proposition 3.2 further reduces a trader’s strategy space to \{$P^*_b(s), P^*_s(s)$\}.

Since trader $s$’s position is either 0 or 1 and uses a trigger strategy, then at the beginning everyone wants to buy, but is financially constrained and some traders are waiting for loans. Once she gets the loan, she will buy to the maximum position 1 and wait for the price to rise to $P^*_s(s)$ and sell all shares and exit the market. If, however, the price rises to $P^*_b(s)$ before she gets the loan, then she no long wants to buy. In either case, she will never re-enter the market. Readers can review the price dynamics under this strategy profile in Figure 8.

Similar to the benchmark model, we have the following results:

**Corollary 3.3.** If all traders use strategy $P^*_b(\cdot)$, then the bubble bursts at $\min\{P^*_b(\theta + \frac{\eta}{2}), \theta + B\}$

**Lemma 3.2.** Any equilibrium strategy $P^*_b(\cdot)$ must be that $P^*_b(s) - s$ is bounded, $\forall s$.

See Appendix B.6 for proof.

### 3.2 Reduced form game

After $T$, the price jumps to $\theta$ and stays there after. If a trader $s$ sells before the crash, she gets the selling price $P^*_s(s)$. Otherwise, she gets the post-crash price $\theta$.

Since the prior belief about $\theta$ is uniform and buyers are uniformly randomly chosen to grant the loan, a trader’s belief about $\theta$, conditioned on her signal $s$ and the fact that she has been chosen, is the uniform distribution between $[\hat{\theta}, s + \frac{\eta}{2}]$, where $\hat{\theta} = \max\{s - \frac{\eta}{2}, P^*_b(p) - \frac{\eta}{2}\}$ and $p$ is the current price. Denote CDF of this belief $\Phi(\theta|s)$ and the corresponding PDF $\phi(\theta|s)$.

Once a trader has bought the asset, the purchase cost is sunk and is irrelevant to the choice of selling price. So we can solve the individual trader’s problem backward, starting with the sale.
3.2.1 Sale stage

We first formulate the expected sale revenue \( E[R|s] \).

\[
E[R|s] = \max_{P_s} \int_{\theta}^{P_s} \left( \theta \phi(\theta|s) d\theta + P_s (1 - \Phi(P_s^{s-1}(P_s) - \frac{\eta}{2}|s)) - c \right)
\]

(6)

Trader \( s \)'s purchase price is sunk and irrelevant now, but other traders' decisions on when to stop buying \((P_b^{s-1}(\cdot))\) matter because they determine the distribution of bursting price \( p_T \). Differentiating \( E[R|s] \) w.r.t \( P_s \) and setting \( \frac{dE[R|s]}{dP_s} = 0 \) gives

\[
\text{FOC: } 1 = \left[ P_s - \theta(P_s) \right] \frac{1}{P_b^{s-1}(P_s)} \frac{\phi(\theta(P_s)|s)}{1 - \Phi(\theta(P_s)|s)}
\]

(7)

where \( \theta(P_s) = P_b^{s-1}(P_s) - \eta \). This FOC has the same marginal benefit-cost interpretation as we discussed in the benchmark model. If \( \text{SOC} < 0 \) and there is an interior solution, the question is well defined and \( \text{FOC} = 0 \) gives the solution \( P_s \). In the proof of Proposition 3.3 we will show that when the transaction cost is small and traders' signals are sufficiently dispersed, \( \text{SOC} \) is indeed negative in equilibrium, so that it is valid to use FOC.

The above static formulation is, again, compatible with the dynamic game. If a trader re-evaluates her choice at any price \( p_e \), given that the bubble has not yet burst, she will find that she has incentive to commit to the initial plan prescribed by the static formulation. The proof is the same as that of Lemma ?? in the benchmark model and hence is omitted.

3.2.2 Purchase stage

Given that all traders sell at \( P_s^*(s) \) and all other traders stop buying at \( P_b^*(s) \), we solve for trader \( s \)'s optimal choice of stop-buy price, \( P_b^*(s) \). Let \( \omega \) denote the profit from purchase and sale. Since the bubble will not burst before \( P_b \) by Corollary 3.1, her expected profit of buying at price \( P_b \) is

\[
E[\omega|s, P_b] = E[R|s, P_b] - (P_b + c)
\]

\[
= \int_{P_b^{s-1}(P_b) - \frac{\eta}{2}}^{P_b^{s-1}(P_b) - \frac{\eta}{2}} \theta \phi(\theta|s, P_b) d\theta + P_b^*(s) \left[ 1 - \Phi(P_b^{s-1}(P_b^*(s)) - \frac{\eta}{2}|s, P_b) \right] - c - (P_b + c)
\]

(8)

If expected profit \( \frac{dE[\omega|s,P_b]}{dP_b} < 0 \) is decreasing in \( P_b \), then as long as \( E[\omega] > 0 \), a trader will try to buy the asset. In this case, a trader \( s \) sets her stop-buy price at \( P_b^*(s) \) defined by

\[
E[\omega|s, P_b] = 0
\]

(9)
We will verify that in equilibrium, \( \frac{dE\omega|s,P_b}{dP_b} \) < 0 in the proof of Proposition 3.3.

Given all other traders stop buying at \( P_b^*(s) \), we know that \( p_T = P_b^*(\theta + \tfrac{\eta}{2}) \). That is, the bursting price is determined by \( P_b^*(s) \) and is not directly determined by sale strategies \( P_s^*(s) \). Equation (8) shows that \( P_b^*(s) \) is determined by \( P_s^*(s) \). Hence, the burst of the bubble is determined by stop-buy price \( P_b^*(s) \), which is in turn determined by \( P_s^*(s) \), therefore \( P_s^*(s) \) indirectly determines the burst. The equilibrium is thus defined by Equations (6) and (9).

### 3.3 Equilibrium

Proposition 3.3 shows that if \( c < \frac{\eta}{4} \), i.e., when transaction cost is small or traders’ signals are sufficiently dispersed, there is a unique trading equilibrium with a bubble, which corresponds to an interior solution of Equation 6. Proposition 3.4 shows that if \( c \geq \frac{\eta}{4} \), the trading equilibrium has no bubble, which corresponds to a corner solution.

#### 3.3.1 Economy with a bubble

**Proposition 3.3.** When \( c < \frac{\eta}{4} \), there exists a unique trading equilibrium in which bubble size is \( B \equiv \eta - 2\sqrt{\epsilon\eta} > 0 \), and the bubble bursts at \( \theta + B \), and a trader \( s \geq \frac{\eta}{2} \) will

\[
\begin{cases}
\text{buy, if price } < P_b^*(s) = s + \frac{\eta}{2} - 2\sqrt{\epsilon\eta}; \\
\text{sell, if price } \geq P_s^*(s) = s + \frac{\eta}{2}; \\
\text{hold, if } P_b^*(s) \leq \text{price } < P_s^*(s).
\end{cases}
\]

For traders with \( s < \frac{\eta}{2} \),

- **if** \( c < \frac{2 - \sqrt{3}}{2\eta} \), a trader

\[
\frac{\eta}{2} - 2\sqrt{\epsilon\eta} < s < \frac{\eta}{2}
\]

\[
\begin{cases}
\text{buy, if price } < P_b^*(s) = H(s); \\
\text{sell, if price } \geq P_s^*(s) = s + \frac{\eta}{2}; \\
\text{hold, if } P_b^*(s) \leq \text{price } < P_s^*(s).
\end{cases}
\]
\( s \leq \frac{\eta}{2} - 2\sqrt{c\eta} \) will buy, if price \( < \eta - 2\sqrt{c\eta} - 2c \);
sell, if price \( \geq \eta - 2\sqrt{c\eta} \);
hold, if \( \eta - 2\sqrt{c\eta} - 2c \leq \) price \( < \eta - 2\sqrt{c\eta} \).

\( \) \[ \]• if \( \frac{2 - \sqrt{3}}{2} \eta \leq c < \frac{\eta}{4} \) a trader
\( \) \[ \]\( \) \[ \]where \( H(s) \equiv \frac{s^2 + s\eta + \frac{5}{4}\eta^2 - 4\eta\sqrt{c\eta} + 2c\eta - 4cs}{2s + \eta} \), and \( S_c \equiv \frac{-\eta + 4c + 2\sqrt{-\eta^2 - 4c\eta + 4c^2 + 4\eta\sqrt{c\eta}}}{2} \).

The equilibrium is depicted in Figure 9.

Figure 9: Equilibrium with bubble
Proposition 3.3 states that when \( c < \frac{\eta}{4} \), a trader \( s > \frac{\eta}{2} \) always stops buying at \( P_b^*(s) = s + \frac{\eta}{2} - 2\sqrt{c\eta} \) and sells at \( P_s^*(s) = s + \frac{\eta}{2} \). Those at the lower boundary behave differently.

When \( c < \frac{2 - \sqrt{3}}{2} \eta \), traders with very low signals \( (s \leq \frac{\eta}{2} - 2\sqrt{c\eta}) \) all stop buying at the same price sell at the same price \( \eta - 2\sqrt{c\eta} \), because they know that the bubble will not burst before trader \( s = \frac{\eta}{2} \) stops buying. Hence, they all sell at \( P_b^*(\frac{\eta}{2}) \) and maintain the minimum gap \( 2c \) between their stop-buy and selling prices. For those whose \( s \in [\frac{\eta}{2} - 2\sqrt{c\eta}, \frac{\eta}{2}] \), it is optimal for them to sell at \( P_s^*(s) = s + \frac{\eta}{2} \), since their purchase costs are sunk and irrelevant. However, it is possible that the bubble bursts before they sell. In addition, since their belief support is truncated, they have gaps between their stop-buy and selling prices that are larger than \( 2c \) but smaller than \( 2\sqrt{c\eta} \).

When \( \frac{2 - \sqrt{3}}{2} \eta \leq c < \frac{\eta}{4} \), traders with very low signals \( (s \leq S_c) \) never buy the asset, because their stop-buy prices \( P_b^*(s) \leq 0 \).

Static analysis gives the following results:

- \( B < \eta \), \( \frac{dB}{dc} < 0 \) and \( \frac{dB}{d\eta} > 0 \)
  The bubble size \( B \) never exceeds \( \eta \), the dispersion of valuation of the asset. But \( B \) is increasing in \( \eta \) and decreasing in \( c \), and hence can be made arbitrarily large if we let \( \eta \rightarrow \infty \) and \( c \rightarrow 0 \). Conversely, if we can effectively reduce the belief dispersion \( \eta \), or increase the transaction cost, the size of bubble will become smaller.

- \( 2c < b < \eta \), \( \frac{db}{dc} > 2 \), and \( 0 < \frac{db}{d\eta} < 1 \), where \( b \equiv 2\sqrt{c\eta} \)
  For a trader whose signal is large enough, her stop-buy price is always \( b \) lower than her selling price, so \( b \) can be regarded as the effective transaction cost. When a trader can buy at some \( P_b \) and sell at some \( P_s \) for sure, \( P_s - P_b > 2c \)
  is a necessary markup for positive profit. When it is possible that the bubble bursts before the trader can sell, this markup must be larger to compensate for the risk, hence \( b > 2c \).
  But larger gap means higher chance of burst in between, which requires selling even earlier. Since bubble size is decreasing as traders’ stop-buy decisions advance, the loss of getting caught in the crash diminishes, and there is a stop-buy price where profit is zero. Therefore a small \( c \) can induce a large \( b \).

When \( c \) is very small, \( b \) is small and traders keep buying until it is almost the price at which they have to sell. This allows the bubble to grow larger before it bursts, hence the bubble size \( B \) is decreasing in \( c \). When \( c \rightarrow 0 \), effective transaction cost \( b \) also converges to zero. \( b \) is increasing in \( c \) at a rate more than twice that of \( c \). \( b \) is also increasing in \( \eta \), but at a lower rate.

\footnote{Recall that purchase and sale each incurs a cost \( c \).}
Figure (10) shows the relationship between $b$ and $c$.

![Figure 10](image_url)  

Figure 10: Relationship between $b$ and $c$ ($\eta = 15$)

Traders keep buying when the price is lower than $P_b^*(s)$, which is exactly when there is no danger of burst, i.e., the price is lower than the bursting price corresponding to the subjective lowest possible $\theta$, which is $s - \frac{\eta}{2}$. They stop buying once the price enters the subjective burst-possible range.

Even with the lowest realization of fundamental $\theta = 0$, where signals are distributed on $[-\frac{\eta}{2}, \frac{\eta}{2}]$, there are traders buying. So there is always a bubble and a crash when $c < \frac{\eta}{4}$. With a $\theta$ that is large enough, upon the burst, traders with signals in the lower range $[\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2} - 2\sqrt{c\eta}]$ have fled the market successfully with positive profit, while the rest (in the higher range $s \in [\theta + \frac{\eta}{2} - 2\sqrt{c\eta}, \theta + \frac{\eta}{2}]$) are caught in the crash and incur a loss.

For a trader $s$, the highest possible realization of $\theta$ is $s + \frac{\eta}{2}$. If price $> s + \frac{\eta}{2}$, then she knows for sure that there is a bubble. Upon the burst, price is $p_T(\theta) = P_b^*(\theta + \eta) = \theta + B$. Hence, upon the burst, traders $s \in [\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2} - 2\sqrt{c\eta}]$ are sure that a bubble already exists. From previous analysis, they are exactly those who have fled the market.

The shape of the price dynamics before burst is determined by the $v(t)$, the rate of loan arrival. There is virtually no restriction on $v(t)$ as long as $v(t) > 0, \forall t$, which ensures that the price is strictly increasing. Hence, our model is flexible enough to replicate a realistic price dynamics. This is in contrast to AB2003, which relies on an exogenous exponential price growth, and DM2012, which relies on an endowment with an exponential growth rate.

### 3.3.2 Economy without a bubble

Proposition 3.4 states that when the transaction cost is large, or equivalently when traders' opinions are sufficiently concentrated, there is a different equilibrium without bubble (which
Proposition 3.4. When $c \geq \frac{\eta}{4}$, there exists a trading equilibrium in which bubble size is $= \frac{\eta}{2} - 2c \leq 0$, and a trader

- $s \leq 2c$ will never buy the asset;
- $s > 2c$ will
  
  \[
  \begin{cases}
  \text{buy, if price} < s - 2c; \\
  \text{hold, if } s - 2c \leq \text{price} < s + \frac{\eta}{2}; \\
  \text{sell, if price} \geq s + \frac{\eta}{2}.
  \end{cases}
  \]

In this equilibrium, there is no bubble riding, and traders stop buying too early, which makes the price stop at a level lower than $\theta$. If a trader sells before the “bubble” bursts, it means that she forgoes the price appreciation that would have occurred had she waited till the “bubble” bursts and the price jumped up to $\theta$. As a result, everyone has incentive after uncertainty is resolved, thus the strategy profile in the previous case is no longer an equilibrium. Given that her expected payoff is her expectation of $\theta$ conditional on her signal, $E[\theta|s]$, she should buy if the price is lower than $E[\theta|s] - 2c$, which gives her a weakly positive expected profit. Notice that when the realized $\theta$ is smaller than $-\eta + 2c$, even the highest type $\theta + \frac{\eta}{2}$ stops buying at price zero, thus no one is buying at all.

3.4 Discussion: overshooting in recession

To some extent, recession can be regarded as the reverse process of the rise of a bubble. When an asset’s fundamental value deteriorates, observing that price is decreasing, rational investors will not buy the asset until they believe that the price has touched the bottom. By doing so, they collectively overact to the recession and there will be a downward price overshoot, which is the reverse of the bubble. This may help explain why some recessions have had a surprisingly huge impact and last for decades.

Consider a model that is same as the bubble model, with the following differences. At $t = 0$, the asset’s fundamental value jumps down from $\theta_0$ to an unobservable value $\theta$. $\theta$ has a distribution over $[\theta_0, \theta]$ with density $\phi(\theta)$. The two types of agents are the same as those in the bubble model, and each rational trader receives the same signal $s$, which has a continuous distribution over a finite support. At $t = 0$, all of the asset is held by rational

\[\text{Assume that the transaction cost is small and traders' signals about the deteriorated asset fundamental value, or economy as a whole, are sufficiently dispersed.}\]
traders and initial price \( p(0) = \theta_0 \). When the fundamental of the asset jumps down, all traders want to sell the asset. But since the passive agent has a limit absorbing rate, at each instant/price only some of the traders successfully sell the asset. As the sale goes on, the price is continuously depressed. Observing that the price keeps falling, a trader has incentive to sell the asset as soon as possible while the price is still high; let the asset depreciate further; and buy back at lower price, ideally at the bottom. Although traders’ positions are still restricted in \([0, 1]\), they have their own funding and do not rely on loans to buy back. That is, they have to wait to sell their shares, but they can buy at any price and are not subject to loan availability. When the price is low enough, a trader \( s \) believes it is the right price and then buys back. Because of the transaction cost, at an earlier moment, a trader with same signal \( s \) who has not been able to sell the asset will no longer seek to sell. When the price is so low that no traders want to sell any more, the price stops decreasing. The fundamental value \( \theta \) is thus perfectly revealed by the fact of no sale, and the price jumps up to \( \theta \).

With this process, it is possible to have V-shaped price dynamics, where the price falls to a low level that is lower than \( \theta \), and then bounces back to \( \theta \). This arises from the fact that when the price is lower than \( \theta \), some low types of traders are still selling due to their uncertainty about \( \theta \). Also, when the price is lower than \( \theta \), some low types of traders who already sold, are reluctant to buy back and are still waiting for a even lower price, which aggravates the recession and postpones the possible recovery.

If we allow \( \theta \) (the lower bound of support of \( \theta \)) to be \(-\infty\), and signal support \([\theta - \eta, \theta]\), and hence for the price to be negative, then the analysis in the bubble model goes through in this recession model, and we have a symmetric result.

4 Conclusion

Through a simplified Abreu and Brunnermeier (2003) model, where rational traders optimally ride the bubble, we show that an individual’s problem is equivalent to a reverse discriminatory price common value auction. A Bubble arises because prices in this auction fail to reflect the asset fundamental. In particular, traders have two incentives to “shade their bids” upwards: 1. to avoid the winner’s curse, where their sale prices could be lower than the asset fundamental (their awarded contract value could be lower than the effort needed to fulfill the contract); 2. the non-vanishing fraction of the ”crash surviving slots”, and consequently the smaller risks, strengthen the incentive to sell at higher prices (the incentive to charge more in discriminatory price vs. uniform price procurement auctions).
Hence, the failure of common value auctions as a trading mechanism to aggregate information dispersed in the population suggests that the bubbles and crashes are the consequences of our inability to coordinate in an environment where private information is dispersed, which prompts further study of this issue from a mechanism design perspective to explore and develop policies that could possibly mitigate the miscoordination and minimize bubbles and economic fluctuations.

In addition, a simple tax/subsidy scheme is devised that could reduce the size of the bubble by half. By subsidizing early sellers and taxing on late sellers, this scheme results in identical payoff for pre-crash sellers and a balanced budget, and reduces the size bubble by half.

In an extended model, we show that, given the uncertainty, traders rationally overact to the fundamental change, collectively and unintentionally created, yet each optimally rides the bubble for a sufficiently large rise of price. We analyze how traders decide to first purchase then sell in the hope of capturing the upturn and unloading before the downturn. When the price is so high that no one wants to buy any more, the price stops increasing and the existence of the bubble becomes commonly known, which triggers the burst. In a unique equilibrium, bubble size decreases in transaction cost and traders maintain a substantial markup between their purchase and sale prices, which is induced by this small transaction cost.

References


Li, Mei, and Frank Milne (2012) ‘The role of a large trader in a dynamic currency attack model.’ *Working Papers*

Appendix A  Benchmark model

A.1  Proof of Lemma 2.1

Recall that $P^*(\cdot)$ is an increasing function. If $\lim_{s \to \infty} P^*(s) - s = \infty$, then $\sqrt{B} > 0$, there exists $\theta$ such that $P^*(\theta - \frac{\eta}{2} + \eta \kappa) - \theta > \sqrt{B}$. Hence $P^*(s)$ cannot be an equilibrium strategy.

A.2  Proof of Proposition 2.1

Step 1: Solve the differential equations

In the uniform prior case, the hazard rate in Equation (2) equals $\frac{1}{\eta \kappa}$. Solving the differential equation gives $P^*(s) = s + \frac{\eta}{2} + Ce^{\frac{1}{\eta \kappa}}$. Since we assumed that $P^*(\cdot) > 0$, then $C$ must be non-negative. If $C > 0$ then $P^*(s) - s$ will grow unboundedly with $s$, which conflits Lemma 2.1. Hence $C$ must be zero. This gives $P^*(s) = s + \frac{\eta}{2}$, which does not depend on $\kappa$. Bubble bursts at price $p_T = P^*(\theta - \frac{\eta}{2} + \eta \kappa) = \theta + \eta \kappa$, and hence the bubble size is $\eta \kappa$. 

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In the exponential prior case, the hazard rate in Equation (2) equals \( \frac{\lambda}{1-e^{-\lambda \eta \kappa}} \). Solve the differential equation, we have \( P^*(s) = s + \frac{\eta}{2} - \eta \kappa + \frac{1-e^{-\lambda \eta \kappa}}{\lambda} + Ce^{\frac{-s}{\lambda}} \), where \( C \) is the constant arising from the differential equation. By the same reason as above, \( C \) must be zero. Then we have \( P^*(s) = s + \frac{\eta}{2} - \eta \kappa + \frac{1-e^{-\lambda \eta \kappa}}{\lambda} \). As a result, the bubble bursts at price \( p_T = P^*(\theta - \frac{\eta}{2} + \eta \kappa) = \theta + \frac{1-e^{-\lambda \eta \kappa}}{\lambda} \), and hence the bubble size is \( \frac{1-e^{-\lambda \eta \kappa}}{\lambda} \).

In both uniform and exponential prior cases, it can be verified that, in equilibrium, the SOC<0 and hence the questions are well defined.

**Step 2: Uniqueness**

We prove the uniqueness in two steps.

1. In any equilibrium, FOC= 0 holds for all traders in \([-\frac{\eta}{2} + \eta \kappa, \infty)\). Assume an (arbitrary) \( P^*_s(\cdot) \) is an equilibrium strategy. Given all others use this strategy, we consider a trader \( s \)'s problem. If she also sells at \( P^*_s(s) \) but her FOC\( \neq 0 \), that means she has a corner solution. Since there is no upper bound for \( P^*_s(s) \), the only possible corner solution would be at the lower boundary. But, as showed earlier, when trader \(-\frac{\eta}{2} + \eta \kappa \) sells at \( P^*_s(-\frac{\eta}{2} + \eta \kappa) \), her FOC= 0, and all traders with higher signal also has FOC= 0.

Let \( P^*(s) \) be the equilibrium strategy for an equilibrium, and \( p^*(s) \equiv P^*(s) - s \). By Lemma 2.1, there exist \( s \in \arg\min p^*(s) \) and \( \bar{s} \in \arg\max p^*(s) \) and suppose \( p^*(\bar{s}) > p^*(s) \).

2. Given FOC= 0, \( \max p^*(s) \) and \( \min p^*(s) \) must coincide. Because \( s \in \arg\min p^*(s) \) and \( P^*(s) = s + p^*(s) \), starting with \( s \), trader above \( s \) are using weakly increasing \( p^*(s) \), or loosely speaking, they wait longer and longer before they sell. By the same logic, trader above \( \bar{s} \) are using weakly decreasing \( p^*(s) \). By continuity and differentiability of \( P^*(\cdot) \), this means \( p^*(\bar{s}) \geq p^*(s) \), and hence \( P^*(\bar{s}) \geq P^*(s) \).

The hazard rate of bursting at \( P^*(s) \) is \( \frac{1}{P^*(s)} \phi_{\bar{s} + \frac{\eta}{2} - \eta \kappa}[\bar{s}] \), and the hazard rate of bursting at \( P^*(\bar{s}) \) is \( \frac{1}{P^*(\bar{s})} \phi_{s + \frac{\eta}{2} - \eta \kappa}[s] \). For both uniform and exponential prior cases, \( \frac{\phi_{\bar{s} + \frac{\eta}{2} - \eta \kappa}[\bar{s}]}{\phi_{s + \frac{\eta}{2} - \eta \kappa}[s]} = \frac{1}{1 - \Phi_{\bar{s} + \frac{\eta}{2} - \eta \kappa}[\bar{s}]} \).

However at \( P^*(s) \), \( \frac{1}{p^*(s)} = \frac{1}{s + p^*(s) - (s + \frac{\eta}{2} - \eta \kappa)} = \frac{1}{s + p^*(s) - (s + \frac{\eta}{2} - \eta \kappa)} = \frac{1}{p^*(\bar{s})} \), with \( \frac{1}{p^*(s)} > \frac{1}{p^*(\bar{s})} \). Hence the FOC in Equation (2) cannot be satisfied for both traders \( s \) and \( \bar{s} \), a contradiction.

**Step 3: Strategies for traders in the lower boundary and their uniqueness**

For a trader with signal \( s \in [-\frac{\eta}{2}, \frac{\eta}{2}] \), the support of her posterior belief about \( \theta \) is \([0, s + \frac{\eta}{2}] \), which is different than \([s - \frac{\eta}{2}, s + \frac{\eta}{2}] \) (the support of a trader with \( s > \frac{\eta}{2} \)). We show that the strategy \( P^*_s(s) = s + \frac{\eta}{2} - \eta \kappa + B \) can still be derived from FOC of the expected payoff. Denote \( \Phi_s(\theta|s) \) the CDF of this posterior belief and \( \phi_s(\theta|s) \) the corresponding p.d.f. In particular, \( \Phi_s(\theta|s) = \frac{\Phi(\theta) - \Phi(0)}{\Phi(s + \frac{\eta}{2}) - \Phi(0)} = \frac{\Phi(\theta)}{\Phi(s + \frac{\eta}{2})} \) and \( \phi_s(\theta|s) = \frac{\phi(\theta)}{\Phi(s + \frac{\eta}{2})} \), then \( \phi_s(\theta|s) = \frac{\phi(\theta)}{\Phi(s + \frac{\eta}{2})} \). Hence the first order condition is exactly the same, which gives the same strategy. But, the bubble cannot burst before trader \( s = -\frac{\eta}{2} + \eta \kappa \) sells (recall that \( P^*_s(\cdot) \) is continuous and strictly increasing). Hence having traders within \([-\frac{\eta}{2}, -\frac{\eta}{2} + \eta \kappa] \) all sell together with trader \(-\frac{\eta}{2} + \eta \kappa \) actually makes the former better off because they are selling at a higher price, without altering the strategy \( P^*_s(s) \) for those in \([-\frac{\eta}{2} + \eta \kappa, \infty) \). Traders within \([-\frac{\eta}{2}, -\frac{\eta}{2} + \eta \kappa + \varepsilon] \) will not sell together at any higher price \( P^*_s(-\frac{\eta}{2} + \eta \kappa + \varepsilon) \), because the trader \( s = -\frac{\eta}{2} \) knows for sure that \( \theta = 0 \) and she will not join the rest. This is because more than \( \eta \kappa \) mass of traders selling together will make her
worse off, so she has incentive to preempt.

Appendix B Extended model

B.1 Anticipated price path \( p(t) \)

From \( D_v(t) = \int_{\tau=0}^{t} \frac{v(\tau)}{p(\tau)} d\tau \) and \( p(t) = \alpha(D_v(t)) \) we have \( \int_{\tau=0}^{t} \frac{v(\tau)}{p(\tau)} d\tau = \alpha^{-1}(p(t)) \). Differentiating w.r.t \( t \) on both sides, we have \( \frac{v(t)}{p(t)} = \frac{dp/dt}{\alpha'(\alpha^{-1}(p))} \Rightarrow v(t)dt = \frac{p}{\alpha'(\alpha^{-1}(p))} dp \). Integrating both sides w.r.t \( t \) and \( p \), respectively, we have \( \int v(t)dt + C = \int \frac{p}{\alpha'(\alpha^{-1}(p))} dp \), where \( C \) is a constant. Let \( G(p) \equiv \int \frac{p}{\alpha'(\alpha^{-1}(p))} dp \), then \( p(t) = G^{-1}(\int v(t)dt + C) \). Combine with boundary condition \( p(0) = 0 \), we can solve for \( p(t) \).

B.2 A general specification of the payoff function

Let \( B(p) = \{ \theta | p_T(\theta) < p \} \), which is the set of \( \theta \) that makes the bubble burst below \( p \). \( B^c(p) \) is its complement, i.e. set of \( \theta \) that makes the bubble burst on and above \( p \).

We know that price is increasing from 0 and will not stop until bubble bursts. Let \( V^1(x^1, p^1|s_i) \) be trader \( s_i \)’s expected value by holding position \( x^1(p^1) \) at price \( p^1 \), till the bubble bursts.

\[
V^1((x^1, p^1)|s_i) = x^1 \int_{\theta \in B^c(p^1)} \theta d\Phi(\theta | s_i, B^c(p^1)) - c
\]

where

\[
\Phi(\theta | s_i, B^c(p^1)) = \frac{\Phi(\theta | s_i) - \Phi(p_{T}^{-1}(p^1)|s_i)}{1 - \Phi(p_{T}^{-1}(p^1)|s_i)} = \frac{\Phi(\theta | s_i) - \Phi(B(p^1)|s_i)}{1 - \Phi(B(p^1)|s_i)}
\]

and \( c \) is the cost of liquidating in the end.

Conditional on \( s_i \) means \( \theta \in \{ s - (1 - a)\eta, s + a\eta \} \), where \( \theta \in \{ p_{T}^{-1}(p^1), s_i + a\eta \} \). Since we have not shown that \( p_{T}(\cdot) \) is strictly monotonic, \( p_{T}^{-1}(\cdot) \) is not defined yet. So \( \Phi(p_{T}^{-1}(p^1)|s_i) \) should be written as \( \Phi(B(p^1)|s_i) \). \( \Phi(B(p^1)|s_i) \) is the probability of bursting below \( p^1 \). \( \Phi(\theta | s_i, B^c(p^1)) \) is trader \( s_i \)’s belief about \( \theta \) at \( p^1 \). Note that trader \( s_i \)’s belief at \( p^1 \) is different from that at the beginning, because of the fact that the bubble has not burst at price \( p^1 \), which suggests that the minimum possible \( \theta \) is no longer \( s_i - (1 - a)\eta \), but \( p_{T}^{-1}(p^1) \).

If there is a different position \( x^2(p^2) \) at price \( p^2(< p^1) \), i.e. the trader sells \( x^2 - x^1 \) at \( p^1 \), then the value at \( p^2 \) is

\[
V^2((x^2, p^2), (x^1, p^1)|s_i) = x^2 \int_{\theta \in B(p^1)\setminus B(p^2)} \theta d\Phi(\theta | s_i, B^c(p^2)) + \left[ 1 - \Phi(B(p^1)|s_i, B^c(p^2)) \right] \left\{ -c + V^1((x^1, p^1)|s_i) + (x^2 - x^1)p^1 - A(p^1|s_i, B^c(p^2)) \right\}
\]

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where
\[ A(p^1|s_i, B^c(p^2)) = (x^2 - x^1) \int_{\theta \in b(p^1)} \alpha_t(\theta)(p^1 - \theta) d\Phi(\theta|s_i, B^c(p^2)) \]
is the loss if bubble bursts right at \( p^1 \).

Proceeding this way, if we start with \( p^K \), position changes \( K - 1 \) times, and the value at the beginning is
\[ -c + V^K((x^K, p^K), \ldots, (x^1, p^1)|s_i) \]
where \( c \) is the transaction cost of initial purchase.

### B.3 Proof of Proposition 3.1

From the general form of payoff function \( V^{k+1}((x^{k+1}, p^{k+1}), (x^k, p^k), \ldots) \), we know that it is linear in \( x^k \). Assume that \( x^k \not\in \{0, 1\} \) and it is optimal. \( V^{k+1} \) must be strictly positive. (The worst case is to hold till burst, then trader gets post crash price \( \theta \)). If \( x^k > x^{k+1} \), it means that even with the transaction cost, buying makes \( V^{k+1} > 0 \), then the trader should buy as much as possible, i.e. \( x^k = 1 \) maximizes \( V^{k+1} \). If \( x^k < x^{k+1} \), it means that even with transaction cost, selling makes \( V^{k+1} > 0 \), then the trader should sell all, i.e. \( x^k = 0 \) maximizes \( V^{k+1} \). The case where \( x^k = x^{k+1} \) means position does not change, then there is no need to put \( x^k \) in \( V^{k+1}(\cdot) \). These contradict \( x^k \not\in \{0, 1\} \). Therefore \( x^k \in \{0, 1\} \).

### B.4 Proof of Lemma 3.1

We first prove the following lemma.

**Lemma B.1.** \( \lim_{s \to \infty} P^*_s(s) = \infty. \)

**Proof.** Assume there exists \( \overline{P} \) such that \( \overline{P} \geq P^*_s(s), \forall s \geq 0 \). For a trader with signal \( s > \overline{P} + \eta \), she knows for sure that the fundamental \( \theta > \overline{P} \). Then selling at price lower than \( \theta \) cannot be an equilibrium strategy. A contradiction. \( \square \)

Note that this lemma does not conflict Lemma 3.2. Lemma 3.2 rules that \( P^*_s(s) = P^*_s(s) - s \) must be bounded.

Recall that \( P^*_b(s) \) is weakly increasing. Suppose that there exists \( \bar{s} < \overline{\pi} \) such that \( P^*_b(\bar{s}) = P^*_b(\overline{\pi}) \). Then the probability of bursting at \( P^*_b(\bar{s}) \) is strictly positive. Since \( P^*_s(s) \) is weakly increasing, continuous and does not have upper bound (by Lemma B.1), there exists \( s \) where \( s \in \{s' : P^*_s(s') = P^*_b(\bar{s}) + \varepsilon\} \). Consider trader \( s \). If \( \varepsilon \) is small enough, the extra benefit, \( \varepsilon \), from selling at \( P^*_b(\bar{s}) + \varepsilon \) compared with selling at \( P^*_b(s) \) will be smaller than the loss from bursting at \( P^*_b(\bar{s}) \). Then trader \( s \) is strictly better off by lowering her selling price from \( P^*_s(s) \) to \( P^*_b(s) \). A contradiction.
B.5 Proof of Proposition 3.2

We first show that the distribution of $p_T$ is atomless.

**Lemma B.2.** Upon selling, agent $s$ believes that the bubble bursts right now with probability zero.

**Proof.** That is, $Pr[p_T = P_s^*(s)|s, B^c(P_s^*(s))] = 0$, i.e. $\Pi(p_T|s, B^c(P_s^*(s))$ is continuous without jump. $\Pi(p_T|s, B^c(P_s^*(s)) = \Phi(p_T^*(\cdot)|s, B^c(P_s^*(s)))$. Since $p_T^*(\cdot)$ is strictly increasing and continuous, then $p_T^{\pi^{-1}}(\cdot)$ is strictly increasing and continuous. Let $\bar{\theta}^\text{supp}(s)$ be the lower bound of the support of trader $s$’s posterior beliefs about $\theta$, upon her selling at $p_s^*(s)$, then

$$\Phi(\theta|s, B^c(P_s^*(s))) = \frac{\Phi(\theta) - \Phi(\bar{\theta}^\text{supp}(s))}{\Phi(s) - \Phi(\bar{\theta}^\text{supp}(s))}$$

The RHS is obviously continuous in $\theta$. Therefore, $Pr[p_T = P_s^*(s)|s, B^c(P_s^*(s))] = 0$. \hfill $\Box$

We show traders use trigger strategy and do not re-enter market by contradiction. Assume that, in equilibrium, trader $s$ sells out at $P_s^*(s)$ and re-enters later. Since transaction cost $> 0$, a trader will sell at $P_s^*(s)$ and re-enter right away and incur the cost only when there is a strictly positive measure of traders selling together at $P_s^*(s)$ so that bubble bursts at price $P_s^*(s)$ with strictly positive probability. From Lemma B.2 we know that the bubble bursts at $P_s^*(s)$ with probability zero. Therefore trader $s$ will not re-enter right away, instead she will wait until trader $s + \varepsilon$ exits and then re-enter first. By Corollary 3.2, trader $s$ cannot return before trader $s + \varepsilon$ does, and trader $s + \varepsilon$ cannot do so before trader $s + 2\varepsilon$ does so. Proceeding this way, trader $s$ stay out of the market until trader $s_T = P_s^*(s_T(\theta + \frac{\eta}{2}))$ exits and return first. Trader $s_T$ sells at $P_s^*(s_T)$ which is exactly the bursting price $P_b^*(\theta + \frac{\eta}{2})$. From trader $s$’s viewpoint, the highest possible price at which the bubble bursts is when $s_T$ sells. Therefore trader $s$ will not return to the market.

B.6 Proof of Lemma 3.2

Recall that $P_b^*(\cdot)$ is an increasing function. If $\lim_{s \to \infty} P_b^*(s) - s = \infty$, then $\forall P > 0$, there exists $\theta$ such that $P_b^*(\theta + \frac{\eta}{2}) - \theta > P$. Hence $P_b^*(s)$ cannot be an equilibrium strategy.

B.7 Proof of Proposition 3.3

**Part 1:** $P_s^*(\cdot)$ and $P_b^*(\cdot)$ define an equilibrium

Suppose that all other traders use strategies prescribed in Proposition 3.3

**Part 1.a: Sale stage**

Then $\theta(P_s) = P_s^{\pi^{-1}}(P_s) - \frac{\eta}{2} = P_s - \theta + 2\sqrt{c\eta}$, and then $\Phi(\theta|s) = \frac{\theta - (s - \frac{\eta}{2})}{\sqrt{c\eta}}$, and $\phi(\theta|s) = \frac{1}{\sqrt{c\eta}}$.

Substitute above expressions in to Equation (7), we have $\frac{dE[R|s]}{dp_s} = \frac{1}{\eta} (-P_s + \frac{\eta}{2} + s)$, and SOC is $\frac{d^2E[R|s]}{dp_s^2} = -1 < 0$. Set $\frac{dE[R|s]}{dp_s} = 0$, we have $P_s^*(s) = s + \frac{\eta}{2}$. Therefore, given all traders stop buying at $P_b^*(s)$, selling at $P_s^*(s) = s + \frac{\eta}{2}$ is an equilibrium.

**Part 1.b: Purchase stage**

The expected profit from buying at price $P_b$ is $E[\omega|s, P_b] = E[R|s, P_b] - (P_b + c)$, where
the belief is $\phi(\theta|s, P_b)$. Given signal $s$, $\theta \in [s - \frac{n}{2}, s + \frac{n}{2}]$. Since $\theta = P_b^{s-1}(p_T) - \frac{n}{2}$, if the bubble bursts at $P^s_b(s)$ then $\theta = P_b^{s-1}(P_b^*(s)) - \frac{n}{2} = s - \frac{n}{2}$. Hence conditional on bubble has not burst below any $P_b \leq P^s_b(s)$ is equivalent to conditional on $s$. So we first calculate optimal stop-buy price $P_b$ with belief $\phi(\theta|s)$, then we verify that this optimal $P_b$ is indeed $\leq P_b^*(s)$, so that our use of belief $\phi(\theta|s)$ is justified. Then we verify that stop buying at any higher price will lead to negative expected profit.

Substitute $P_b^*(\cdot)$, $P_b^{s-1}(\cdot)$ in Equation (8), but replace belief with $\phi(\theta|s)$, we have

$$E[\omega|s] = \int_{P_b^* - \eta \sqrt{C}}^{P_b^* + \eta \sqrt{C}} \theta d\theta + (s + \frac{n}{2}) \left(1 - \frac{s - \frac{n}{2} + \eta \sqrt{C}}{s + \frac{n}{2}} - \frac{s - \frac{n}{2} - \eta \sqrt{C}}{s + \frac{n}{2}} \right) - c - (P_b + c) = 0 \implies P_b(s) = s + \frac{n}{2} - 2\eta \sqrt{C} = P_b^*(s).$$

Now we show that $\frac{dE[\omega|s, P_b]}{dP_b}|_{P_b^*} \leq 0$. When price $P_b > P_b^*(s)$, we have $\theta \in [P_b^{s-1}(P_b) - \frac{n}{2}, s + \frac{n}{2}]$. $\Phi(\theta|s, P_b) = \frac{\theta - P_b^{s-1}(P_b) - \frac{n}{2}}{s + \frac{n}{2} - \frac{n}{2}}$ and $\phi(\theta|s, P_b) = \frac{1}{s - P_b + \frac{n}{2} \eta - \frac{n}{2} \sqrt{C}} \eta 0$ all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$ have incentive to sell at $P_b^*(\frac{n}{2}) = \eta - 2\sqrt{C}$. However $\eta - 2\sqrt{C} - 2c \leq 0$ all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$ cannot burst before trader $s = \frac{n}{2}$ stops buying, hence all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$ have incentive to sell at $P_b^*(\frac{n}{2}) = \eta - 2\sqrt{C}$. Hence $\eta - 2\sqrt{C} - 2c \leq 0$ all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$.

Part 2: Strategies at lower boundary

Part 2a. $\frac{n}{16} \leq c < \frac{n}{2}$

Bubble cannot burst before trader $s = \frac{n}{2}$ stops buying, hence all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$ have incentive to sell at $P_b^*(\frac{n}{2}) = \eta - 2\sqrt{C}$. However $\eta - 2\sqrt{C} - 2c \leq 0$ all traders $s \in [-\frac{n}{2}, \frac{n}{2}]$

Part 2b. $c < \frac{n}{16}$

Part 3: Equilibrium uniqueness

Assume traders use strategies $p_b^*(s)$ and $p_b^*(s)$. By Lemma 3.2, there exist $s \in \argmin p_b^*(s)$ and $s \in \argmax p_b^*(s)$ and suppose $\max p_b^*(s) > \min p_b^*(s)$.

Part 3a: $\max p_b^*(s)$ and $\min p_b^*(s)$ must coincide

Since $P_b^*(s)$ is weakly increasing, continuous and does not have upper bound (by Lemma B.1), there exist $s \in \{s|P_b^*(s) = P_b^*(\hat{s})\}$ and $s \in \{s|P_b^*(s) = P_b^*(\hat{s})\}$. When trader $s$ sells at $P_b^*(\hat{s}) = P_b^*(s)$, trader $\hat{s}$ stops buying. Because $s \in \argmin p_b^*(s)$ and $P_b^*(s) = s + p_b^*(s)$, starting with $s$, trader above $s$ are using weakly increasing $p_b^*(s)$, or loosely speaking, they wait longer and longer before they stop buying. By the same logic, trader above $\hat{s}$ are using weakly decreasing $p_b^*(s)$. Hence the signals of traders who stop buying between price $P_b^*(s)$ and $P_b^*(s) + \Delta$ must have a weakly smaller span than the signals of those who stop buying between price $P_b^*(\hat{s})$ and $P_b^*(\hat{s}) + \Delta$. Since the price change from $P_b^*(\hat{s})$ and $P_b^*(\hat{s}) + \Delta$ covers more types, this change must makes CDF of bursting increase faster. Since $\Pi(P_b^*(s)|s, B^c(P_b^*(s))) = \Pi(P_b^*(s)|\hat{s}, B^c(P_b^*(s))) = 0$, there exists $\Delta$, such that $\forall 0 < \Delta < \Delta$,

$$\Pi(P_b^*(s) + \Delta|s, B^c(P_b^*(s))) \leq \Pi(P_b^*(s) + \Delta|\hat{s}, B^c(P_b^*(s)))$$

This implies that, if $P_b^*(\cdot)$ is differentiable, the right derivative of $\Pi(P_b^*(\cdot) + \Delta|s, B^c(P_b^*(s)))$ is smaller than the right derivative of $\Pi(P_b^*(\cdot) + \Delta|\hat{s}, B^c(P_b^*(\cdot)))$. But even with $P_b^*(\cdot)$ not differentiable, we still have

$$h(P_b^*(\hat{s})|s, B^c(P_b^*(s))) = \pi(P_b^*(\hat{s})|s, B^c(P_b^*(s))) \leq \pi(P_b^*(s)|\hat{s}, B^c(P_b^*(s))) = h(P_b^*(s)|\hat{s}, B^c(P_b^*(s)))$$
However when trader $s$ sells at $P^*_s(s) = P^*_b(s)$, $\frac{1}{P^*_b(s)-\theta} = \frac{1}{s+p^*_b(s)-(s-\eta)} = \frac{1}{p^*_b(s)+\eta}$. When trader $\tilde{s}$ sells at $P^*_s(\tilde{s}) = P^*_b(s)$, $\frac{1}{P^*_b(s)-\theta} = \frac{1}{\tilde{s}+p^*_b(\tilde{s})-(\tilde{s}-\eta)} = \frac{1}{p^*_b(\tilde{s})+\eta}$, with $\frac{1}{\frac{1}{p^*_b(s)+\eta}} < \frac{1}{\frac{1}{p^*_b(\tilde{s})+\eta}}$. Hence the FOC in Equation (7) cannot be satisfied for both traders $s$ and $\tilde{s}$, a contradiction.

B.8 Proof of Proposition 3.4

When everyone else stops buying at price $s - 2c$, bubble bursts at $p_T = \theta + \frac{\eta}{2} - 2c$, which is $< \theta$ when $c \geq \frac{\eta}{4}$. As the bubble size is 0, it is optimal for traders to sell after the burst, i.e. they will try selling as late as possible. The largest possible $\theta$ is $s + \frac{\eta}{2}$, so selling at any price $\geq s + \frac{\eta}{2}$ is justified and is equivalent. We assume that in this case traders sell at $s + \frac{\eta}{2}$. Since the probability of bursting at or after $s$ is zero, they will actually sell after the burst, then everyone simply gets the after burst price $\theta$, which means their expected sale revenue $E[R] = E[\theta|s]$. Then the best response in purchase stage is to stop buying at $P^*_b(s) = E[\theta|s] - 2c = s - 2c$. Then for traders with $s \leq 2c$, their stop-buy price is zero or negative, so they will never buy the asset.