ECON 222
Macroeconomic Theory I
Fall Term 2010/11

Assignment 3 - ANSWER KEY

Question 1: Capital, Population, TFP and Growth Accounting (20 Marks)

From growth accounting we know that output growth in both Countries (M being Madagascar, and P Poland) can be expressed as follows:

$$
\begin{aligned}
\frac{\Delta Y_t^i}{Y_t^i} &= \frac{\Delta A_t^i}{A_t^i} + \alpha_i \frac{\Delta K_t^i}{K_t^i} + (1-\alpha_i)\frac{\Delta N_t^i}{N_t^i}, i = M, P \\
& \quad \textit{or} \\
g_Y &= g_A^i + \alpha_i g_K + (1-\alpha_i) g_N, i = M, P
\end{aligned}
$$

a) In order to get the output growth rates for Madagascar (g_Y^M) and Poland (g_Y^P), we just need to exploit the growth accounting equation and plug in the data:

$$
\begin{aligned}
g_Y^M &= g_A^M + \alpha_M g_K^M + (1-\alpha_M) g_N^M \\
g_Y^M &= 0.02 + 0.3 * 0.01 + 0.7 * 0.01 \\
g_Y^M &= 3\%
\end{aligned}
$$

$$
\begin{aligned}
g_Y^P &= g_A^P + \alpha_P g_K^P + (1-\alpha_P) g_N^P \\
g_Y^P &= 0.01 + 0.4 * 0.02 + 0.6 * 0.02 \\
g_Y^P &= 3\%
\end{aligned}
$$

Which highlights how the same output growth rate can come from very different combinations of inputs growth and technological change.

b) Taking the difference in the growth rates for the two Countries gets:

$$
\begin{aligned}
g_Y^M - g_P^M &= g_A^M - g_A^P + \alpha_P g_K + (1-\alpha_P) g_N - \alpha_M g_K - (1-\alpha_M) g_N \\
0.01 &= g_A^M - g_A^P + \alpha_P g + (1-\alpha_P) g - \alpha_M g - (1-\alpha_M) g \\
0.01 &= g_A^M - g_A^P + g - g = g_A^M - g_A^P
\end{aligned}
$$

The two *TFP* growth rates differed by one percentage point, with Madagascar showing a faster growth in its TFP.

Question 2: Quasi-Solow Growth (30 Marks)

a) Differently from the Neoclassical production function, here there are no diminishing returns to capital and labor.

In order to get total output Y_t as a function of aggregate capital K_t and aggregate labor N_t we simply have to multiply the definition of the per capita production function by the size of the labor force:

$$Y_t = y_t N_t = k_t^2 N_t = \left(\frac{K_t}{N_t}\right)^2 N_t = K_t^2 \frac{1}{N_t}$$

b) The condition for a steady-state is the usual one (*Savings=Investment*), which leads to:

$$sf\left(k^*\right) = (n+d)\,k^*$$

The steady-state capital-labor ratio k^* is equal to:

$$\begin{aligned} s\left(k^*\right)^2 &= (n+d)\,k^* \\ sk^* &= n+d \\ k^* &= \frac{n+d}{s} \end{aligned}$$

The last formula shows that the capital-labor ratio is going to be higher for economies with high fertility and depreciation rates, and lower for economies with a high saving rate. This seems to be counter-intuitive: the lower the saving rate, the higher the capital stock. This is why we always consider production functions that are concave.

In order to get y^*, c^* and i^* we simply substitute the value of k^* into the definition of output per worker, consumption per worker and investment per worker. This gets:

$$\begin{aligned} y^* &= \left(k^*\right)^2 = \left(\frac{n+d}{s}\right)^2 \\ c^* &= (1-s)\,y^* = (1-s)\left(\frac{n+d}{s}\right)^2 \\ i^* &= sy^* = s\left(\frac{n+d}{s}\right)^2 \end{aligned}$$

c) The condition for a steady-state is the same:

$$sf\left(k^*_{new}\right) = (n+d)\,k^*_{new}$$

However, the steady-state capital-labor ratio k^*_{new} is now equal to:

$$\begin{aligned} s\left(k^*_{new}\right)^{\frac{1}{2}} &= (n+d)\,k^*_{new} \\ s &= (n+d)\left(k^*_{new}\right)^{\frac{1}{2}} \\ k^*_{new} &= \left(\frac{s}{n+d}\right)^2 \end{aligned}$$

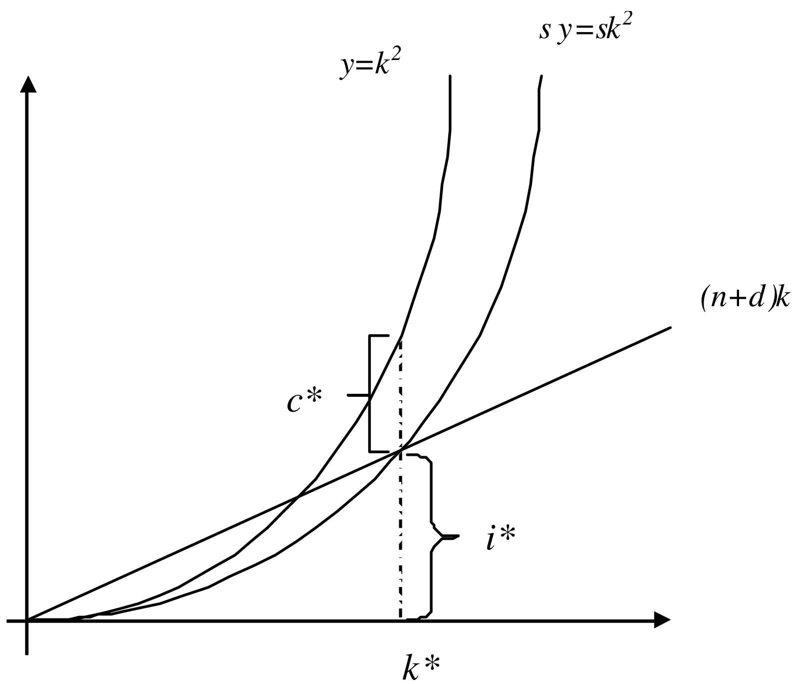
The last formula shows that the capital-labor ratio is going to be higher for economies with a high saving rate, and lower for economies with high fertility and depreciation rates. This case seems to be more plausible.

Notice that $k^* = [k^*_{new}]^{-2}$, because $\alpha = [\alpha_{new}]^{-2}$.

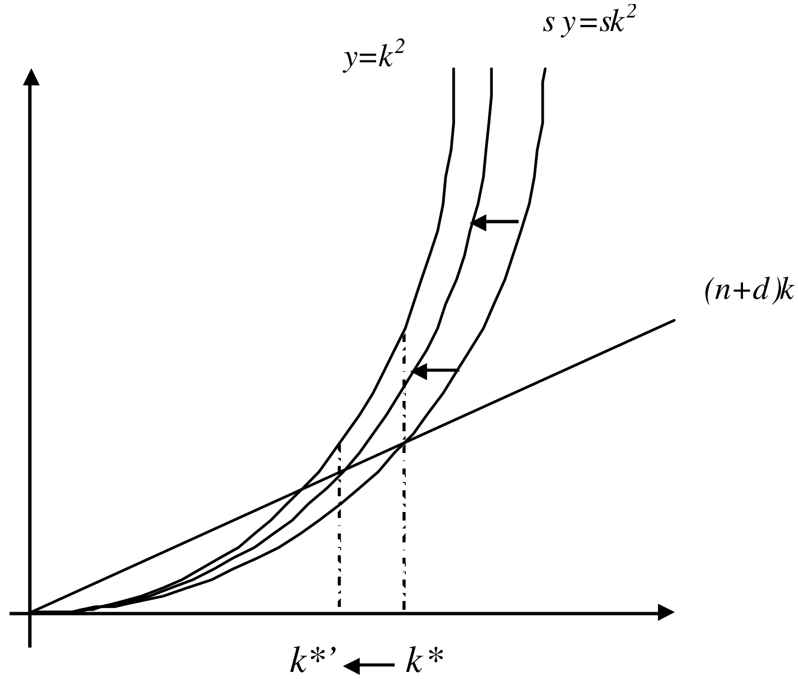
The corresponding values of y^*, c^* and i^* are now:

$$\begin{aligned}
y^*_{new} &= (k^*_{new})^{\frac{1}{2}} = \left[\left(\frac{s}{n+d}\right)^2\right]^{\frac{1}{2}} = \frac{s}{n+d} \\
c^*_{new} &= (1-s)\, y^*_{new} = \frac{(1-s)\, s}{n+d} \\
i^*_{new} &= s y^*_{new} = s\left(\frac{s}{n+d}\right) = \frac{s^2}{n+d}
\end{aligned}$$

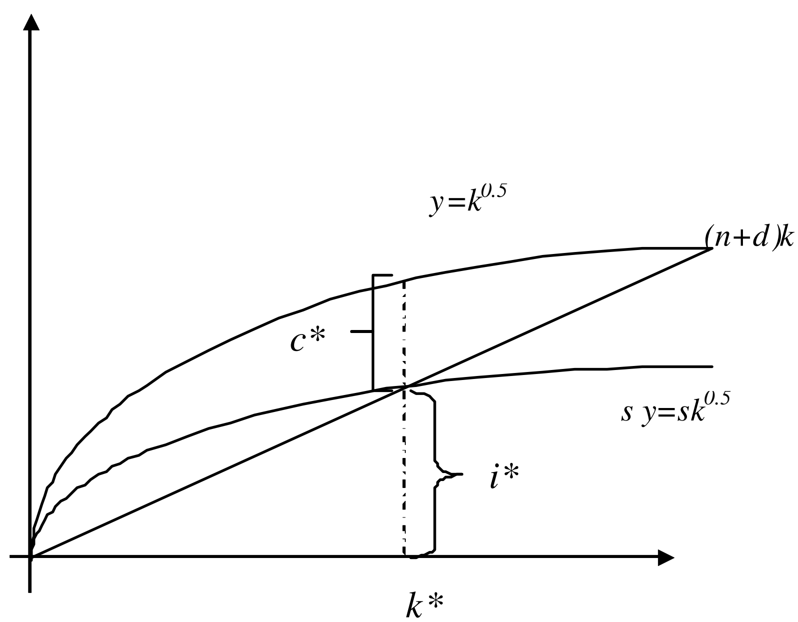
d) The graphical representation of the steady-state for $\alpha = 2$ is as follows:



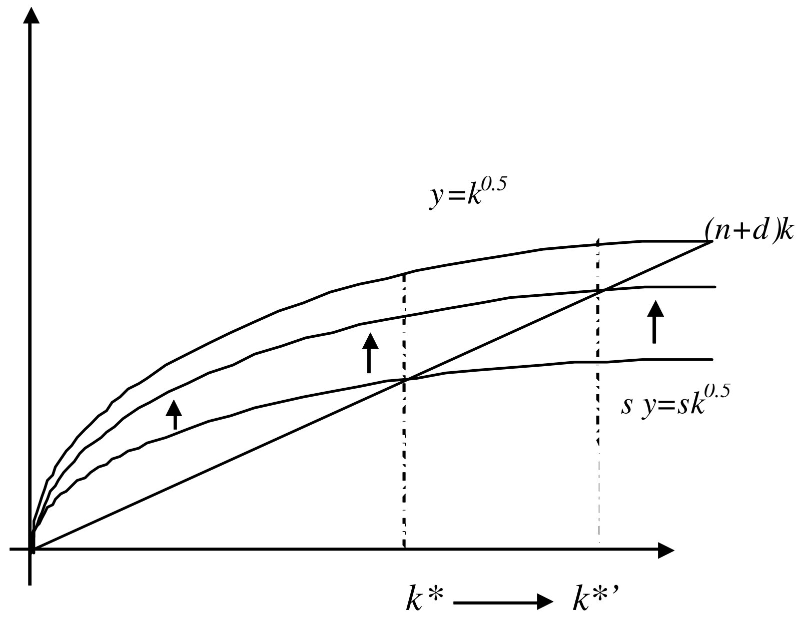
If s increases to s' there is a long run effect on the growth rate of output per-capita, and the economy does not converge to the new steady-state $k^{*\prime}$, which would imply a lower capital and output per-capita. The economy does have long run growth in per-capita terms. Graphically:



The graphical representation of the steady-state for $\alpha = \frac{1}{2}$ is as follows:



If s increases to s' there is a no long run effect on the growth rate of output per-capita, and the economy does converge to the new steady-state $k^{*\prime}$, which implies a higher capital and output per-capita. The economy does not have long run growth in per-capita terms, though. Graphically:



e) For $\alpha = \frac{1}{2}$ the economy is stable. If some events were to move it away from k^* there would forces that would lead it back to k^*. The dynamics of capital are such that we would see capital decreasing if it were to start above k^*, while it would increase if it were to start below k^*. Eventually, the economy ceases to grow. For the case $\alpha = 2$ the economy is unstable. Any event moving it away from k^* has drastic effects. If capital starts from below k^* the economy shrinks over time, until it vanishes, when it reaches a capital per capita equal to zero. If capital starts from above k^* the economy expands over time, never ceasing to grow.

Question 3: Fertility and Growth (25 Marks)

a) The steady state condition for this economy is similar to the "textbook" one we relied on in class. We just have to understand that now the investment per worker is different and that the production function is linear in k. The fertility rate depends on the capital stock, so the steps to get the steady state condition become the following:

$$\begin{aligned}
K_{t+1} &= (1-d)K_t + I_t \\
K_{t+1} &= (1-d)K_t + sY_t \\
K_{t+1} &= K_t + sY_t \\
\frac{K_{t+1}}{N_{t+1}} &= \frac{K_t}{N_t}\frac{N_t}{N_{t+1}} + s\frac{Y_t}{N_t}\frac{N_t}{N_{t+1}} \\
k_{t+1} &= k_t\frac{N_t}{N_{t+1}} + sAk_t\frac{N_t}{N_{t+1}}
\end{aligned}$$

The population evolves according to:

$$\begin{aligned}
N_{t+1} &= (1+n(k_t))N_t \\
\frac{N_t}{N_{t+1}} &= \frac{1}{1+nk_t}
\end{aligned}$$

Substituting the last result in the law of motion of capital per capita gets:

$$\begin{aligned}
k_{t+1} &= \frac{k_t}{1+nk_t} + \frac{sAk_t}{1+nk_t} \\
k_{t+1}(1+nk_t) &= k_t + sAk_t \\
k_{t+1}(1+nk_t) - k_t &= sAk_t
\end{aligned}$$

In the steady-state capital does not change over time, so $k_{t+1} = k_t = k^*$, and:

$$\begin{aligned}
sAk^* &= k^*(1+nk^*) - k^* \\
sAk^* &= k^* + n(k^*)^2 - k^* \\
sAk^* &= n(k^*)^2
\end{aligned}$$

The expression for the steady-state capital-labor ratio k^* is obtained by solving the steady state condition for k^*:

$$\begin{aligned}
sAk^* &= n(k^*)^2 \\
sA &= nk^* \\
k^* &= \frac{sA}{n}
\end{aligned}$$

b) The requested plot is as follows.

The curves representing output per worker and savings per worker are similar to the standard case: they are increasing, but now they are just straight lines. Another new feature here is represented by the investment per worker, which is no longer a straight line but a parabola, that is it's increasing and convex. As usual, the steady state is obtained when the savings per worker are equal to the investment per worker.

c) If there is a decrease in the fertility parameter, the schedule representing the investment per capita is decreasing its slope for every possible value of k. This leads to an increase in the steady-state capital-labor ratio $\underline{k^*}$. For some time the growth rate of output per worker is going to be positive, unlike in the long run where the savings are just enough to offset the depreciation of capital.

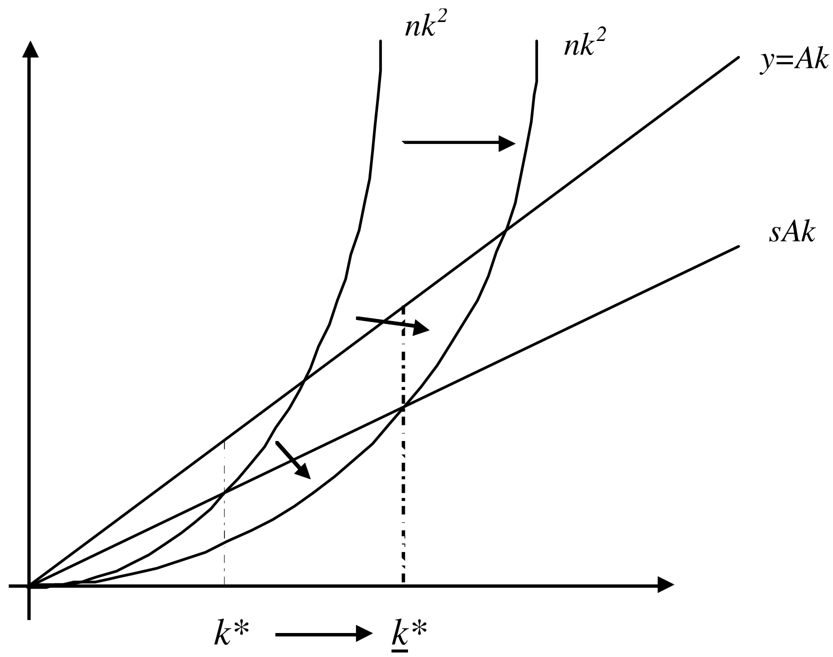


Figure 1:

Question 4: Savings, Investment, and Current Accounts (25 Marks)

a) Imposing the equilibrium condition gets:

$$\begin{aligned} S^d(r) &= I^d(r) \\ 325 + 500r &= 410 - 500r \\ r^* &= 0.085 \end{aligned}$$

An interest rate equal to 8.5% clears the goods market: r^* is such that the savings and investments are exactly identical (and equal to $S(r^*) = I(r^*) = 367.5$). The graph is trivial.

b) We have to follow the same procedure as above for each Country separately. However, there is an additional step involved as we have to obtain the desired savings first. For Canada we obtain:

$$\begin{aligned} S^d_{Ca}(r) &= Y_{Ca} - C^d_{Ca}(r) - G_{Ca} \\ &= Y_{Ca} - (320 + c_{Ca}(Y_{Ca} - T_{Ca}) - 200r) - G_{Ca} \\ &= 1000 - (320 + 0.4(1000 - 200) - 200r) - 275 \\ &= 85 + 200r \end{aligned}$$

Since the economy is closed, we rely on the same equilibrium condition as in part a):

$$
\begin{aligned}
S^d_{Ca}(r) &= I^d_{Ca}(r) \\
85+200r &= 150-200r \\
r^*_{Ca} &= 0.1625 \\
S_{Ca}(r^*_{Ca}) &= I_{Ca}(r^*_{Ca}) = 117.5 \\
C_{Ca}(r^*_{Ca}) &= 320+0.4(1000-200)-200*0.1625 = 607.5
\end{aligned}
$$

Similarly, the desired savings for the US are:

$$
\begin{aligned}
S^d_{US}(r) &= Y_{US}-C^d_{US}(r)-G_{US} \\
&= Y_{US}-(480+c_{US}(Y_{US}-T_{US})-300r)-G_{US} \\
&= 1500-(480+0.4(1500-300)-300r)-300 \\
&= 240+300r
\end{aligned}
$$

And the interest rate the clears the goods market is such that:

$$
\begin{aligned}
S^d_{US}(r) &= I^d_{US}(r) \\
240+300r &= 260-300r \\
r^*_{US} &= 0.0\overline{3} \\
S_{US}(r^*_{US}) &= I_{US}(r^*_{US}) = 250 \\
C_{US}(r^*_{US}) &= 480+0.4(1500-300)-300*0.0\overline{3} = 950
\end{aligned}
$$

An interest rate equal to 16.25% ($3.\overline{3}$%) clears the goods market in Canada (US). Quite predictably, investments and consumption are higher in the US, given their higher income.

c) We have to compute the current accounts for each Country.
For Canada:

$$
\begin{aligned}
CA_{Ca}(r) &= NX_{Ca}(r) = S^d_{Ca}(r)-I^d_{Ca}(r) \\
&= 85+200r-150+200r = -65+400r \\
&\quad \textit{or alternatively} \\
CA_{Ca}(r) &= Y_{Ca}-\left(C^d_{Ca}(r)+I^d_{Ca}(r)+G_{Ca}\right) \\
&= 1000-(320+0.4(1000-200)-200r+150-200r+275) \\
&= -65+400r
\end{aligned}
$$

For the US:

$$
\begin{aligned}
CA_{US}(r) &= NX_{US}(r) = S^d_{US}(r)-I^d_{US}(r) \\
&= 240+300r-260+300r = -20+600r \\
&\quad \textit{or alternatively} \\
CA_{US}(r) &= Y_{US}-\left(C^d_{US}(r)+I^d_{US}(r)+G_{US}\right) \\
&= 1500-(480+0.4(1500-300)-300r+260-300r+300) \\
&= -20+600r
\end{aligned}
$$

Imposing the equilibrium condition for the case of two large economies gets:

$$
\begin{aligned}
CA_{Ca}(r) &= -CA_{US}(r) \\
-65+400r &= 20-600r \\
r_W^* &= 0.085 \\
CA_{Ca}(r_W^*) &= -CA_{US}(r_W^*) = -31 \\
C_{Ca}(r_W^*) &= 623 \\
C_{US}(r_W^*) &= 934.5
\end{aligned}
$$

The world interest rate r_W^* is in between its two closed economy counterparts, that is $r_{US}^* < r_W^* < r_{Ca}^*$. The reason is simple. When the Countries open their borders and start having access to the international market for borrowing and lending, Canadian firms can use the cheaper American funds to finance their investment. At the same time, American savers want to lend their money to Canadian firms, because they are willing to pay a higher rate of return. These forces decrease the interest rate for Canada and increase it for the US, up the point where there are no longer "arbitrage" opportunities, or until $r_{US,new}^* = r_W^* = r_{Ca,new}^*$.

The decrease in the interest rate allows Canadians to increase their consumption, while the opposite happens to the Americans.

d) The world desired saving and investment are obtained by simply adding up the desired saving and investment in the two Countries (remember that graphically we are adding *horizontally* the S^d and I^d schedules).

$$
\begin{aligned}
S_{Ca}^d(r) + S_{US}^d(r) &= 85+200r+240+300r \\
S_W^d(r) &= 325+500r
\end{aligned}
$$

$$
\begin{aligned}
I_{Ca}^d(r) + I_{US}^d(r) &= 150-200r+260-300r \\
I_W^d(r) &= 410-500r
\end{aligned}
$$

$S_W^d(r)$ and $I_W^d(r)$ correspond to the ones we used in part a). This tells us that in our framework, in order to get the world interest rate when there are only two large economies, it is equivalent to consider the desired saving and investment of one big economy (the world economy) or to consider the current accounts of the two large economies. Formally, it's easy to show that you can move from one definition to the other, when we are in equilibrium:

$$
\begin{aligned}
S_W^d(r) &= S_{Ca}^d(r) + S_{US}^d(r) \\
I_W^d(r) &= I_{Ca}^d(r) + I_{US}^d(r)
\end{aligned}
$$

so that at the equilibrium:

$$
\begin{aligned}
S_W^d(r) &= I_W^d(r) \\
S_{Ca}^d(r) + S_{US}^d(r) &= I_{Ca}^d(r) + I_{US}^d(r) \\
S_{Ca}^d(r) - I_{Ca}^d(r) &= -\left(S_{US}^d(r) - I_{US}^d(r)\right) \\
CA_{Ca}(r) &= -CA_{US}(r)
\end{aligned}
$$

This result is mainly due to the absence of frictions and imperfections in the international transactions.