ECON 222
Macroeconomic Theory I
Winter Term 2009/10

Assignment 3 - ANSWER KEY

Question 1:

a) We know that:

$$\begin{aligned} Y_{08} &= A_{08} K_{08}^{0.5} N_{08}^{0.5} \\ 2000 &= A_{08} 1600^{0.5} 100^{0.5} \\ A_{08} &= \frac{2000}{400} = 5 \end{aligned}$$

and that

$$\begin{aligned} Y_{09} &= A_{09} K_{09}^{0.5} N_{09}^{0.5} \\ 2200 &= A_{09} 1681^{0.5} 121^{0.5} \\ A_{09} &= \frac{2200}{451} = 4.88 \end{aligned}$$

TFP had a negative growth rate, since it decreased by 2.4%:

$$\frac{A_{09} - A_{08}}{A_{08}} = \frac{4.88 - 5}{5} = -0.024$$

If you were to use the growth accounting equation you would get a slightly different answer:

$$\begin{aligned} \frac{\Delta Y}{Y} &= \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1 - \alpha) \frac{\Delta N}{N} \\ \frac{\Delta A}{A} &= \frac{\Delta Y}{Y} - \alpha \frac{\Delta K}{K} - (1 - \alpha) \frac{\Delta N}{N} \\ \frac{\Delta A}{A} &= 0.1 - 0.5 * 0.0506 - (1 - 0.5) * 0.21 = -0.030315 \end{aligned}$$

According to this method, TFP still had a negative growth rate, but it decreased by 3.03%.

For most applications the two methods are going to deliver asnwers that are very close to each other. However, it is worth stressing that the growth accounting equation is valid only as an approximation. Namely, it's going to be accurate only when dealing with small growth rates.

b) We are looking for the level of capital K_{10} which solves the following equation:

$$\begin{aligned} Y_{10} &= A_{10} K_{10}^{0.5} N_{10}^{0.5} = 4.88 K_{10}^{0.5} 144^{0.5} \\ 2300 &= 58.56 K_{10}^{0.5} 144_{10}^{0.5} \\ K_{10}^{0.5} &= \frac{2300}{58.56} \rightarrow K_{10} = \left[\frac{2300}{58.56} \right]^2 = 1542.60 \end{aligned}$$

Notice that the we need to decrease the capital stock to get the "target" production.

c) We know that:

$$\begin{aligned}
\frac{\Delta Y}{Y} &= \frac{\Delta A}{A} + \alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta N}{N} \\
g_Y &= g_A + \alpha g_K + (1-\alpha)\, g_N \\
0.04 &= 0.01 + \alpha * 0.03 + (1-\alpha) * 0.03
\end{aligned}$$

In this particular economy, $g_K = g_N = 0.03$, hence we cannot say anything about the value of α.

Question 2:

To write the function in intensive form, divide each side by N. Then:

$$\begin{aligned}
\frac{Y}{N} &= \frac{AK^{\alpha}N^{1-\alpha}}{N} = AK^{\alpha}N^{-\alpha} = A\frac{K^{\alpha}}{N^{\alpha}} = A\left(\frac{K}{N}\right)^{\alpha} \\
y &= Ak^{\alpha}
\end{aligned}$$

Start by noting that in the steady state all the variables of interest are growing at a constant rate n. This in turn implies that the various per capita measures are constant. For the capital output ratio to be constant, investment must grow by enough to supply each new worker with a unit of capital as well as to take care of depreciation of the existing capital stock. Then:

$$I = (n+d)K$$

In equilibrium, saving must equal investment and in this case saving is a constant fraction of output: $S = sAK^{\alpha}N^{1-\alpha}$. Setting $S = I$ and writing everything in per capita terms yields:

$$\begin{aligned}
\frac{S}{N} &= \frac{I}{N} \\
sy &= (n+d)k \\
sAk^{\alpha} &= (n+d)k
\end{aligned}$$

b) The key here is to show that, in the case of the productivity shift, the production shifts up and the economy now has a higher level of income as well as consumption.

In the case of the rise in depreciation, the $(n+d)k$ line becomes steeper and output per capita will fall as more resources need to be devoted to maintaining the K/N constant in the steady state.

c) Consumption is maximised at the point of greatest distant between the investment requirement line $(n+d)k$ and the production function. A simple way to show this is to start from the consumption relationship:

$$c = Ak^{\alpha} - (n+d)k$$

The maximum level can be derived by setting the first derivative with respect to k equal to zero and solving for k. Then:

$$\frac{dc}{dk} = \alpha Ak^{\alpha-1} - (n+d) = 0$$

Solving for k yields:

$$k_{GR} = \left(\frac{\alpha A}{n+d}\right)^{\frac{1}{1-\alpha}}$$

When the values of the various parameters are substituted into this expression the “Golden Rule” level of capital is $k_{GR} = 32$.

d): The growth accounting equation takes the following form:

$$\begin{aligned}\frac{\Delta A}{A} &= \frac{\Delta Y}{Y} - \alpha \frac{\Delta K}{K} - (1-\alpha)\frac{\Delta N}{N} \\ \frac{\Delta A}{A} &= 0.028 + 0.3 * 0.026 + (1-0.3) * 0.015 = 0.0097\end{aligned}$$

Based on the information we have, we conclude that TFP grew by 0.97%.

Question 3:

a) From growth accounting we know that output growth in both Countries (I being Italy, and F France) can be expressed as follows (since $\frac{\Delta N_t^i}{N_t^i} = 0$):

$$\frac{\Delta Y_t^i}{Y_t^i} = \frac{\Delta A_t^i}{A_t^i} + 0.25\frac{\Delta K_t^i}{K_t^i} + 0.75\frac{\Delta N_t^i}{N_t^i} = \frac{\Delta A_t^i}{A_t^i} + 0.25\frac{\Delta K_t^i}{K_t^i}, i = I, F$$

Using the information that $\frac{\Delta K_t^I}{K_t^I} = \frac{\Delta K_t^F}{K_t^F}$, the difference in the output growth rates is obtained as follows:

$$\begin{aligned}\frac{\Delta Y_t^I}{Y_t^I} - \frac{\Delta Y_t^F}{Y_t^F} &= \frac{\Delta A_t^I}{A_t^I} + 0.25\frac{\Delta K_t^I}{K_t^I} - \frac{\Delta A_t^F}{A_t^F} - 0.25\frac{\Delta K_t^F}{K_t^F} = \\ &= \frac{\Delta A_t^I}{A_t^I} - \frac{\Delta A_t^F}{A_t^F} = a_I - a_F = 0.04\end{aligned}$$

The difference in the two Countries' output growth rates is just the difference in the *TFP* growth rates, or 4%.

b) The law of motion for the capital-labor ratio k_t is obtained as follows: one starts by dividing both sides of the capital (in levels) law of motion by N_{t+1}

$$\begin{aligned}K_{t+1} &= K_t + sY_t \\ \frac{K_{t+1}}{N_{t+1}} &= \frac{K_t + sY_t}{N_{t+1}}\end{aligned}$$

Then the expression about population growth, $N_{t+1} = (1+n)N_t$, is used to substitute for N_{t+1} in the RHS:

$$\begin{aligned}\frac{K_{t+1}}{N_{t+1}} &= k_{t+1} = \frac{K_t + sY_t}{(1+n)N_t} \\ k_{t+1} &= \frac{1}{(1+n)}\left[\frac{K_t}{N_t} + \frac{sY_t}{N_t}\right]\end{aligned}$$

To conclude with, one relies on the definition of the capital-labor ratio and the expression found in part b) for the per worker production function, obtaining:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{(1+n)}\left[k_t + s y_t\right] \\
k_{t+1} &= \frac{1}{(1+n)}\left[k_t + s k_t^{0.25}\right]
\end{aligned}
$$

c) We know that the steady state capital-labor ratio k_I^* satisfies the following condition: $k_{t+1} = k_t = k_I^*$. Imposing this requirement in the capital-labor ratio law of motion and some simple algebra gets:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{(1+n)}\left[k_t + s k_t^{0.25}\right] \\
k_I^* &= \frac{1}{(1+n)}\left[k_I^* + s\left(k_I^*\right)^{0.25}\right] \\
\left(1 - \frac{1}{(1+n)}\right) k_I^* &= \frac{1}{(1+n)} s\left(k_I^*\right)^{0.25} \\
\left(\frac{1+n-1}{(1+n)}\right) &= \frac{1}{(1+n)} s\left(k_I^*\right)^{-0.75} \\
\left(\frac{n}{s}\right) &= \left(k_I^*\right)^{-0.75} \\
k_I^* &= \left(\frac{s}{n}\right)^{\frac{1}{0.75}} \\
k_I^* &= \left(\frac{0.5}{0.04}\right)^{1.\overline{3}} = 29.01
\end{aligned}
$$

Substituting the steady state capital-labor ratio k_I^* into the per worker production function and using the definition of output in a closed economy gets:

$$
\begin{aligned}
y_I^* &= \left(k_I^*\right)^{0.25} = 2.32 \\
c_I^* &= y_I^* - s y_I^* = 0.5 * 2.32 = 1.16
\end{aligned}
$$

d) We are told that $\frac{y_F^*}{y_I^*} = 1.1$. Hence, we can obtain the value of k_F^* simply by using the expressions of the per worker production functions in the two economies:

$$
\begin{aligned}
\frac{y_F^*}{y_I^*} &= \frac{\left(k_F^*\right)^{0.25}}{\left(k_I^*\right)^{0.25}} = 1.1 \\
k_F^* &= 1.46 k_I^* = 42.47
\end{aligned}
$$

Question 4:

a) Imposing the equilibrium condition gets:

$$
\begin{aligned}
S^d(r) &= I^d(r) \\
650 + 1000r &= 820 - 1000r \\
r^* &= 0.085
\end{aligned}
$$

An interest rate equal to 8.5% clears the goods market: r^* is such that the savings and investments are exactly identical (and equal to $S(r^*) = I(r^*) = 735$). The graph is trivial.

b) We have to follow the same procedure as above for each Country separately. However, there is an additional step involved as we have to obtain the desired savings first. For Canada we obtain:

$$\begin{aligned} S^d_{Ca}(r) &= Y_{Ca} - C^d_{Ca}(r) - G_{Ca} \\ &= Y_{Ca} - (640 + c_{Ca}(Y_{Ca} - T_{Ca}) - 200r) - G_{Ca} \\ &= 2000 - (640 + 0.4(2000 - 400) - 400r) - 550 \\ &= 170 + 400r \end{aligned}$$

Since the economy is closed, we rely on the same equilibrium condition as in part a):

$$\begin{aligned} S^d_{Ca}(r) &= I^d_{Ca}(r) \\ 170 + 400r &= 300 - 400r \\ r^*_{Ca} &= 0.1625 \\ S_{Ca}(r^*_{Ca}) &= I_{Ca}(r^*_{Ca}) = 235 \\ C_{Ca}(r^*_{Ca}) &= 640 + 0.4(2000 - 400) - 400 * 0.1625 = 1215 \end{aligned}$$

Similarly, the desired savings for the US are:

$$\begin{aligned} S^d_{US}(r) &= Y_{US} - C^d_{US}(r) - G_{US} \\ &= Y_{US} - (960 + c_{US}(Y_{US} - T_{US}) - 600r) - G_{US} \\ &= 3000 - (960 + 0.4(3000 - 600) - 600r) - 600 \\ &= 480 + 600r \end{aligned}$$

And the interest rate the clears the goods market is such that:

$$\begin{aligned} S^d_{US}(r) &= I^d_{US}(r) \\ 480 + 600r &= 520 - 600r \\ r^*_{US} &= 0.0\overline{3} \\ S_{US}(r^*_{US}) &= I_{US}(r^*_{US}) = 500 \\ C_{US}(r^*_{US}) &= 960 + 0.4(3000 - 600) - 600 * 0.0\overline{3} = 1900 \end{aligned}$$

An interest rate equal to 16.25% ($3.\overline{3}$%) clears the goods market in Canada (US). Quite predictably, investments and consumption are higher in the US, given their higher income.

c) We have to compute the current accounts for each Country.
For Canada:

$$\begin{aligned}
CA_{Ca}(r) &= NX_{Ca}(r) = S^d_{Ca}(r) - I^d_{Ca}(r) \\
&= 170 + 400r - 300 + 400r = -130 + 800r \\
& \quad \textit{or alternatively} \\
CA_{Ca}(r) &= Y_{Ca} - \left(C^d_{Ca}(r) + I^d_{Ca}(r) + G_{Ca}\right) \\
&= 2000 - (6400 + 0.4\,(2000 - 400) - 400r + 300 - 400r + 550) \\
&= -130 + 800r
\end{aligned}$$

For the US:

$$\begin{aligned}
CA_{US}(r) &= NX_{US}(r) = S^d_{US}(r) - I^d_{US}(r) \\
&= 480 + 600r - 520 + 600r = -40 + 1200r \\
& \quad \textit{or alternatively} \\
CA_{US}(r) &= Y_{US} - \left(C^d_{US}(r) + I^d_{US}(r) + G_{US}\right) \\
&= 3000 - (960 + 0.4\,(3000 - 600) - 600r + 520 - 600r + 600) \\
&= -40 + 1200r
\end{aligned}$$

Imposing the equilibrium condition for the case of two large economies gets:

$$\begin{aligned}
CA_{Ca}(r) &= -CA_{US}(r) \\
-130 + 800r &= 40 - 1200r \\
r^*_W &= 0.085 \\
CA_{Ca}(r^*_W) &= -CA_{US}(r^*_W) = -62 \\
C_{Ca}(r^*_W) &= 1246 \\
C_{US}(r^*_W) &= 1869
\end{aligned}$$

The world interest rate r^*_W is in between its two closed economy counterparts, that is $r^*_{US} < r^*_W < r^*_{Ca}$. The reason is simple. When the Countries open their borders and start having access to the international market for borrowing and lending, Canadian firms can use the cheaper American funds to finance their investment. At the same time, American savers want to lend their money to Canadian firms, because they are willing to pay a higher rate of return. These forces decrease the interest rate for Canada and increase it for the US, up the point where there are no longer "arbitrage" opportunities, or until $r^*_{US,new} = r^*_W = r^*_{Ca,new}$.

The decrease in the interest rate allows Canadians to increase their consumption, while the opposite happens to the Americans.

d) The world desired saving and investment are obtained by simply adding up the desired saving and investment in the two Countries (remember that graphically we are adding *horizontally* the S^d and I^d schedules).

$$\begin{aligned}
S^d_{Ca}(r) + S^d_{US}(r) &= 170 + 400r + 480 + 600r \\
S^d_W(r) &= 650 + 1000r
\end{aligned}$$

$$\begin{aligned}
I^d_{Ca}(r) + I^d_{US}(r) &= 300 - 400r + 520 - 600r \\
I^d_W(r) &= 820 - 1000r
\end{aligned}$$

$S_W^d(r)$ and $I_W^d(r)$ correspond to the ones we used in part a). This tells us that in our framework, in order to get the world interest rate when there are only two large economies, it is equivalent to consider the desired saving and investment of one big economy (the world economy) or to consider the current accounts of the two large economies. Formally, it's easy to show that you can move from one definition to the other, when we are in equilibrium:

$$\begin{aligned}
S_W^d(r) &= S_{Ca}^d(r) + S_{US}^d(r) \\
I_W^d(r) &= I_{Ca}^d(r) + I_{US}^d(r)
\end{aligned}$$

so that at the equilibrium:

$$\begin{aligned}
S_W^d(r) &= I_W^d(r) \\
S_{Ca}^d(r) + S_{US}^d(r) &= I_{Ca}^d(r) + I_{US}^d(r) \\
S_{Ca}^d(r) - I_{Ca}^d(r) &= -\left(S_{US}^d(r) - I_{US}^d(r)\right) \\
CA_{Ca}(r) &= -CA_{US}(r)
\end{aligned}$$

This result is mainly due to the absence of frictions and imperfections in the international transactions.