Math for 222

Rules of Exponents

1. $x^{-a} = \frac{1}{x^a}$, i.e. $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2. $x^a \cdot x^b = x^{a+b}$, i.e. $2^2 \cdot 2^3 = 2^5 = 32$
3. $\frac{x^a}{x^b} = x^a \cdot x^{-b} = x^{a-b}$, i.e. $\frac{2^2}{2^3} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
4. $(x^a)^b = x^{a \cdot b}$, i.e. $(2^2)^3 = 2^6 = 64$
5. $x^0 = 1$, i.e. $2^0 = 1$

Marginal Products with Calculus

Our production function is: $Y = AK^\alpha N^{1-\alpha}$ and we want to find the MPK. Recall that the MPK is the change in output for a one unit change in capital ($\frac{\Delta Y}{\Delta K}$ when $\Delta K = 1$) keeping everything else constant. As explained besides Figure 3.2 in the text, the MPK is the slope of the tangent line to the production function. But the slope of the tangent line of a function is equal to the derivative of the function.

Example: $Y = 135.36K^{0.3}$. Find the MPK when $K = 500$.

$$\begin{aligned} \frac{dY}{dK} &= 0.3 \cdot 135.36K^{0.3-1} = 40.61K^{-0.7} \\ &= \frac{40.61}{K^{0.7}} \underset{K=500}{\longrightarrow} = 0.524 \end{aligned}$$

Example: $Y = AK^{\alpha}N^{1-\alpha}$. Find a formula for the MPK.

$$\frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}N^{1-\alpha} = \frac{\alpha AN^{1-\alpha}}{K^{1-\alpha}} = \text{MPK}$$

In the last example we are told now that: $\alpha = 0.3$, A = 20.43, N = 14.9. What the formula becomes? MPK = $\frac{40.61}{K^{0.7}}$.

Note: the sign ∂ indicates that we are differentiating "partially" the function: as $Y = AK^{\alpha}N^{1-\alpha}$ is a function of two variables, K and N, we differentiate it with respect to one variable (K for MPK or N for MPN, MPK = $\frac{\partial Y}{\partial K}$ and MPN = $\frac{\partial Y}{\partial N}$). When there is only one variable we use the letter "d" by convention (i.e. if $Y = 135.36K^{0.3}$, MPK = $\frac{dY}{dK}$).

Finally, note that we can apply that technique to apparently more complicated Cobb-Douglas functions.

Example: $Y = AK^{\alpha}N^{\beta}L^{1-\alpha-\beta}$, where L = land . Find a formula for the MPN.

$$\frac{\partial Y}{\partial N} = \beta AK^{\alpha}N^{\beta-1}L^{1-\alpha-\beta} = \frac{\beta AK^{\alpha}L^{1-\alpha-\beta}}{N^{1-\beta}} = \text{MPN}$$

Example: $Y = AK^{0.5}N^{0.5}$. Find the MPN.

$$\frac{\partial Y}{\partial N} = 0.5 \cdot AK^{0.5}N^{-0.5} = \frac{AK^{0.5}}{2N^{0.5}} = \text{MPN}$$

As A,K,N are all positive, the MPN is always positive (extra labor never reduce the output produced).

Suppose A = 1, K = 4: MPN = $\frac{1 \cdot \sqrt{4}}{2N^{0.5}} = \frac{1}{\sqrt{N}}$. As N increases, the MPN decreases: more and more labor do not add much keeping capital fixed.

Suppose now that the technology improves: A = 2 (and K=4). The new MPN is now: $\frac{2}{\sqrt{N}}$. The new MPN is twice higher at every level of N.

Suppose now that the capital stock is increased: K = 9 (and A=1). The new MPN is now: $\frac{3}{2\sqrt{N}}$. The new MPN is again higher at every level of N.

You can verify that all those properties hold true for the MPK.