Dropbox Rules

Policy for the dropbox

- (i) The assignment boxes on the second floor of Dunning Hall will only be available to students from 9:00 am until 4:00 pm.
- (ii) Every day at 4:00 pm the boxes will be emptied. (iii) Any assignments submitted into these boxes after 4:00 pm (and before 9:00 am the next morning) will be recycled.
- (iii) The staff in the main office (Dun 209) and the undergrad studies office (Dun 221) will NOT accept assignments at any time (particularly after 4:00 pm).
- (iv) Instructors who wish to use these boxes must inform their students of this new policy before their first assignments are due.
Outline

- Exponents
- Logarithms
- Growth rates
- Differentiation
- Elasticities
- Examples
General Properties of Exponents

\[ a^n \cdot a^m = a^{n+m} \]
\[ 2^2 \cdot 2^3 = 2^5 = 32 \]

\[ a^n / a^m = a^{n-m} \]
\[ \frac{2^3}{2^2} = 2^{3-2} = 2 \]

\[ (a^n)^m = a^{n \cdot m} \]
\[ (2^2)^3 = 2^6 = 64 \]
General Properties of Exponents

\[(a \cdot b)^2 = a^n \cdot b^n\]
\[(2 \cdot 3)^2 = 2^2 \cdot 3^2 = 4 \cdot 9 = 36\]

\[\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\]
\[\left(\frac{2}{3}\right)^2 = \left(\frac{2^2}{3^2}\right) = \left(\frac{4}{9}\right)\]

Example: Simplify the following:

\[
\frac{x^n \cdot x^2}{x^{n-1}} = x^{n+2-n-(-1)} = x^3
\]
Useful Rules for logs

Let a be any positive number

\[ \exp \ln a = a \]

The exponential and logarithm functions are the inverse of one another (i.e. they appear to cancel each other out)

\[ \ln(xy) = \ln x + \ln y \] (logarithm additive property) (1)

\[ \ln \frac{x}{y} = \ln x - \ln y \] (2)

\[ \ln x^p = p \ln x \] (3)

\[ \ln 1 = 0 \] (4)

\[ \ln \exp = 1 \] (5)
Useful Rules for logs

Example - Simplify:

\[ \exp \left[ \ln(x^2) - 2 \ln y \right] \]

\[ = \exp[\ln(x^2) - 2 \ln y] \]
\[ = \exp[\ln(x^2) - \ln y^2] \]
\[ = \frac{\exp[\ln(x^2)]}{\exp[\ln(y^2)]} \]
\[ = \left( \frac{x}{y} \right)^2 \]

Previously there was an incorrect solution to this problem. My thanks to the student who pointed it out.

\[ \exp^{2[\ln x - \ln y]} = \exp[\ln\left( \frac{x}{y} \right)^2] \neq \exp^2 \left( \frac{x}{y} \right) \]
Growth Rate Formulas

- Let $X$ and $Z$ be any two variables, not necessarily related by a function, that are changing over time.
- Let $\Delta X/X$ and $\Delta Z/Z$ represent the growth rates (percentage changes).
- You know that the two-period growth rate of $X$ is
  \[
  \frac{X_{t+1}-X_t}{X_t} = \frac{\Delta X}{X_t}
  \]
- **Rule 1.** The growth rate of the product of $X$ and $Z$
  \[
  \left(\frac{\Delta X}{X}\right) \left(\frac{\Delta Y}{Y}\right)
  \] equals the growth rate of $X$ plus the growth rate of $Z$. 

Proof of Rule 1

Proof. Suppose that \( X \) increases by \( \Delta X \) and \( Z \) increases by \( \Delta Z \). Then the absolute increase in the product of \( X \) and \( Z \) is \((X + \Delta X)(Z + \Delta Z) - XZ\), and the growth rate of the product of \( X \) and \( Z \) is:

\[
\frac{\Delta(XY)}{XY} = \frac{(X_{\text{new}})(Z_{\text{new}}) - XZ}{XZ} = \frac{(X + \Delta X)(Z + \Delta Z) - XZ}{XZ} = \frac{XZ - (\Delta Z)Z + (\Delta X)X + \Delta X\Delta Z - XZ}{XZ} = \frac{\Delta X}{X} + \frac{\Delta Z}{Z} + \frac{\Delta X\Delta Z}{XZ}
\]

The last term can be ignored if both term deviations are small (which they usually are).
Rule 2. The growth rate of the ratio of $X$ to $Z$ is the growth rate of $X$ minus the growth rate of $Z$.

Proof. Let $W$ be the ratio of $X$ to $Z$, so $W = X/Z$. Then $X = ZW$. By Rule 1, as $X$ equals the product of $Z$ and $W$, the growth rate of $X$ equals the growth rate $Z$ plus the growth rate of $W$:

\[
\frac{\Delta X}{X} = \frac{\Delta Z}{Z} + \frac{\Delta W}{W}
\]
\[
\frac{\Delta W}{W} = \frac{\Delta X}{X} - \frac{\Delta Z}{Z}
\]
Growth Rates and Logarithms

- For small values of $x$, the following approximation holds:

$$\ln(1 + x) \approx x$$

- From that property let us show that the growth rate is also equal to the difference in logarithms:

$$\frac{X_{t+1} - X_t}{X_t} \approx \ln X_{t+1} - \ln X_t$$

Proof:

$$\ln X_{t+1} - \ln X_t = \ln \left( \frac{X_{t+1}}{X_t} \right)$$

$$= \ln \left( \frac{X_{t+1} - X_t + X_t}{X_t} \right)$$

$$= \ln \left( 1 + \frac{X_{t+1} - X_t}{X_t} \right)$$

$$\approx \frac{X_{t+1} - X_t}{X_t}$$
Single-variable Differentiation

- $\frac{\Delta Y}{\Delta X}$ measures how $Y$ changes if we increase $X$ by one unit.
- With derivatives it is possible to find the instantaneous rate of change.
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- several ways to denote derivatives
  - $f'(x)$ partial derivative with respect to (wrt) $x$
  - $f'_x(x, y)$ also the partial derivative wrt $x$
  - $\frac{dy}{dx}$ total derivative
  - $\frac{\partial y}{\partial x}$ partial derivative
Rules:

General Notation

\[ f(x) = A \implies f'(x) = 0 \]
\[ y = A + f(x) \implies y' = f'(x) \]
\[ y = Af(x) \implies y' = Af'(x) \]

How to take a partial derivative

\[ f(x) = x^a \implies f'(x) = ax^{a-1} \text{ (this is vital)} \]
\[ f(x) = \ln x \implies f'(x) = \frac{1}{x} \text{ (extremely important)} \]
More Rules

Take the derivative wrt each term separately

\[ F(x) = f(x) + g(x) \implies F'(x) = f'(x) + g'(x) \]
\[ F(x) = f(x) - g(x) \implies F'(x) = f'(x) - g'(x) \]

Product Rule:

\[ F(x) = f(x)g(x) \implies F'(x) = f'(x)g(x) + f(x)g'(x) \]

Quotient Rule:

\[ F(x) = \frac{f(x)}{g(x)} \implies F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \]

The Chain Rule:

\[ \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} \quad (6) \]

Example) \[ F(x) = u(f(x)) \quad (7) \]
\[ F'(x) = u'(f(x))f'(x) \quad (8) \]
Finding the MPK of a Cobb-Douglas production:

\[Y = AK^\alpha N^{1-\alpha}\]

- MPK is the change in output for a one unit change in capital \((\frac{\Delta Y}{\Delta K} \text{ when } \Delta K = 1)\) keeping everything else equal.
- The MPK is the slope of the tangent line to the production function and is equal to the derivative of the function.
- Example: \(Y = 4K^{0.5}\) Find the MPK when \(K = 100\)

\[
\text{MPK} = \frac{\partial Y}{\partial K} = 2k^{-0.5}
\]

\[
= \frac{2}{\sqrt{100}} = \frac{2}{10} = \frac{1}{5}
\]

- Apply the same method to find the marginal product of labour
Maximization and Minimization

- \( \pi(q) = 1000q - cq^2 \) where \( 1000 > c > 0 \)
- to maximize the above objective function, find the partial derivative w.r.t. \( q \) and set it equal to zero

\[
\pi'(q) = 1000 - 2cq = 0
\]

\[
q^* = \frac{1000}{2c}
\]

- the procedure for maximization and minimization is the same
- a function has been maximized if \( \pi''(q) < 0 \)
- why? At the max the slope of the objective function is zero. An increase or decrease in \( q \) decreases \( \pi \).
- a function has been minimized if \( \pi''(q) > 0 \)
Simple Constrained Maximization

- $\pi(p, q) = pq - cq$ where $c > 0$
- subject to $p(q) = 1000 - 4q$
- substitute the constraint ($p(q)$) directly into the $\pi(p, q)$ objective function and set the partial derivative equal to zero

$$\begin{align*}
\pi(q) &= (1000 - 4q)q - cq \\
&= 1000q - 4q^2 - cq \\
\pi'(q) &= 1000 - c - 8q = 0 \\
q^* &= \frac{1000 - c}{8}
\end{align*}$$

- the procedure for maximization and minimization is the same
- a function has been maximized if $\pi''(q) < 0$
- a function has been minimized if $\pi''(q) > 0$
Elasticities

- The *elasticity* of $Y$ with respect to $N$ is defined to be the percentage change in $Y$, $\Delta Y/Y$, divided by the percentage change in $N$, $\Delta N/N$:

  \[ \eta_{Y,N} = \frac{\Delta Y/Y}{\Delta N/N} \]

- Or if we use derivatives:

  \[ \eta_{Y,N} = \frac{N}{Y} \frac{\partial Y}{\partial N} = \frac{\partial \ln Y}{\partial \ln N} \]

- How so? (Hint: exploit the chain rule):

  \[
  \frac{N}{Y} \frac{\partial Y}{\partial N} = \frac{\partial \ln Y}{\partial \ln N} \frac{\partial Y}{\partial N} = \frac{\partial \ln Y}{\partial \ln N} \]
Elasticity of substitution

- Marginal rate of substitution between \(y\) and \(x\) is:
  \[
  MRS_{y,x} = \frac{F_y'(x,y)}{F_x'(x,y)}
  \]
- When \(F(x, y) = c\), the elasticity of substitution between \(y\) and \(x\) is:
  \[
  \sigma_{yx} = \eta \left( \frac{y}{x} \right), MRS_{y,x}
  \]
- Example: Find \(\sigma_{K,N}\) for the following Cobb-Douglas production function:
  \[
  Y = AK^\alpha N^{1-\alpha}
  \]

\[
MRS_{K,N} = \frac{MPK}{MPN} = \frac{A\alpha K^{\alpha-1} N^{1-\alpha}}{A(1-\alpha)K^{\alpha} N^{-\alpha}}
\]

\[
= \frac{1 - \alpha}{\alpha} \frac{K}{N}
\]

\[
\frac{K}{N} = \left( \frac{\alpha}{1-\alpha} \right) MRS_{K,N}
\]

\[
\sigma_{K,N} = \frac{MRS_{K,N}}{\alpha \frac{\alpha}{1-\alpha}} \cdot \frac{\alpha}{1-\alpha} = 1
\]
Example 1

Find the first and second order partial derivative

\[ f(x) = x^2 \]
\[ f'(x) = 2x \]
\[ f''(x) = 2 \]
Example 2

Find the first and second order partial derivative

\[ f(x) = 3x \]
\[ f'(x) = 2 \]
\[ f''(x) = 0 \]
Example 3

Find the first and second order partial derivative

\[ f(x) = yx^3 \]
\[ f'(x) = 3yx^2 \]
\[ f''(x) = 6yx \]
Example 4

Find the first and second order partial derivative

\[ f(x) = \ln(x) \]
\[ f'(x) = \frac{1}{x} \]
\[ f''(x) = -\frac{1}{x^2} \]
Example (kinda) 5

\[ f(x) = e^x \]
\[ f'(x) = e^x \]

why?

\[ y = e^x \]
\[ \ln(y) = \ln e^x \]
\[ \ln(y) = x \]
\[ \frac{1}{y} \frac{dy}{dx} = 1 \]
\[ \frac{dy}{dx} = y \]
\[ \frac{dy}{dx} = e^x \]
Example 6

Find the first order partial derivative

\[ f(x) = \frac{e^x}{x^2} \]

\[ f'(x) = \frac{e^x x^2 - e^x 2x}{(x^2)^2} \]

\[ f''(x) = \frac{e^x (x^2 - 2x)}{x^4} \]
Example 7

Find the first order partial derivative

\[ f(x) = \ln(x^5 - 5x^3 - 100) \]

\[ f'(x) = \frac{5x^4 - 15x^2}{x^5 - 5x^3 - 100} \]
Rule 3. Suppose that $Y$ is a variable that is a function of two other variables $X$ and $Z$. Then $\frac{\Delta Y}{Y} = \eta_{Y,X} \frac{\Delta X}{X} + \eta_{Y,Z} \frac{\Delta Z}{Z}$

Proof.: informal. The overall effect on $Y$ is approximately equal to the sum of the individual effects on $Y$ of the change in $X$ and the change in $Z$.

Rule 4. The growth rate of $X$ raised to the power $a$, or $X^a$, is $a$ times the growth rate of $X$, growth rate of $(X^a) = a \frac{\Delta X}{X}$

Proof.: Let $Y = X^a$. Using Rule 3, we find the elasticity of $Y$ with respect to $X$ equals $a$. 

Back to growth rate rules
References