Math Review for ECON 222

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Outline

▶ Exponents
▶ Logarithms
▶ Growth rates
▶ Differentiation
▶ Elasticities
General Properties of Exponents

\[ a^n \cdot a^m = a^{n+m} \quad (1) \]
\[ a^n / a^m = a^{n-m} \quad (2) \]
\[ (a^n)^m = a^{n \cdot m} \quad (3) \]
\[ (a \cdot b)^n = a^n \cdot b^n \quad (4) \]
\[ \left( \frac{a}{b} \right)^n = \frac{a^n}{b^n} \quad (5) \]

Example: Simplify \( \frac{x^n \cdot x}{x^{n-1}} \)

\[ \frac{x^n \cdot x}{x^{n-1}} = \frac{x^{n+1}}{x^{n-1}} = x^2 \]
Useful Rules for logs

\[
\exp \ln a = a \quad (a \text{ is any positive number})
\]

\[
\ln(x \cdot y) = \ln x + \ln y \quad (x \text{ and } y \text{ are positive}) \quad (6)
\]

\[
\ln \left( \frac{x}{y} \right) = \ln x - \ln y \quad (7)
\]

\[
\ln x^p = p \ln x \quad (8)
\]

\[
\ln 1 = 0 \quad (9)
\]

\[
\ln \exp = 1 \quad (10)
\]

Example: Simplify \( \exp \left[ \ln(x^2) - 2 \ln y \right] \)

\[
= \exp[\ln x^2 - 2 \ln y]
\]

\[
= \exp[2 \ln x - 2 \ln y]
\]

\[
= \exp[2 \ln \left( \frac{x}{y} \right)]
\]

\[
= e^{2 \left( \frac{x}{y} \right)}
\]
Growth Rate Formulas

- Let $X$ and $Z$ be any two variables, not necessarily related by a function, that are changing over time.
- Let $\Delta X/X$ and $\Delta Z/Z$ represent the growth rates (percentage changes).
- **Rule 1.** The growth rate of the product of $X$ and $Z$ equals the growth rate of $X$ plus the growth rate of $Z$. 
Proof of Rule 1

Proof. Suppose that $X$ increases by $\Delta X$ and $Z$ increases by $\Delta Z$. Then the absolute increase in the product of $X$ and $Z$ is $(X + \Delta X)(Z + \Delta Z) - XZ$, and the growth rate of the product of $X$ and $Z$ is:

\[
\frac{(X+\Delta X)(Z+\Delta Z)-XZ}{XZ} = \frac{(\Delta X)Z+(\Delta Z)X+\Delta X\Delta Z}{XZ} = \frac{\Delta X}{X} + \frac{\Delta Z}{Z} + \frac{\Delta X\Delta Z}{XZ}
\]

The last term can be ignored if both term deviations are very small.
Rule 2. The growth rate of the ratio of $X$ to $Z$ is the growth rate of $X$ minus the growth rate of $Z$.

Proof. Let $W$ be the ratio of $X$ to $Z$, so $W = X/Z$. Then $X = ZW$. By Rule 1, as $X$ equals the product of $Z$ and $W$, the growth rate of $X$ equals the growth rate of $Z$ plus the growth rate of $W$:

$$\frac{\Delta X}{X} = \frac{\Delta Z}{Z} + \frac{\Delta W}{W}$$

$$\frac{\Delta W}{W} = \frac{\Delta X}{X} - \frac{\Delta Z}{Z}$$
Growth Rates and Logarithms

- For small values of $x$, the following approximation holds:

$$\ln(1 + x) \approx x$$

- From that property let us show that the growth rate is also equal to the difference in logarithms:

$$\frac{X_{t+1} - X_t}{X_t} \approx \ln X_{t+1} - \ln X_t$$

Solution:

$$\ln X_{t+1} - \ln X_t = \ln \left( \frac{X_{t+1}}{X_t} \right)$$

$$= \ln \left( \frac{X_{t+1} - X_t + X_t}{X_t} \right)$$

$$= \ln \left( 1 + \frac{X_{t+1} - X_t}{X_t} \right)$$

$$\approx \frac{X_{t+1} - X_t}{X_t}$$
Single-variable Differentiation

- $\frac{\Delta Y}{\Delta X}$ measures how $Y$ changes if we increase $X$ by one unit.
- With derivatives it is possible to find the instantaneous rate of change.
- $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Rules:

- $f(x) = A \Rightarrow f'(x) = 0$
- $y = A + f(x) \Rightarrow y' = f'(x)$
- $y = Af(x) \Rightarrow y' = Af'(x)$
- $f(x) = x^a \Rightarrow f'(x) = ax^{a-1}$
- $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
More Rules

\[ F(x) = f(x) + g(x) \implies F'(x) = f'(x) + g'(x) \]
\[ F(x) = f(x) - g(x) \implies F'(x) = f'(x) - g'(x) \]
\[ F(x) = f(x)g(x) \implies F'(x) = f'(x)g(x) + f(x)g'(x) \]
\[ F(x) = \frac{f(x)}{g(x)} \implies F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \]

- The Chain Rule: \[ \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} \]
Finding the MPK of a Cobb-Douglas production:

\[ Y = AK^\alpha N^{1-\alpha} \]

MPK is the change in output for a one unit change in capital \( \Delta Y / \Delta K \) when \( \Delta K = 1 \) keeping everything else equal.

The MPK is the slope of the tangent line to the production function and is equal to the derivative of the function.

Example: \( Y = 4K^{0.5} \) Find the MPK when \( K = 100 \)

\[
\frac{\partial Y}{\partial K} = 2k^{-0.5} \\
= \frac{2}{\sqrt{K}} = \frac{2}{10}
\]
Elasticities

- The elasticity of $Y$ with respect to $N$ is defined to be the percentage change in $Y$, $\Delta Y / Y$, divided by the percentage change in $N$, $\Delta N / N$: $\eta_{Y,N} = \frac{\Delta Y / Y}{\Delta N / N}$

- Or if we use derivatives: $\eta_{Y,N} = \frac{N \partial Y}{Y \partial N} = \frac{\partial \ln Y}{\partial \ln N}$

- How so? (Hint: apply the chain rule): $\frac{\partial \ln Y}{\partial \ln N} = \frac{\partial \ln Y}{\partial N} \frac{\partial Y}{\partial N} \frac{\partial N}{\partial \ln N}$

- Example 1.: Find $\eta_{Y,N}$ for the following Cobb-Douglas production function: $Y = AK^{\alpha}N^{1-\alpha}$

  $\eta_{Y,N} = \frac{N}{AK^{\alpha}N^{1-\alpha}}(1 - \alpha)AK^{\alpha}N^{-\alpha} = 1 - \alpha$

- Example 2.: Suppose that the real money demand function is: $L(Y, r + \pi^e) = \frac{0.01Y}{(r+\pi^e)}$. What is the income elasticity of money demand?

  $\eta_{L,Y} = \frac{Y}{0.01Y} \cdot \frac{0.01}{r + \pi^e} = 1$
Elasticity of substitution

- Marginal rate of substitution between $y$ and $x$ is:
  \[ MRS_{y,x} \frac{F_1'(x,y)}{F_2'(x,y)} \]

- When $F(x, y) = c$, the elasticity of substitution between $y$ and $x$ is:
  \[ \sigma_{yx} = \eta\left(\frac{y}{x}\right), MRS_{y,x} \]

- Example: Find $\sigma_{K,N}$ for the following Cobb-Douglas production function:
  \[ Y = AK^\alpha N^{1-\alpha} \]

\[
MRS_{K,N} = \frac{MPN}{MPK} = \frac{A(1 - \alpha)K^\alpha N^{-\alpha}}{A\alpha K^{\alpha - 1}N^{1-\alpha}} = \frac{1 - \alpha}{\alpha} K \div \frac{\alpha}{N}
\]

\[
\frac{K}{N} = \left(\frac{\alpha}{1 - \alpha}\right) MRS_{K,N}
\]

\[
\sigma_{K,N} = \frac{MRS_{K,N}}{\frac{\alpha}{1 - \alpha} MRS_{K,N}} \cdot \frac{\alpha}{1 - \alpha} = 1
\]
Rule 3. Suppose that $Y$ is a variable that is a function of two other variables $X$ and $Z$. Then \[ \frac{\Delta Y}{Y} = \eta_{Y,X} \frac{\Delta X}{X} + \eta_{Y,Z} \frac{\Delta Z}{Z} \]

Proof.: informal. The overall effect on $Y$ is approximately equal to the sum of the individual effects on $Y$ of the change in $X$ and the change in $Z$.

Rule 4. The growth rate of $X$ raised to the power $a$, or $X^a$, is $a$ times the growth rate of $X$, growth rate of $(X^a) = a \frac{\Delta X}{X}$

Proof.: Let $Y = X^a$. Using Rule 3, we find the elasticity of $Y$ with respect to $X$ equals $a$. 
To Wrap Up: A Last Exercise

- Another type of production function is called the constant elasticity of substitution (CES) production function. (In fact, it can be shown that a Cobb-Douglas function is a subcase of the CES function)

\[ Y = \left( \omega \cdot K^{-\epsilon} + (1 - \omega) \cdot N^{-\epsilon} \right)^{-\frac{1}{\epsilon}} \]

- Let us find \( MPK \) and \( MPN \) and then the elasticity of substitution between \( K \) and \( N \).

\[
\begin{align*}
MPK &= \frac{\partial Y}{\partial K} = -\frac{1}{\epsilon} \left( \omega K^{-\epsilon} + (1 - \omega) N^{-\epsilon} \right)^{-\frac{1}{\epsilon}-1} \omega(-\epsilon)K^{-\epsilon-1} \\
MPN &= \frac{\partial Y}{\partial N} = -\frac{1}{\epsilon} \left( \omega K^{-\epsilon} + (1 - \omega) N^{-\epsilon} \right)^{-\frac{1}{\epsilon}-1} (1 - \omega)(-\epsilon)N^{-\epsilon-1}
\end{align*}
\]
Solution to the Last Exercise

\[ \frac{MRS_{K,N}}{\omega K^{\epsilon-1} N} = \frac{(1 - \omega) N^{\epsilon-1}}{(\omega K)^{\epsilon-1}} = \frac{(1 - \omega)}{\omega} \left( \frac{K}{N} \right)^{\epsilon+1} \]

\[ \frac{K}{N} = \left( \frac{\omega}{1 - \omega} \right)^{\frac{1}{\epsilon+1}} (MRS_{K,N})^{\frac{1}{\epsilon+1}} \]

\[ \sigma_{K,N} = \frac{MRS_{K,N}}{\left( \frac{\omega}{1 - \omega} \right)^{\frac{1}{\epsilon+1}} MRS_{K,N}^{\epsilon+1}} \left( \frac{\omega}{1 - \omega} \right)^{\frac{1}{\epsilon+1}} \frac{1}{\epsilon + 1} (MRS_{K,N})^{\frac{1}{\epsilon+1} - 1} \]

\[ \sigma_{K,N} = \frac{1}{\epsilon + 1} \]
References