Math Review for ECON 222

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Outline

- Exponents
- Logarithms
- Growth rates
- Differentiation
- Elasticities

General Properties of Exponents

$$\begin{aligned} a^n \cdot a^m &= a^{n+m} && (1)\\ a^n / a^m &= a^{n-m} && (2)\\ (a^n)^m &= a^{n \cdot m} && (3)\\ (a \cdot b)^n &= a^n \cdot b^n && (4)\\ \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} && (5) \end{aligned}$$

Example: Simplify $\frac{x^n \cdot x}{x^{n-1}}$

$$\frac{x^n \cdot x}{x^{n-1}} = \frac{x^{n+1}}{x^{n-1}} = x^2$$

Useful Rules for logs

$$\exp \ln a = a \text{ (a is any positive number)}$$

$$\begin{aligned}
\ln(xy) &= \ln x + \ln y \text{ (x and y are positive)} && (6) \\
\ln \frac{x}{y} &= \ln x - \ln y && (7) \\
\ln x^p &= p \ln x && (8) \\
\ln 1 &= 0 && (9) \\
\ln \exp &= 1 && (10)
\end{aligned}$$

Example: Simplify $\exp\left[\ln(x^2) - 2\ln y\right]$

$$\begin{aligned}
&= \exp[\ln x^2 - 2\ln y] \\
&= \exp[2\ln x - 2\ln y] \\
&= \exp[2\ln\left(\frac{x}{y}\right)] \\
&= e^2\left(\frac{x}{y}\right)
\end{aligned}$$

Growth Rate Formulas

- Let X and Z be any two variables, not necessarily related by a function, that are changing over time.
- Let $\Delta X/X$ and $\Delta Z/Z$ represent the growth rates (percentage changes)
- **Rule 1.** The growth rate of the product of X and Z equals the growth rate of X plus the growth rate of Z.

Proof of **Rule 1**

- *Proof.* Suppose that X increases by ΔX and Z increases by ΔZ. Then the absolute increase in the product of X and Z is $(X + \Delta X)(Z + \Delta Z) - XZ$, and the growth rate of the product of X and Z is:

$$
\begin{aligned}
&= \frac{(X+\Delta X)(Z+\Delta Z)-XZ}{XZ} \\
&= \frac{(\Delta X)Z+(\Delta Z)X+\Delta X \Delta Z}{XZ} \\
&= \frac{\Delta X}{X} + \frac{\Delta Z}{Z} + \frac{\Delta X \Delta Z}{XZ}
\end{aligned}
$$

- The last term can be ignored if both term deviations are very small.

Growth Rate Formulas

- **Rule 2.** The growth rate of the ratio of X to Z is the growth rate of X minus the growth rate of Z.
- *Proof.* Let W be the ratio of X to Z, so $W = X/Z$. Then $X = ZW$. By Rule 1, as X equals the product of Z and W, the growth rate of X equals the growth rate Z plus the growth rate of W:

$$\begin{aligned} \frac{\Delta X}{X} &= \frac{\Delta Z}{Z} + \frac{\Delta W}{W} \\ \frac{\Delta W}{W} &= \frac{\Delta X}{X} - \frac{\Delta Z}{Z} \end{aligned}$$

Growth Rates and Logarithms

- For small values of x, the following approximation holds :

$$\ln(1 + x) \approx x$$

- From that property let us show that the growth rate is also equal to the difference in logarithms:

$$\frac{X_{t+1} - X_t}{X_t} \approx \ln X_{t+1} - \ln X_t$$

Solution:

$$\begin{aligned}
\ln X_{t+1} - \ln X_t &= \ln\left(\frac{X_{t+1}}{X_t}\right) \\
&= \ln\left(\frac{X_{t+1} - X_t + X_t}{X_t}\right) \\
&= \ln\left(1 + \frac{X_{t+1} - Xt}{X_t}\right) \\
&\approx \frac{X_{t+1} - X_t}{X_t}
\end{aligned}$$

Single-variable Differentiation

- $\frac{\Delta Y}{\Delta X}$ measures how Y changes if we increase X by one unit.
- With derivatives it is possible to find the instantaneous rate of change.
- $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Rules:

$$\begin{aligned}
f(x) = A &\quad \Rightarrow \quad f'(x) = 0 \\
y = A + f(x) &\quad \Rightarrow \quad y' = f'(x) \\
y = Af(x) &\quad \Rightarrow \quad y' = Af'(x) \\
f(x) = x^a &\quad \Rightarrow \quad f'(x) = ax^{a-1} \\
f(x) = \ln x &\quad \Rightarrow \quad f'(x) = \frac{1}{x}
\end{aligned}$$

More Rules

$$
\begin{aligned}
F(x) = f(x) + g(x) \quad &\Rightarrow \quad F'(x) = f'(x) + g'(x) \\
F(x) = f(x) - g(x) \quad &\Rightarrow \quad F'(x) = f'(x) - g'(x) \\
F(x) = f(x)g(x) \quad &\Rightarrow \quad F'(x) = f'(x)g(x) + f(x)g'(x) \\
F(x) = \frac{f(x)}{g(x)} \quad &\Rightarrow \quad F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}
\end{aligned}
$$

- The Chain Rule: $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x}$

Marginal Products with Calculus

- Finding the MPK of a Cobb-Douglas production: $Y = AK^{\alpha}N^{1-\alpha}$
- MPK is the change in output for a one unit change in capital $\left(\frac{\Delta Y}{\Delta K} \text{ when } \Delta K = 1\right)$ keeping everything else equal.
- The MPK is the slope of the tangent line to the production function and is equal to the derivative of the function.
- Example: $Y = 4K^{0.5}$ Find the MPK when $K = 100$

$$\begin{aligned} &= \frac{\partial Y}{\partial K} = 2k^{-0.5} \\ &= \frac{2}{\sqrt{K}} = \frac{2}{10} \end{aligned}$$

Elasticities

- The *elasticity* of Y with respect to N is defined to be the percentage change in Y, $\Delta Y/Y$, divided by the percentage change in N, $\Delta N/N$: $\eta_{Y,N} = \frac{\Delta Y/Y}{\Delta N/N}$
- Or if we use derivatives: $\eta_{Y,N} = \frac{N}{Y}\frac{\partial Y}{\partial N} = \frac{\partial \ln Y}{\partial \ln N}$
- How so? (Hint: apply the chain rule): $\frac{\partial \ln Y}{\partial \ln N} = \frac{\partial \ln Y}{\partial N}\frac{\partial Y}{\partial N}\frac{\partial N}{\partial \ln N}$

- Example 1.: Find $\eta_{Y,N}$ for the following Cobb-Douglas production function: $Y = AK^{\alpha}N^{1-\alpha}$

$$\eta_{Y,N} = \frac{N}{AK^{\alpha}N^{1-\alpha}}(1-\alpha)AK^{\alpha}N^{-\alpha} = 1-\alpha$$

- Example 2.: Suppose that the real money demand function is: $L(Y, r+\pi^e) = \frac{0.01Y}{(r+\pi^e)}$. What is the income elasticity of money demand?

$$\eta_{L,Y} = \frac{Y}{\frac{0.01Y}{r+\pi^e}} \cdot \frac{0.01}{r+\pi^e} = 1$$

Elasticity of substitution

- Marginal rate of substitution between y and x is: $MRS_{y,x}\frac{F_1'(x,y)}{F_2'(x,y)}$
- When $F(x,y) = c$, the elasticity of substitution between y and x is: $\sigma_{yx} = \eta_{\left(\frac{y}{x}\right),MRS_{y,x}}$
- Example: Find $\sigma_{K,N}$ for the following Cobb-Douglas production function: $Y = AK^{\alpha}N^{1-\alpha}$

$$
\begin{aligned}
MRS_{K,N} &= \frac{MPN}{MPK} = \frac{A(1-\alpha)K^{\alpha}N^{-\alpha}}{A\alpha K^{\alpha-1}N^{1-\alpha}} \\
&= \frac{1-\alpha}{\alpha}\frac{K}{N} \\
\frac{K}{N} &= \left(\frac{\alpha}{1-\alpha}\right)MRS_{K,N} \\
\sigma_{K,N} &= \frac{MRS_{K,N}}{\frac{\alpha}{1-\alpha}MRS_{K,N}} \cdot \frac{\alpha}{1-\alpha} = 1
\end{aligned}
$$

Back to growth rate rules

- **Rule 3.** Suppose that Y is a variable that is a function of two other variables X and Z. Then $\frac{\Delta Y}{Y} = \eta_{Y,X}\frac{\Delta X}{X} + \eta_{Y,Z}\frac{\Delta Z}{Z}$
- *Proof.*: informal. The overall effect on Y is approximately equal to the sum of the individual effects on Y of the change in X and the change in Z.

- **Rule 4.** The growth rate of X raised to the power a, or X^a, is a times the growth rate of X, growth rate of $(X^a) = a\frac{\Delta X}{X}$
- *Proof.*: Let $Y = X^a$. Using Rule 3, we find the elasticity of Y with respect to X equals a.

To Wrap Up: A Last Exercise

- Another type of production function is called the constant elasticity of substitution (CES) production function. (In fact, it can be shown that a Cobb-Douglas function is a subcase of the CES function)

$$Y = \left(\omega \cdot K^{-\epsilon} + (1-\omega) \cdot N^{-\epsilon}\right)^{-\frac{1}{\epsilon}}$$

- Let us find MPK and MPN and then the elasticity of substitution between K and N.

$$\begin{aligned} MPK &= \frac{\partial Y}{\partial K} = -\frac{1}{\epsilon}\left(\omega K^{-\epsilon} + (1-\omega) N^{-\epsilon}\right)^{-\frac{1}{\epsilon}-1} \omega(-\epsilon) K^{-\epsilon-1} \\ MPN &= \frac{\partial Y}{\partial N} = -\frac{1}{\epsilon}\left(\omega K^{-\epsilon} + (1-\omega) N^{-\epsilon}\right)^{-\frac{1}{\epsilon}-1} (1-\omega)(-\epsilon) N^{-\epsilon-1} \end{aligned}$$

Solution to the Last Exercise

$$
\begin{aligned}
MRS_{K,N} &= \frac{(1-\omega)N^{-\epsilon-1}}{\omega K^{-\epsilon-1}} = \frac{(1-\omega)}{\omega}\left(\frac{K}{N}\right)^{\epsilon+1} \\
\frac{K}{N} &= \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\epsilon+1}} (MRS_{K,N})^{\frac{1}{\epsilon+1}} \\
\sigma_{K,N} &= \frac{MRS_{K,N}}{\left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\epsilon+1}} MRS_{K,N}^{\frac{1}{\epsilon+1}}} \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\epsilon+1}} \frac{1}{\epsilon+1} (MRS_{K,N})^{\frac{1}{\epsilon+1}-1} \\
\sigma_{K,N} &= \frac{1}{\epsilon+1}
\end{aligned}
$$

References

- Abel, A. B., B. S. Bernanke and R. D. Kneebone, *Macroeconomics*, Pearson, 5th Canadian Edition, 2009.
- Sydsaeter, K. and P. J. Hammond, *Mathematics for Economic Analysis*, Prentice Hall, 1995.