Liquidity Effects of Open-Market Operations

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Abstract

I construct a micro-founded model of monetary economy, in which different government-issued assets provide liquidity services. Assets in the model differ in terms of liquidity services they provide, and money is the most liquid asset. The central bank can implement policies by changing the supply of money and other assets. I show that the central bank can change the overall liquidity and welfare in the economy by changing the relative supply of assets with different liquidity characteristics. My model also enables me to study the welfare effects of a restriction on trade with government bonds.

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1 Introduction

What are the liquidity effects of open-market operations (OMO)? To answer this question, I construct a microfounded model of monetary economy where households can trade goods with different type of assets. I use the theoretical model to show that within a specific set of parameters OMO can affect welfare. In these cases, the central bank can affect the amount of produced goods in the economy by trading less liquid assets with money. There is an optimum supply of bonds that maximizes welfare in the economy. In an economy with two types of government-issued assets with different liquidity characteristics, the central bank is able to use OMO to change the liquidity characteristics of agents’ portfolio. OMO can improve welfare by increasing (decreasing) liquidity in periods of low (high) liquidity.

During the period 2008-2011 many central banks implemented a series of unconventional monetary policies in response to the financial crisis. A major part of these policies has been large scale asset purchase programs (known as quantitative easing). Bank of Japan implemented similar policies during 2000-2006. These programs are basically open-market operations that change the size or the composition of the central bank’s balance sheet. Similarly the Fed implemented two sets of policies in response to the financial crisis: 1-Quantitative easing: expanding the asset size of the central bank’s balance sheet by purchasing conventional assets\(^1\) and issuing reserves on the liability side. 2-Credit easing: changing the composition of the Fed’s balance sheet by selling conventional assets and buying unconventional assets\(^2\). Policy makers point to two channels that quantitative easing can affect the real economy (e.g. Bernanke and Reinhart (2004)): 1-Signaling lower interest rate in the long term. 2-Increasing demand for other assets in the economy and decreasing yield on these assets\(^3\).

The literature on OMO and quantitative easing falls in two categories. First, there are earlier papers who show that OMO are irrelevant for the real economy. Wallace (1981) uses

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\(^1\)Mainly treasuries in US and bonds and term purchase and resale agreements for private sector in Canada

\(^2\)Credit easing is also called asset sterilizing program or operation twist

\(^3\)Agents rebalanced their portfolios towards other assets in the economy
a Modigliani-Miller argument to show that OMO are irrelevant. Many papers follow Wallace (1981) and show that OMO do not affect the real economy. Second, papers that show OMO can affect the real economy. Andres et al. (2004) study the portfolio rebalancing channel by building a New-Kaynsian model of monetary economy. Woodford and Curdia (2010) study the effects of size of the central bank’s balance sheet in a New Kaynsian model. They find out that if we do not take into account the signaling channel, pure quantitative easing has no real effects. Auerbach and Obstfeld (2005) study the replacement of the interest-bearing government debt with non-interest-bearing currency or reserves on central bank’s balance sheet and find out that quantitative easing has real effects. While these papers answer some of the questions concerning OMO and quantitative easing, they use reduced form models (e.g. money in the utility function and sticky prices).

Kiyotaki and Moore (2008) study a model of monetary economy with differences in liquidity across assets. They show that OMO are effective when central bank purchases the asset with partial resaleability and a substantial liquidity premium during negative liquidity shocks. Illiquid assets in their model are mainly capital and securities that are issued based on capital. Salas (2009) uses a similar framework to show that Kiyotaki and Moore (2008) can qualitatively explain some policy implications, but quantitatively, accounts for a small variation due to aggregate shocks.

I expand the existing literature on the effects of OMO by building a micro-founded model of monetary economy. The basic model is a variation of Shi (2008), who uses a similar framework to study the legal restriction on trade with nominal bonds. Agents can trade with different government-issued assets that provide different liquidity services. Contrary to the argument in Shi (2008), here the argument is not based on parameters in the utility function. He assumes that agents can use bonds to trade certain types of goods that yield a higher utility when consumed. Here, consumption of different goods yield the same amount of utility.

In section 2, I develop a micro-founded model of monetary economy. I then study the
optimal choices and discuss different equilibria and welfare effects of different policies. In section 3, I study the model with two types of government issued assets. Section 4 offers concluding remarks and possible extensions.

2 Model environment

2.1 Households

Time is discrete, and there are different types of households \((H \geq 3)\). Each household consumes a good which is produced by some other type of household, type \(h\) household consumes good \(h\), but produces good \(h + 1\). There is no double coincidence of wants, and goods are perishable. Each household consists of a large number of members (measure one). There is perfect consumption insurance between household members, members of a household share consumption and regard utility of the household as the common objective\(^4\).

Household divides its members to three groups: sellers/ producers (measure \(\sigma\)), buyers (measure \(n - \sigma\)) and leisure seekers (measure \(n\)). Household chooses \(n\) and \(\sigma\) is fixed\(^5\).

2.2 Markets and matching

There are two markets in this economy, a centralized market for assets and a decentralized market for goods. In the centralized market for assets government bond is sold for money. In the decentralized market for goods there exists search frictions. Buyers and sellers of different households are randomly matched in pairs. The number of matches for each household is \(\alpha N\), where \(\alpha\) is a parameter of the environment and \(N\) is the aggregate number of traders in the market (Lower case letters are choices of the household under consideration and capital letters are per capita variables that individual households cannot affect). According to this matching technology, the matching rate for buyers is \(\frac{\alpha N}{N-\sigma}\) and the matching rate for the

\(^4\)The large household structure and the assumptions of perfect consumption insurance make the distribution of assets among agents degenerate.

\(^5\)This assumption is for simplicity, other papers have tried ...
The matching process is shown in figure 1. Because of the assumed structure of the environment a successful match is between buyers of household "h" and sellers of household "h + 1".

Figure 1: Matching process in a 3 household economy

2.3 Trade

In the centralized market for assets households trade government bond for money. In the decentralized market for goods household members trade goods for money or government bond. Trade history is private information, therefore there is no credit. After household members are matched a matching shock determines whether they can trade only with money (with probability $1 - l$, this trade is indexed by “m”), or money along with bond (with probability $l$, this trade is indexed by “b”).

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6I assume $\alpha$ is low enough that matching rates are less than 1

7Different types of matches can be interpreted as “monitored” and “non-monitored” matches as in Williamson (2011)
2.4 Timing

The timing of the events is shown in figure 2.

At the beginning of each period asset market opens. Households redeem nominal bond from the previous period for 1 unit of money, trade assets for money, receive transfer $T$ and adjust their portfolios to $(m, b)$. The asset market is closed until the beginning of the next period. Households choose the amount of total traders $n$ and give buyers instructions on how to trade in different trades (Goods: $q_i$, assets: $x_i, i \in \{m,b\}$). Buyers and sellers search in the goods market and match according to the linear matching function. Matched sellers produce and trade and then bring goods and assets back to the household and members of the household share consumption.

The representative household solves the following maximization problem:

$$v(m, b) = \max_{c, q, x, n, m+1, b+1} \left\{ u(c_m) - \alpha N(1 - l)\psi(Q_m) + u(c_b) - \alpha N l \psi(Q_b) + h(1 - n) + \beta v(m+1, b+1) \right\}$$

(1)

Households choose consumption, terms of trade, number of traders and the asset portfolio for the next period to maximize the above value function. The utility from trade is the
sum of the net utility in each type of trade. In each trade household share the utility from consumption of the purchased goods and the cost of production of the sold goods\footnote{Since buyers have all the bargaining power the sold amount is shown by capital letters $Q_i$.}. Consumption in each type of trade is the matching rate times the total amount of goods bought by the buyers in that trade:

$$c_b = \alpha N \frac{(n - \sigma)(N - \sigma)}{(N - \sigma)} lq_b$$

Matching rate

$$c_m = \alpha N (n - \sigma)(1 - l) \frac{(N - \sigma)}{q_m}$$

In each type of trade, buyers are constrained by the portfolio of assets that they have. In a money trade, buyers are constrained by the amount of money they have:

$$x_m \leq \frac{m}{n - \sigma} \quad (2)$$

In a money and bond trade, buyers are constrained by the total portfolio of assets they carry:

$$x_b \leq \frac{m + b}{n - \sigma} \quad (3)$$

Lets define $\omega_i, \ i \in \{m, b\}$ as the marginal value of assets:

$$\omega_m = \frac{\beta}{\gamma} \frac{\partial}{\partial m_{+1}} v(m, b)$$

$$\omega_b = \frac{\beta}{\gamma} \frac{\partial}{\partial b} v(m, b)$$

$\Omega_m$ is the per capita value of money in the economy. In each trade, sellers sell goods for a portfolio of assets, which has a marginal value of $\Omega_m$. Seller’s surplus is $x_i \Omega_m - \psi(q_b) \ i \in \{m, b\}$. Since buyers have all the bargaining power the offer sets sellers’ surplus to 0. Thus,
the participation constraint is:

\[ x_i = \frac{\psi(q_i)}{\Omega_i} \quad i \in \{m, b\} \]  \hspace{1cm} (4)

Household face the following law of motion for assets:

\[
m^{+1} + s^{+1}b_{+1} + T^{+1} = \frac{1}{\gamma} \begin{bmatrix}
m + b + \alpha N L X^b + \alpha N (1 - l) X^m - \frac{\alpha N (n - \sigma)}{N - \sigma} l x^b - \frac{\alpha N (n - \sigma)}{N - \sigma} (1 - l) x^m \\
\text{Sellers' assets} \\
\text{Buyers' assets}
\end{bmatrix}
\]  \hspace{1cm} (5)

Money balance plus the amount spent on the assets in the next period and the tax and transfers is equal to the portfolio of assets in the current period plus the assets that the sellers bring back minus the assets that buyers have spent on their purchases of goods.

### 2.5 Optimal choices

The first order condition for \( q_i \) is:

\[
u'(c_i) = (\omega^m + \lambda^i) \frac{\psi'(q_i)}{\Omega} \quad i \in \{b, m\} \]  \hspace{1cm} (6)

I can solve for bond prices by taking the first order conditions with respect to \( b \):

\[ b^{+1} : s^{+1} = \frac{\omega_b}{\omega_m} \]  \hspace{1cm} (7)

Households’ choice of the measure of traders (\( n \)) solves the following:

\[
k'(1 - n) = \frac{\alpha N}{N - \sigma} \left[ l u'(c_b)(q_b - \frac{\psi(q_b)}{\psi'(q_b)}) + (1 - l) u'(c_m)(q_m - \frac{\psi(q_m)}{\psi'(q_m)}) \right] \]  \hspace{1cm} (8)

The envelope conditions for \( m^{+1}, b^{+1} \) are:
\[
m^{+1} : \frac{\gamma}{\beta} \omega_{m} = \omega_{m} + \frac{\alpha N l}{N - \sigma} \lambda^{b} + \frac{\alpha N(1 - l)}{N - \sigma} \lambda^{m}
\]

\[
b^{+1} : \frac{\gamma}{\beta} \omega_{b} = \omega_{m} + \frac{\alpha N l}{N - \sigma} \lambda^{b}
\]

At the end of each period each unit of bond is redeemed for a unit of money, therefore the value of an asset is the value of money in the next period plus the transaction services that the asset provides accounting for discounting and inflation. Money provides transaction service in all types of trades, but bond in certain types of trade.

### 2.6 Definition of the equilibrium

**Definition 1** An equilibrium is households' choices \((c_{i} \in \{m, b\}, q_{i} \in \{m, b\}, x_{i} \in \{m, b\}, n, m^{+1}, b^{+1}, b^{+1})\), the value function \((v(m, b))\), shadow value of assets \((\omega_{m}, \omega_{b})\), asset price \((s)\), and other HHs' choices such that:

1. Household choices are optimal
2. The choices and shadow prices are the same across households.
3. Bonds market clear
4. Positive and finite values of assets
5. Stationarity

### 2.7 Welfare analysis

The envelope conditions show that the only point that all of the constraints are non-binding is where \(\gamma = \beta\).

**Lemma 1** At Friedman rule \((\gamma = \beta)\), \(\lambda_{m} = \lambda^{b} = 0\). For \(\gamma > \beta \exists i \in \{m, b\}\) such that \(\lambda_{i} > 0\).
In order to study open-market operations, let’s define the ratio of stock of bonds to stock of money as:

\[ z = \frac{B}{M} \]

I define the welfare function as the utility function of a representative household:

\[ w = u(c_b) - \alpha N l \psi(Q_b) + u(c_m) - \alpha N (1 - l) \psi(Q_m) \] (11)

By using the above measure of welfare I can study the welfare effects of policies. We have 4 types of equilibrium based on the set of liquidity constraints that are binding. The only case where all of the liquidity constraints are non-binding is at the Friedman rule. Friedman rule is shown to be optimal in a wide variety of models. As the next theorem shows Friedman rule is optimal in this framework.

**Theorem 2** Friedman rule is optimal.

Since buyers’ bargaining power is 1, comparing to planner’s choice, households send too many buyers. Increasing \( \gamma \) punishes unmatched buyers and the representative households. On the other hand inflation decreases the amount of goods in a specific trade. The former effect is known as the extensive margin of trade and the later the intensive margin of trade. Both intensive margin (\( q \)) and extensive margin (\( n \)) decrease with \( \gamma \). Planner chooses the lowest possible level for \( \gamma \) to maximize welfare.

Based on the set of liquidity constraints that are binding, we can have 4 types of equilibria. As shown before, an equilibrium where both of the liquidity constraints are non-binding can only happen in the Friedman rule and it is efficient. In the appendix I have characterized different types of equilibria and proved the following theorem:

**Theorem 3** open-market operations can only have welfare effects when both of the liquidity
Table 1: Results

<table>
<thead>
<tr>
<th>Case</th>
<th>$\lambda^m$</th>
<th>$\lambda^b$</th>
<th>$s$</th>
<th>$\frac{\partial W}{\partial z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>0</td>
<td>$\frac{\beta}{s}$</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>0</td>
<td>+</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>III</td>
<td>+</td>
<td>+</td>
<td>$\frac{\beta}{s} &lt; s &lt; 1$</td>
<td>+, −</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

constraints are binding. The properties of the equilibrium is shown in table 1.

2.8 Numerical example

Using the following functional forms and parameters, I have simulated the model. The calculations of the different types of equilibria is in the appendix and the results of the simulation are reported in figures 2.8 and 2.8:

$$u(c) = \log(c); \psi(q) = \frac{q^2}{2}; h(n) = 2a(n)^{1/2}$$

Table 2 shows the properties of the equilibrium for different amounts of liquidity parameter ($l$).

Table 2: Results

<table>
<thead>
<tr>
<th>Case</th>
<th>$\lambda^m$</th>
<th>$\lambda^b$</th>
<th>Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>0</td>
<td>$l &gt; \bar{l} = \frac{(\frac{\beta}{s} - 1)(N-\sigma) + \alpha N}{(2+z)\alpha N}$</td>
</tr>
<tr>
<td>II</td>
<td>0</td>
<td>+</td>
<td>$\underline{l} &lt; l &lt; \bar{l}$</td>
</tr>
<tr>
<td>III</td>
<td>+</td>
<td>+</td>
<td>$\underline{l} \leq l \leq \bar{l}$</td>
</tr>
</tbody>
</table>

Figure 3 shows the range of parameters for different types of equilibrium. For high enough values of $l$ the liquidity constraint for money binds, and the constraint on trade with bond is slack. For low enough $l$ the constraint on bond binds, and for $\underline{l} < l < \bar{l}$ both of the constraints are binding.

As I have proven in theorem 3, increasing $z$ will only affect welfare when we are in type
Figure 3: Range of parameters for different types of equilibrium
III equilibrium with both liquidity constraints binding. Figure 4 shows the welfare properties of the equilibrium when both for values of $l$ that causes both liquidity constraints binding.

![Figure 4: Welfare](image)

### 2.8.1 The case for legal restriction on trading with bond

As shown in figure 2.8, for a range of parameters increasing $z$ from zero increases the overall welfare. Similar to the argument in Shi (2008) an increase in $z$ can be interpreted as imposing legal restriction on trade with bonds. An economy with zero supply of bond is a pure monetary model. When $z$ rises from zero its exactly as we have imposed legal restriction on trade with bond. As figure 2.8 shows for a range of parameters this can improve welfare.

Contrary to the argument in Shi (2008), the argument here is not based on parameters
in the utility function. Shi (2008) assumes that agents can trade certain types of goods that yield a higher utility when consumed. Here, consumption of different goods yield the same amount of utility.

3 Model with two bonds

The matching shock works as follows:

- **Shock n**: With probability $l$ agents can trade with money, short term bond and long term bond

- **Shock s**: With probability $k$ agents can trade with money and short term bond

- **Shock l**: With probability $1 - l - k$ agents can trade only with money

In each trade buyers make take-it-or-leave-it offer on the amount of goods $q_{i\in\{n,s,l\}}$ and the portfolio of assets to be traded for goods $x_{i\in\{n,s,l\}}$. Note that the portfolio of assets could be a combination of money, short-term bond and long-term bond depending on what type of trade.

Household solves the following maximization problem:

$$v(m, b_l, b_s) = \max_{c_{i\in\{n,s,l\}}, q_{i\in\{n,s,l\}}, x_{i\in\{n,s,l\}}, n, m^{+1}, b_l^{+1}, b_s^{+1}} \left\{ u(c_l) - \alpha N (1 - l - k) \psi(Q_l) \right\}$$

Money trade

$$+ \ u(c_s) - \alpha N k \psi(Q_s)$$

Money+short term bond

$$+ \ u(c_n) - \alpha N l \psi(Q_n)$$

Money+short term+long term bond

$$+ h(1 - n) + \beta v(m^{+1}, b_l^{+1}, b_s^{+1}) \right\}$$

(12)

$$c_n = \frac{\alpha N}{(N - \sigma)} (n - \sigma) l_q_r$$

Matching rate
\[ c_s = \frac{\alpha N(n - \sigma)k}{(N - \sigma) - q_s} \]

\[ c_i = \frac{\alpha N(n - \sigma)(1 - k - l)}{(N - \sigma)}q_i \]

In each type of trade, buyers are constrained by the portfolio of assets that they have:

\[ x_n \leq \frac{m + b_l + b_s}{n - \sigma} \]  \hspace{1cm} (13)

\[ x_s \leq \frac{m + b_s}{n - \sigma} \]  \hspace{1cm} (14)

\[ x_l \leq \frac{m}{n - \sigma} \]  \hspace{1cm} (15)

Lets define \( \omega_i \quad i \in \{m, b_s, b_l\} \) as the marginal value of assets:

\[ \omega_m = \frac{\beta}{\gamma} \frac{\partial}{\partial m_{+1}} v(m, b_l, b_s) \]

\[ \omega_{b_s} = \frac{\beta}{\gamma} \frac{\partial}{\partial b_{s+1}} v(m, b_l, b_s) \]

\[ \omega_{b_l} = \frac{\beta}{\gamma} \frac{\partial}{\partial b_{l+1}} v(m, b_l, b_l) \]

Since buyers have all the bargaining power the offer sets sellers’ surplus to 0. Thus, the participation constraint is:

\[ x_i = \psi(q_i)/\Omega_m \quad i \in \{l, s, n\} \]  \hspace{1cm} (16)

Household face the following law of motion for assets:
\[ m^{+1} + s^{+1}_l b^{+1}_l + s^{+1}_s b^{+1}_s + T^{+1} = \]
\[
\frac{1}{\gamma}[m + b_l + b_s + \alpha NlX^n + \alpha NkX^s + \alpha N(1 - k - l)X^l]
\]
\[
= \frac{\alpha N(n - \sigma)}{N - \sigma} l x^n - \frac{\alpha N(n - \sigma)}{N - \sigma} (1 - l - k)x^l
\]

Buyers’ assets

\[
- \frac{\alpha N(n - \sigma)}{N - \sigma} k x^s - \frac{\alpha N(n - \sigma)}{N - \sigma} (1 - l - k)x^l
\]

Sellers’ assets

\[ (17) \]

3.1 Optimal choices

The first order condition for \( q_i \) is:

\[ u'(c_i) = (\omega^m + \lambda^i) \frac{\psi'(q_i)}{\Omega} \quad i \in l, s, n \]

(18)

I can solve for bond prices by taking the first order conditions with respect to \( b_l^+, b_s^+ \):

\[ b_l^{+1} : s_l^{+1} = \frac{\omega_b}{\omega_m} \]

(19)

\[ b_s^{+1} : s_s^{+1} = \frac{\omega_b}{\omega_m} \]

(20)

The envelope conditions for \( m^{+1}, b_s^{+1}, b_l^{+1} \) are:

\[ m^{+1} : \frac{\gamma}{\beta} \omega_m^{-1} = \omega_m + \frac{\alpha Nl}{N - \sigma} \lambda^n + \frac{\alpha Nk}{N - \sigma} \lambda^s + \frac{\alpha N(1 - l - k)}{N - \sigma} \lambda^l \]

(21)

\[ b_s^{+1} : \frac{\gamma}{\beta} \omega_b^{-1} = \omega_m + \frac{\alpha Nl}{N - \sigma} \lambda^n + \frac{\alpha Nk}{N - \sigma} \lambda^s \]

(22)

\[ b_l^{+1} : \frac{\gamma}{\beta} \omega_l^{-1} = \omega_m + \frac{\alpha Nl}{N - \sigma} \lambda^n \]

(23)

At the end of each period each asset is redeemed for a unit of money, therefore the value of
an asset is the value of money in the next period plus the transaction services of each asset accounting for discounting and inflation. Money provides transaction service in all types of trades, but bonds in certain types of trade.

**Theorem 4** Friedman rule ($\gamma = \beta$) is optimal.

In order to study different equilibria and welfare effects of policy, I will focus on the log-utility and quadratic cost functions:

$$u(c) = \log(c)$$

$$\psi(q) = \frac{q^2}{2}$$

**Lemma 2** With $u(c) = \log(c)$ and $\psi(q) = \frac{q^2}{2}$, $N = n$ is the same for different cases of equilibrium.

With log-utility and quadratic cost functions first order condition for $n$ becomes:

$$h'(1 - N) = \frac{3}{2(N - \sigma)}$$

In all of the cases. These functional forms shut down variations in the extensive margin of trade. The proof to this lemma is provided in the appendix.

In the next theorem I characterize these different types of equilibrium:

**Theorem 5** There exists 8 types of equilibrium depending on the set of binding liquidity constraints. The properties of these equilibriums are summarized in table 1.

The proof for the above theorem is provided in the appendix. As the theorem shows, in equilibriums with at least two binding liquidity constraints, there exists a set of parameters that a policy of quantitative easing is welfare improving. In these cases replacing less liquid bonds in household portfolio with liquid money would increase the intensive margin of trade, and welfare\(^9\).

\(^9\)Note that with these specific functional forms the extensive margin of trade remains constant.
Table 3: Results

<table>
<thead>
<tr>
<th>Case</th>
<th>Prices</th>
<th>$\frac{\partial q}{\partial z_i}$</th>
<th>$\frac{\partial n}{\partial z_i}$</th>
<th>$\frac{\partial W}{\partial z_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_s = 0$</td>
<td>$\beta/\gamma &lt; s_t = s_s &lt; 1$</td>
<td>$\frac{\partial q}{\partial z_i} = 0$</td>
<td>$\frac{\partial n}{\partial z_i} &gt; 0$</td>
<td>$\frac{\partial W}{\partial z_i} &gt; = &lt; 0$</td>
</tr>
<tr>
<td>$\lambda^n = 0$</td>
<td>$s_t = \beta/\gamma &lt; s_s &lt; 1$</td>
<td>$\frac{\partial q}{\partial z_i} = 0$</td>
<td>$\frac{\partial n}{\partial z_i} &lt; 0$</td>
<td>$\frac{\partial W}{\partial z_i} &gt; = &lt; 0$</td>
</tr>
<tr>
<td>$\lambda^l = 0$</td>
<td>$\beta/\gamma &lt; s_t &lt; s_s = 1$</td>
<td>$\frac{\partial q}{\partial z_i} = 0$</td>
<td>$\frac{\partial n}{\partial z_i} &lt; 0$</td>
<td>$\frac{\partial W}{\partial z_i} &gt; = &lt; 0$</td>
</tr>
<tr>
<td>$\lambda^s = \lambda^l = 0$</td>
<td>$s_t = s_s = 1$</td>
<td>$\frac{\partial q}{\partial z_i} = 0$</td>
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<tr>
<td>$\lambda^l = \lambda^n = 0$</td>
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</tr>
<tr>
<td>$\lambda^i &gt; 0$</td>
<td>$\beta/\gamma &lt; s_t &lt; s_s &lt; 1$</td>
<td>$\frac{\partial q}{\partial z_i} &gt; = &lt; 0$</td>
<td>$\frac{\partial n}{\partial z_i} &gt; = &lt; 0$</td>
<td>$\frac{\partial W}{\partial z_i} &gt; = &lt; 0$</td>
</tr>
</tbody>
</table>

4 Concluding remarks and possible extensions

Adding illiquid nominal bonds to a microfounded model of monetary economy allows me to study the welfare effects of open-market operations (OMO). I show that the central bank can change the overall liquidity and welfare in the economy by changing the relative supply of assets with different liquidity characteristics. My model also enables me to study the welfare effects of a restriction on trade with government bonds. I show that in a non-empty set of parameters restricting trade with government bonds can affect welfare.

A possible extension to this model is to study the same framework with a high enough sellers bargaining power. In this setup the optimal monetary policy would be greater than the Friedman rule ($\gamma^* > \beta$)$^{10}$. In this framework OMO will have welfare effects at the optimal monetary policy.

$^{10}$ $\left(\frac{\partial q}{\partial \gamma}\right) < 0$ and $\left(\frac{\partial n}{\partial \gamma}\right) > 0$
Appendix

A Model with one bond

I characterize 3 cases of the equilibria based on the set of liquidity constraints that are binding. In all of these cases at least one of the constraints are binding. The case where none of them are binding only happens at the Friedman rule, and it is shown to be efficient.

Case I: \( \lambda^m > 0 \) and \( \lambda^b = 0 \)

The first order conditions are:

\[
u'(c_b) = \psi'(q_b) \tag{24}\]

\[
u'(c_m) = \psi'(q_m) + \frac{\psi'(q_m)}{\Omega^m} \lambda^m \tag{25}\]

I can rewrite the above equation as:

\[\lambda^m = \frac{u'(c_m)}{\psi(q_b)} - 1)\Omega^m\]

Envelope condition gives:

\[
\frac{\gamma}{\beta} \omega^m = \omega^m + \frac{\alpha N(1 - l)}{N - \sigma} \left[\frac{u'(c_m)}{\psi(q_m)} - 1\right]\Omega^m
\]

By applying stationarity, I can write the envelope as:

\[
\frac{\gamma}{\beta} - 1 = \frac{\alpha N(1 - l)}{N - \sigma} \left[\frac{u'(c_m)}{\psi(q_m)} - 1\right]
\tag{26}
\]

The price for nominal bond is:

\[
s = \frac{\omega^b}{\omega^m} = \frac{\beta}{\gamma}
\]
Case II: $\lambda^m = 0$ and $\lambda^b > 0$

The first order conditions are:

$$u'(c_m) = \psi'(q_m) \quad (27)$$

$$\lambda^b = \left(\frac{u'(c_b)}{\psi'(q_b)} - 1\right)\Omega^b$$

Envelope condition gives:

$$\frac{\gamma}{\beta} \omega^m_1 = \omega^m + \frac{\alpha N l}{N - \sigma} \left[ \frac{u'(c_b)}{\psi'(q_b)} - 1 \right] \Omega^b$$

The price for the nominal bond is 1. By applying stationarity, I can write the envelope as:

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N l}{N - \sigma} \left[ \frac{u'(c_b)}{\psi'(q_b)} - 1 \right] \quad (28)$$

It is straightforward to see that changing $z$ would not affect households’ decision and welfare when at least one of the liquidity constraints is not binding.

Case III: $\lambda^m > 0$ and $\lambda^b > 0$

The first order conditions are:

$$\lambda^m = \left(\frac{u'(c_m)}{\psi'(q_m)} - 1\right)\Omega^m \quad (29)$$

$$\lambda^b = \left(\frac{u'(c_b)}{\psi'(q_b)} - 1\right)\Omega^m \quad (30)$$

Envelope conditions give:
\[
\frac{\gamma}{\beta} \omega_{-1}^m = \omega^m + \frac{\alpha N l}{N - \sigma} \left[ \left( \frac{u'(c_b)}{\psi'(q_b)} - 1 \right) \Omega^m \right] + \frac{\alpha N(1 - l)}{N - \sigma} \left[ \left( \frac{u'(c_m)}{\psi'(q_m)} - 1 \right) \Omega^m \right]
\]

\[
\frac{\gamma}{\beta} \omega_{-1}^m = \omega^m + \frac{\alpha N l}{N - \sigma} \left[ \left( \frac{u'(c_b)}{\psi'(q_b)} - 1 \right) \Omega^m \right]
\]

By applying stationarity, I can write the envelope as:

\[
\frac{\gamma}{\beta} s - 1 = \frac{\alpha N l}{N - \sigma} \left[ \frac{u'(c_b)}{\psi'(q_b)} - 1 \right] \quad (31)
\]

\[
\frac{\gamma}{\beta} - 1 = \frac{\alpha N l}{N - \sigma} \left[ \frac{u'(c_b)}{\psi'(q_b)} - 1 \right] + \frac{\alpha N(1 - l)}{N - \sigma} \left[ \frac{u'(c_m)}{\psi'(q_m)} - 1 \right] \quad (32)
\]

**B  Numerical example for model with one bond**

I solve the model for the following functional forms:

\[u(c) = \log(c)\]

\[\psi(q) = \frac{q^2}{2}\]

\[h(n) = 2an^{1/2}\]

Now I solve the model for 3 different cases of liquidity constrains:

**Case I:**

\[q_b = \frac{1}{(\alpha N l)^{1/2}}\]

\[q_m = \frac{1}{\left( \frac{\gamma}{\beta} - 1 \right)(N - \sigma) + \alpha N(1 - l) \right)^{1/2}}\]
\[ \frac{a}{(1 - N)^{1/2}} = N - \sigma \]

By using the constraints it is straightforward to show that this equilibrium happens for high enough \( l \):

\[ l > \bar{l} = \frac{(\frac{2}{3} - 1)(N - \sigma) + \alpha N}{(2 + z)\alpha N} \]

**Case II:**

\[
q_m = \frac{1}{(\alpha N(1 - l))^{1/2}}
\]

\[
q_b = \frac{1}{\left(\left(\frac{2}{3} - 1)(N - \sigma) + \alpha Nl\right)^{1/2}}
\]

\[ \frac{a}{(1 - N)^{1/2}} = N - \sigma \]

By using the constraints it is straightforward to show that this equilibrium happens for low enough \( l \):

\[ l < \bar{l} = \frac{\alpha N - (\frac{2}{3} - 1)(N - \sigma)(1 + z)}{\alpha N(2 + z)} \]

**Case III:**

\[
q_m = \frac{1}{\left(\left(\frac{2}{3}(1 - s)(N - \sigma) + \alpha N(1 - l)\right)^{1/2}}
\]

\[
q_b = \frac{1}{\left(\left(\frac{2}{3}s - 1)(N - \sigma) + \alpha Nl\right)^{1/2}}
\]

\[
s = \frac{\frac{2}{3}(N - \sigma) + \alpha N(1 - l) + (1 + z)(N - \sigma - \alpha Nl)}{\frac{2}{3}(2 + z)(N - \sigma)}
\]

\[ \frac{a}{(1 - N)^{1/2}} = N - \sigma \]
C Model with two bonds

Define:

\[ \zeta(q_i) = \psi'(q_i)q_i - \psi(q_i) \quad i \in \{l, s, n\} \]

In what follows I solve the problem in different cases of equilibrium.

Cases:

I: \( \lambda^s = 0 < \lambda^n, \lambda^l \)

From the envelope conditions it follows:

\[ s_s = s_l < 1 \]

The first order conditions are:

\[ u'(c_s) = \psi'(q_s) \quad (33) \]

\[ \frac{\gamma}{\beta} s_s - 1 = \frac{\alpha N l}{N - \sigma} \left[ \frac{u'(c_n)}{\psi'(q_n)} - 1 \right] \quad (34) \]

\[ \frac{\gamma}{\beta} (1 - s_s) = \frac{\alpha N (1 - l - k)}{N - \sigma} \left[ \frac{u'(c_l)}{\psi'(q_l)} - 1 \right] \quad (35) \]

\[ h'(1 - N) = \left( \frac{\gamma}{\beta} s_s - 1 + \frac{\alpha N l}{N - \sigma} \right) \zeta(q_n) + \frac{\alpha N k}{N - \sigma} \zeta(q_s) \]

\[ \left( \frac{\gamma}{\beta} (1 - s_s) + \frac{\alpha N (1 - k - l)}{N - \sigma} \right) \zeta(q_l) \quad (36) \]

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):

\[ 1/q_s^2 = \alpha N k \]

\[ 1/q_n^2 = \left( \frac{\gamma}{\beta} s_s - 1 \right) (N - \sigma) + \alpha N l \]
\[ 1/q_t^2 = \frac{\gamma}{\beta}(1 - s_s)(N - \sigma) + \alpha N(1 - l - k) \]

\[ h'(1 - N) = \frac{3}{2(N - \sigma)} \]

The solution to the above equations is:

\[ s_s = 1 + \frac{\beta \alpha N(1 - l - k) + \beta/(1 + z_s + z_l)(1 - \alpha N)}{2 + z_l + z_s} \]

\[ 1/q_t^2 = \frac{\alpha N(1 - l - k) + \gamma/\beta(N - \sigma) + \alpha N l - (1 + z_l + z_s)(N - \sigma)}{2 + z_l + z_s} \]

\[ 1/q_l^2 = \frac{1 + z_l + z_s}{2 + z_l + z_s}(\alpha N(1 - k) + (N - \sigma)(\gamma/\beta - 1)) \]

Using some algebra I can solve for the criteria for this equilibrium:

\[ \frac{1 + z_l + z_s}{(1 + z_l + z_s)(2 + z_s) + 1 + z_s} \left(1 + \frac{N - \sigma}{\alpha N}(\gamma/\beta - 1)\right) \leq k \] (37)

\[ \frac{(1 + z_l + z_s)(1 + \frac{N - \sigma}{\alpha N}(\gamma/\beta - (1 + z_s + z_l))))}{(1 + z_l + z_s)(2 + z_s) + 1 + z_s} \leq k \] (38)

The left hand side of the second equation is greater than the first.

\( \Pi: \lambda^n = 0 < \lambda^s, \lambda^l \)

\[ s_l = \beta/\gamma < s_s < 1 \]

\[ u'(c_n) = \psi'(q_n) \] (39)

\[ \frac{\gamma}{\beta}s_s - 1 = \frac{\alpha N k}{N - \sigma} \left[ u'(c_s) - 1 \right] \] (40)

\[ \frac{\gamma}{\beta}(1 - s_s) = \frac{\alpha N(1 - l - k)}{N - \sigma} \left[ u'(c_l) - 1 \right] \] (41)
\[ h'(1 - N) = \frac{\alpha N l}{N - \sigma} \zeta(q_n) + (\frac{\gamma}{\beta} s_s - 1 + \frac{\alpha N k}{N - \sigma}) \zeta(q_s) + (\frac{\gamma}{\beta} (1 - s_s) + \frac{\alpha N (1 - k - l)}{N - \sigma}) \zeta(q_l) \]  

(42)

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):

\[
1/q_s^2 = (\frac{\gamma}{\beta} s_s - 1)(N - \sigma) + \alpha N k
\]

\[
1/q_n^2 = \alpha N l
\]

\[
1/q_l^2 = \frac{\gamma}{\beta} (1 - s_s)(N - \sigma) + \alpha N (1 - l - k)
\]

\[
h'(1 - N) = \frac{3}{2(N - \sigma)}
\]

The solution to the above equations is:

\[
s_s = \frac{1 + \frac{\alpha N (1 - l - k)}{N - \sigma} \beta/\gamma - (\alpha N k - \beta/\gamma)(1 + z_s)}{2 + z_s}
\]

\[
1/q_s^2 = \frac{\alpha N (1 - l) + (1 + z_s)(\alpha N K(1 - \gamma/\beta(N - \sigma)) - (N - \sigma)) - (N - \sigma)(\gamma/\beta - 1)}{2 + z_s}
\]

\[
1/q_l^2 = \frac{1 + z_s}{2 + z_s} \frac{\gamma/\beta(N - \sigma)(1 - \alpha N k) + \alpha N (1 - l - k) - (N - \sigma)}
\]

Using some algebra I can solve for the criteria for this equilibrium:

\[
k \leq \frac{(1 + z_s + z_l)(2 + z_s) + 1 + z_s)(1 + z_s)(1 - \frac{N - \sigma}{\alpha N} (\gamma/\beta - 1))}{(1 + z_s)^2(1 - \gamma/\beta(N - \sigma))}
\]

(43)

\[
\frac{(1 + \frac{N - \sigma}{\alpha N} (\gamma/\beta - 1))(1 + z_s) - ((2 + z_s)(1 + z_l + z_s) + 1 + z_s)}{(1 + \gamma/\beta(N - \sigma))(1 + z_s)} \leq k
\]

(44)

III: \( \lambda' = 0 < \lambda^o, \lambda^s \)
\[ s_t < s_s = 1 \]

\[ u'(c_t) = \psi'(q_t) \quad (45) \]

\[ \frac{\gamma}{\beta} s_t - 1 = \frac{\alpha N l}{N - \sigma} \left[ u'(c_n) - 1 \right] \quad (46) \]

\[ \frac{\gamma}{\beta} (1 - s_t) = \frac{\alpha N K}{N - \sigma} \left[ u'(c_s) - 1 \right] \quad (47) \]

\[ h'(1 - N) = \frac{\alpha N(1 - k - l)}{N - \sigma} \zeta(q_t) + \frac{\gamma}{\beta} s_t - 1 + \frac{\alpha N l}{N - \sigma} \zeta(q_n) + \frac{\gamma}{\beta} (1 - s_t) + \frac{\alpha N k}{N - \sigma} \zeta(q_s) \quad (48) \]

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):

\[ 1/q_s^2 = \frac{\gamma}{\beta} (1 - s_s)(N - \sigma) + \alpha N k \]

\[ 1/q_n^2 = \frac{\gamma}{\beta} s_s - 1)(N - \sigma) + \alpha N l \]

\[ 1/q_l^2 = \alpha N(1 - l - k) \]

\[ h'(1 - N) = \frac{3}{2(N - \sigma)} \]

And by some algebra:

\[ s_s = \frac{(1 + z_s)(1 + \frac{\alpha N k}{N - \sigma} \beta/\gamma - \beta/\gamma (\frac{\alpha N k}{N - \sigma} - 1))(1 + z_s + z_l)}{2 + 2z_s + z_l} \]

\[ 1/q_l^2 = \gamma/\beta(N - \sigma) + \alpha N k - \frac{(1 + z_s)(\gamma/\beta(N - \sigma) + \alpha N k) - (\alpha N l - (N - \sigma))(1 + z_l + z_s)}{2 + 2z_s + z_l} \]
\[ \frac{1}{q_n^2} = \frac{\alpha N(1 + z_s)(k + l) - (N - \sigma)(1 + z_t + z_s)(\gamma/\beta - 1)}{2 + 2z_s + z_t} \]

Using some algebra I can solve for the criteria for this equilibrium:

\[ k + l \leq \frac{(2 + 2z_s + z_t) + \frac{N - \sigma}{\alpha N}(1 + z_s + z_t)^2(\gamma/\beta - 1)}{(1 + z_s)(1 + z_s + z_t) + (2 + 2z_s + z_t)} \quad (49) \]

\[ ((1 + z_s)(1 + z_s + z_t) + 2 + 2z_s + z_t)k + (-(1 + z_s)(1 + z_s + z_t) + 2 + 2z_s + z_t)l \leq 2 + 2z_s + z_t - (1 + z_s + z_t)(1 + z_s)(\frac{N - \sigma}{\alpha N}(\gamma/\beta + 1)) \quad (50) \]

**IV:** \( \lambda^s = \lambda^l = 0 < \lambda^n \)

\[ s_t = s_s = 1 \]

\[ u'(c_l) = \psi'(q_l) \quad (51) \]

\[ u'(c_s) = \psi'(q_s) \quad (52) \]

\[ \frac{\gamma}{\beta} - 1 = \frac{\alpha Nl}{N - \sigma} \left[ \frac{u'(c_n)}{\psi'(q_n)} - 1 \right] \quad (53) \]

\[ h'(1 - N) = \frac{\alpha Nk}{N - \sigma} \zeta(q_s) + (\frac{\gamma}{\beta} - 1 + \frac{\alpha Nl}{N - \sigma}) \zeta(q_n) + \frac{\alpha N(1 - k - l)}{N - \sigma} \zeta(q_l) \quad (54) \]

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):
\[
\frac{1}{q_s^2} = \alpha N k
\]
\[
\frac{1}{q_n^2} = \left(\frac{\gamma}{\beta} - 1\right)(N - \sigma) + \alpha N l
\]
\[
\frac{1}{q_l^2} = \alpha N (1 - l - k)
\]
\[
h'(1 - N) = \frac{3}{2(N - \sigma)}
\]

Using some algebra I can solve for the criteria for this equilibrium:

\[
(\frac{\gamma}{\beta} - 1)(\frac{N - \sigma}{\alpha N})(1 + z_s + z_l) \leq (1 + z_s) k - (1 + z_s + z_l) l
\]  
(55)

\[
(2 + z_s + z_l) l + k \leq 1 - (1 + z_s + z_l)(\frac{\gamma}{\beta} - 1)\frac{N - \sigma}{\alpha N}
\]  
(56)

**V:** \(\lambda^s = \lambda^n = 0 < \lambda^l\)

\[
s_l = s_s = \frac{\beta}{\gamma} < 1
\]

\[
u'(c_n) = \psi'(q_n)
\]  
(57)

\[
u'(c_s) = \psi'(q_s)
\]  
(58)

\[
\frac{\gamma}{\beta} - 1 = \frac{\alpha N (1 - k - l)}{N - \sigma} \left[\frac{u'(c_n)}{\psi'(q_n)} - 1\right]
\]  
(59)
\[ h'(1 - N) = \frac{\alpha N l}{N - \sigma} \zeta(q_n) + \left(\frac{\gamma}{\beta} - 1 + \frac{\alpha N (1 - k - l)}{N - \sigma}\right) \zeta(q_l) + \frac{\alpha N k}{N - \sigma} \zeta(q_s) \]  

(60)

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):

\[
\begin{align*}
1/q_s^2 &= \alpha N k \\
1/q_n^2 &= \alpha N l \\
1/q_l^2 &= \left(\frac{\gamma}{\beta} - 1\right)(N - \sigma) + \alpha N (1 - l - k) \\
h'(1 - N) &= \frac{3}{2(N - \sigma)}
\end{align*}
\]

Using some algebra I can solve for the criteria for this equilibrium:

\[
1 + \left(\frac{\gamma}{\beta} - 1\right) \frac{N - \sigma}{\alpha N} \leq (2 + z_s + z_l)l + k 
\]

(61)

\[
1 + \left(\frac{\gamma}{\beta} - 1\right) \frac{N - \sigma}{\alpha N} \leq (2 + z_s)k + l 
\]

(62)

\textbf{VI:} \( \lambda^a = \lambda^l = 0 < \lambda^s \)

\[
s_l = \frac{\beta}{\gamma} < s_s = 1
\]

(63)

\[
u'(c_n) = \psi'(q_n)
\]

\[
u'(c_l) = \psi'(q_l)
\]

(64)
\[
\frac{\gamma}{\beta} - 1 = \frac{\alpha N k}{N - \sigma} \frac{u'(c_s)}{\psi'(q_s)} - 1 \tag{65}
\]

\[
h'(1 - N) = \frac{\alpha N l}{N - \sigma} \zeta(q_n) + \left(\frac{\gamma}{\beta} - 1 + \frac{\alpha N k}{N - \sigma}\right) \zeta(q_s) + \frac{\alpha N (1 - k - l)}{N - \sigma} \zeta(q_l) \tag{66}
\]

Solution for \(u(c) = \log(c)\) and \(\psi(q) = q^2/2\):

\[
1/q_s^2 = \left(\frac{\gamma}{\beta} - 1\right)(N - \sigma) + \alpha N k
\]

\[
1/q_n^2 = \alpha N l
\]

\[
1/q_l^2 = \alpha N (1 - l - k)
\]

\[
h'(1 - N) = \frac{3}{2(N - \sigma)}
\]

Using some algebra I can solve for the criteria for this equilibrium:

\[
(2 + z_s)k + l \leq 1 - (\gamma/\beta - 1) \frac{N - \sigma}{\alpha N} (1 + z_s) \tag{67}
\]

\[
(\gamma/\beta - 1) \frac{N - \sigma}{\alpha N} (1 + z_s) \leq (1 + z_s + z_l)l - (1 + z_s)k \tag{68}
\]

VII: \(0 < \lambda^s, \lambda^l, \lambda^n\)

\[
s_l < s_s < 1
\]

\[
\frac{\gamma}{\beta} s_l - 1 = \frac{\alpha N l}{N - \sigma} \frac{u'(c_n)}{\psi'(q_n)} - 1 \tag{69}
\]
\begin{align}
\frac{\gamma}{\beta}(s_s - s_l) &= \frac{\alpha N k}{N - \sigma} \left[ u'(c_s) - 1 \right] \\
\frac{\gamma}{\beta} (1 - s_s) &= \frac{\alpha N (1 - l - k)}{N - \sigma} \left[ u'(c_l) - 1 \right]
\end{align}

\begin{align}
h'(1 - N) &= \left( \frac{\gamma}{\beta} s_t - 1 + \frac{\alpha N l}{N - \sigma} \right) \zeta(q_t) + \left( \frac{\gamma}{\beta} (s_s - s_t) + \frac{\alpha N k}{N - \sigma} \right) \zeta(q_s) + \\
&\quad \left( \frac{\gamma}{\beta} (1 - s_s) + \frac{\alpha N (1 - l - k)}{N - \sigma} \right) \zeta(q_l)
\end{align}

Solution for \( u(c) = \log(c) \) and \( \psi(q) = q^2/2 \):

\begin{align}
\frac{1}{q_s^2} &= \frac{\gamma}{\beta} (s_s - s_l)(N - \sigma) + \alpha N k \\
\frac{1}{q_n^2} &= \left( \frac{\gamma}{\beta} s_t - 1 \right)(N - \sigma) + \alpha N l \\
\frac{1}{q_l^2} &= \frac{\gamma}{\beta} (1 - s_s)(N - \sigma) + \alpha N (1 - l - k) \\
h'(1 - N) &= \frac{3}{2(N - \sigma)}
\end{align}
References


Salas S. A., Liquidity and monetary policy, working paper (2009).

