Spatial competition with heterogeneous firms

Jonathan Vogel*

October 5, 2006

Abstract

I model horizontal and vertical differentiation with heterogeneous firms. The model nests the standard Hotelling model on the circumference of a circle. Firms play a three-stage game in which they enter and exit in the first stage, locate and choose quality in the second stage, and set their prices in the third stage. All relevant characteristics of firms (quality, price, market share, and profit) are uniquely determined in strict subgame perfect Nash equilibria to the post-entry subgame. The distance between two neighbors is greater than average if and only if the average productivity of the two neighbors is greater than the average productivity in the market. A firm's profit depends only on its own cost parameters and the "softness" of the market in which it sells. Market softness is a simple function of the number of firms in the market and their average productivity. Because of a selection effect, denser markets are associated with more productive firms and tougher competition.

1 Introduction

A defining feature of perfect competition is that products are identical. But markets are rarely perfectly competitive; firms differentiate their products in order to enhance their profitability. In this paper I argue that differences in firm productivities impact the manner in which firms choose to differentiate their products.

Firm productivities differ significantly both within and across industries. Moreover, a large fraction of industries are characterized by differentiated products.[1] Economists have made great progress in studying

*I thank Vasco Curdia, Wioletta Dziuda, Marc Melitz, Esteban Rossi-Hansberg, Chris Sims, and especially Faruk Gul and my advisor Gene Grossman for helpful comments and the National Science Foundation for support through NSF grant no. SES 0211748.

[1] According to the classification in Rauch (1999), a lower bound of over 55% of commodities at the three-digit level of the Standard International Trade Classification are differentiated goods. For my purposes this estimate is a lower bound of lower bounds, because even homogeneous-good firms are differentiated by location.

separately the effects of firm heterogeneity and imperfect competition.[2] However, it is only recently that the spotlight has been turned on the interaction between firm heterogeneity and imperfect competition.[3] The resulting models, however, abstract from product positioning and product choice; these models do not provide a framework in which product A is more different from product B than it is from product C. That is, current frameworks for interacting differentiation and firm heterogeneity model differentiation in reduced form.

Spatial competition models are ideally suited to answer questions about firm positioning and product choice. The basic concern in spatial economics is determining how firms choose the characteristics of the products that they sell. This literature dates back to Hotelling (1929), who considered the position and prices of two vendors selling a homogeneous good on a town's main street. Hotelling's model is one of Bertrand competition in which location differentiates otherwise homogeneous goods: consumers value both lower prices and a shorter walk to the vendor. Given the location of her competitor, A, a vendor, B, faces a trade-off between two forces when choosing her location. As B's location approaches that of A, B is able to obtain a larger market share. This is true because all consumers between A and B become relatively closer to B as B moves towards A. On the other hand, competition becomes fiercer as B moves towards A because the two firms become less differentiated. In the limit, when B locates on top of A, the outcome is Bertrand competition with undifferentiated goods.

The Hotelling model and subsequent models of spatial competition focus on how firms locate in geographic space or how firms place their products in product characteristics space. Such questions are most relevant and interesting in the realistic case in which firms' productivities differ. A model of location in the context of firm asymmetries provides intuition into how a firm's choice of location or product characteristics is affected by its own productivity and the productivity and number of firms against which it competes. This intuition helps provide answers to interesting questions: Will a firm locate closer to its relatively less productive neighbor? Does opening the black box of differentiation yield new insight into how productivity affects a firm's market share and profit?

Unfortunately, finding equilibria to games with endogenous locations has proven to be difficult even when firms are symmetric. (Hotelling's proposed solution fails the test of subgame perfection.) A realistic model of spatial competition involves at least two stages: one in which firms choose locations and another in which

[2] Seminal models of firm heterogeneity in homogenous-good industries include Hopenhayn (1992) and Jovanovic (1982). Seminal models of imperfect competition include Spence (1976), Dixit and Stiglitz (1977), and Salop (1979).

[3] Recent models fusing imperfect competition and firm heterogeneity include Melitz (2003), Syverson (2004), and Melitz and Ottaviano (2005).

they choose prices. A technical hurdle arises in a model in which location decisions precede price decisions. d'Aspremont, Gabszewicz, and Thisse (1979) investigate the Hotelling model with only two firms. They prove that no pure-strategy equilibrium exists to a price-stage game (one in which locations are fixed) in which the two firms are located "too closely." Their result extends to multiple firms and to the case in which firms locate on the circumference of a circle rather than on a line. If price-stage subgames exist for which no equilibrium is found, the location stage cannot be addressed.

In order to obtain a pure-strategy equilibrium to the price stage, many authors have added additional assumptions to the model. For example, Salop (1979) assumes fixed locations; Lancaster (1979) assumes that firms select prices and locations simultaneously (amongst other assumptions); and d'Aspremont, Gabszewicz, and Thisse (1979) and Economides (1989) assume convex rather than linear transport costs.[4] Each of these authors obtains a pure-strategy equilibrium. However, they either abstract from endogenous locations (Salop (1979)) or create frameworks that are not sufficiently tractable to model firm heterogeneity and endogenous locations.

Rather than adding assumptions, I allow firms to randomize over prices.[5] In this framework, I prove that there exists an equilibrium in which firms use pure strategies *along the equilibrium path*. That is, although firms may mix over prices off the equilibrium path, equilibrium strategies are pure. Moreover, because of the tractability of my framework, I am able to introduce asymmetric firms into a model of spatial competition with not only a horizontal, but also a vertical, or quality dimension.

Firms are asymmetric in that they differ in two cost parameters. Each firm has a firm-specific constant marginal cost of producing zero-quality goods and a firm-specific constant marginal cost of increasing the quality of a unit of output. Consumers are uniformly distributed along the circumference of a circle, as in Salop (1979) and Lancaster (1979). A consumer buys from the firm that minimizes a quality-and-location adjusted price. After an entry-and-exit stage, the number of competitors and all cost parameters are common knowledge amongst firms in a market. Given this knowledge, all firms simultaneously locate along the circumference of the circle and choose the quality at which they will produce. Location can refer either to the physical location of the firm or the characteristics of the product that it produces. In the subsequent price stage, all firms simultaneously choose the distributions of prices at which they sell as much as is demanded given the observable locations, qualities, and cost parameters of all competitors. When firms are symmetric,

[4] For an excellent and complete survey of spatial competition models with ex-ante symmetric firms, see Anderson, de Palma, and Thisse (1992).

[5] I build on the work of Dasgupta and Maskin (1986a) and Dasgupta and Maskin (1986b), who prove that a mixed-strategy equilibrium exists to all price-stage subgames.

consumers place no value on quality, and firms pay no "shipping costs," the model reduces to the standard Hotelling model on the circumference of a circle.

I prove that under an equilibrium refinement, all relevant firm characteristics (price, quality, market share, and profit) are uniquely determined in any strict[6] post-entry subgame perfect equilibrium.[7] For each firm, each of these outcomes is a deterministic function of only the firm's own cost parameters and a measure of "market softness." A market is softer if fewer firms compete, holding average productivity fixed. A market is also softer if average productivity declines, holding the number of competing firms fixed.

Because there is a unique characterization of outcomes, the model can be brought to the data. Indeed, the equilibrium characterization admits relatively straightforward comparative statics that provide new and testable implications. Within a market, more productive firms are more isolated, all else equal, because a competitor chooses to locate farther away from a more productive firm than from a less productive firm. That is, the competitors of a very productive firm sell goods that substitute relatively poorly for its goods. To my knowledge there is no comparable result in the literature. This result provides new insight into how differences in productivity beget differences in profit. Within a market, more productive firms have larger market shares for two reasons. First, they charge lower quality-adjusted prices. This is a standard result. Second, more productive firms also have larger market shares because their direct competitors offer consumers relatively poor substitutes. This is a novel result. Moreover, I also find a new rationale for why productive firms earn higher variable profit per sale: they exert greater market power because consumers find it more difficult to substitute away to competitors.

In addition to explaining product placement and firm location, this paper contributes to other strands of literature. The first is the relation between market size and competition.[8] More firms enter in markets with greater demand density. As more firms compete, consumers can more easily substitute away from firms charging high prices. Because entry decreases market power, markups are lower in markets with greater demand density. Moreover, because competition is fiercer in denser market, less productive firms are unable to survive.[9]

Another literature to which this paper contributes is the relationship between productivity and firm size. It is a standard conclusion that firm size increases with productivity.[10] However, this paper provides a novel

[6] I define a strict equilibrium below.

[7] Given the productivity of a firm and its neighbor, the distance between these two firms is also uniquely determined. However, the productivity of a firm's neighbor is not uniquely determined. I discuss this further below.

[8] Recent contributions includes Campbell and Hopenhayn (2002), Syverson (2004), and Melitz and Ottaviano (2005).

[9] Campbell and Hopenhayn (2002) and Syverson (2004) find empirical support for these predictions.

[10] Although this literature is too large to provide an exhaustive list of citations, contributors include Viner (1932), Lucas (1978), and Campbell and Hopenhayn (2002).

mechanism to explain this relationship. Competitors choose to locate farther away from highly productive firms in order to mitigate competitive pressures. This implies that productive firms are larger not only because they charge lower prices, but also because they are more isolated from their competition.

The remainder of the paper is in four sections. Section 2 provides the setup of the model, describing consumer preferences and the manner in which firms interact strategically. In Section 3, I describe the equilibrium to the post-entry game and outline the structure of the proof, leaving the details for the appendix. Section 4 adds the entry-and-exit stage. Section 5 concludes.

2 The model

There is a unit mass of ex-ante identical product markets. Each market is represented by a circle with unit circumference. Within a given market there is a continuum of potential sites at which firms can locate. I assume a continuum of markets rather than a single market in order to abstract from aggregate uncertainty. This assumption, and the fact that it has no effect on qualitative results, is explained below.

2.1 Consumers

Each market has a mass L of consumers uniformly distributed along the circumference of the circle. Each consumer inelastically demands one unit of one good from the given market if the quality-and-location adjusted price to the consumer is less than her reservation value. A consumer's location represents her ideal variety. A given consumer, z, in a market with a finite set of producers N, purchases one good from firm $i \in N$ if

$$\begin{gathered} i = \arg\min_{j \in N} \left\{ p_j + t \left\| z - j \right\| - q_j^{\gamma} \right\} \\ \text{and } p_i + t \left\| z - i \right\| - q_i^{\gamma} \leq v, \end{gathered} \tag{1}$$

where $\gamma \in [0, 1)$. $\|z - j\|$ is the shortest arc-length separating firm j from consumer z. The parameter $t > 0$ is the "transport" cost per unit of distance incurred by the consumer, implying that $t \|z - j\|$ is the transport cost that a consumer at a distance $\|z - j\|$ from firm j incurs either to ship the good or to travel to firm j to shop. q_j is the quality of good j. Finally, v represents the consumer's reservation value. I assume throughout that reservation values are sufficiently high such that all consumers in a given market purchase a good in equilibrium. These preferences nest the standard Hotelling-style preferences as the special case in which consumers do not value quality ($\gamma = 0$).

2.2 Firms

Suppose that there are $n \geq 2$ firms in a given market, each of which is characterized by two cost parameters, k and c. The parameter k denotes the marginal cost at which the firm can produce a unit of zero-quality output. The parameter c denotes the marginal cost of producing each unit of quality in each unit of output. A firm with cost vector (k, c) that chooses quality level q has a constant marginal cost of production equal to $k + cq$. Figure 1 graphs the marginal cost of production for two firms, i and j, as a function of the qualities at which they produce:

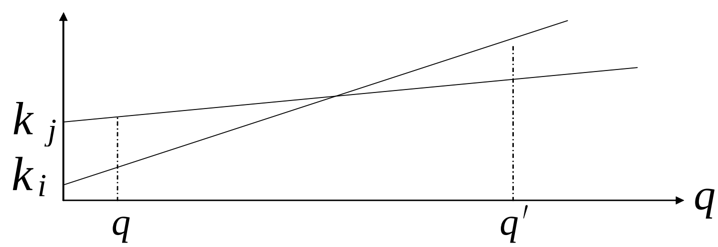


Fig. 1: Marginal cost of production as a function of quality for firms i and j

In Figure 1, firm i can produce a good with zero quality more efficiently than firm j; i.e. $k_i < k_j$. However, firm j can increase quality more cheaply than firm i; i.e. $c_j < c_i$. To fix ideas, suppose that the two firms choose to produce at the same level of quality. If they both produce at a sufficiently low level of quality, such as q, then firm i's marginal cost of production will be lower than firm j's, because $k_i + qc_i$ is strictly less than $k_j + qc_j$ for sufficiently low q. If they produce at a sufficiently high level of quality, such as q', then firm j's marginal cost of production will be lower than firm i's. In equilibrium, each firm chooses the level of quality to produce to maximize profits. Additionally, each firm incurs a "shipping cost" of $2\tau d$ (common to all firms) to ship a good to a consumer located at a distance d from its location, where $\tau \in [0, t)$. Firm i's cost of producing and supplying a consumer located at a distance d from itself is given by

$$k_i + c_i q_i + 2\tau d. \tag{2}$$

The shipping cost τ is included to generate an equilibrium refinement: I obtain a uniqueness result if $\tau > 0$, no matter how small. However, all results hold, excluding uniqueness, if $\tau = 0$. It is important that firms cannot price discriminate even if $\tau > 0$. Hence, a firm cannot observe the distance between itself and a consumer before transacting. Moreover, the firm cannot pass the shipping cost along to the consumer. The assumption that firms pay a shipping cost is straightforward when the model is applied to homogeneous-good firms that differentiate themselves through geographic locations. One interpretation is that consumers and firms share the cost of transporting the good and that firms cannot observe how far they'll ship the good until after the sale is complete. When the model is applied to differentiated-good firms, the shipping cost can be interpreted as a cost of customer service. A consumer who purchases a good that is farther from her ideal variety may be more likely to require service. The firm must not be able to charge for service[11] or judge how far the good is from a consumer's ideal before determining the price. I emphasize again that the shipping cost is used only to select among equilibria and, for this purpose, it can be arbitrarily small.

Firms play a three-stage game. The first stage entails entry and exit. The following two stages comprise the post-entry spatial-competition subgame. Before entry, there is an unbounded set of ex-ante identical entrepreneurs. In the first, *entry-and-exit* stage, those entrepreneurs who choose to enter pay a fixed cost, $f_e > 0$. Upon entry, an entrepreneur draws her cost parameters from the continuous joint density function $\tilde{g}(k,c)$, where $\tilde{g}(k,c)$ has positive support over $[k_{\min}, k_{\max}] \times [c_{\min}, c_{\max}]$, where $j_{\max} \geq j_{\min} \geq 0$ for $j = k, c$. $\tilde{G}(k,c)$ denotes the continuous distribution associated with $\tilde{g}(k,c)$.[12] After drawing her cost parameters, an entrepreneur decides whether to exit or to set up a firm and produce. If an entrepreneur chooses to produce, she must pay another fixed cost, $f_p > 0$, to set up her firm. Each firm simultaneously chooses the market in which to compete. Choosing a market in which to compete is equivalent to choosing on which of the continuum of circles a firm chooses to produce output.

I make two assumptions for reasons of tractability. First, I assume that there are a continuum of markets. Second, I ignore the requirement that the number of firms within a market must be an integer.[13] Under these two assumptions, I abstract from aggregate uncertainty, leaving only idiosyncratic uncertainty. No entrepreneur knows how productive she will be before she enters; but she can predict with certainty how tough competition will be in each market. I explain below that although these assumptions provide tractability, they do not change any qualitative results.

In the following two stages, which comprise the post-entry spatial-competition game, a firm only interacts with the other firms in its chosen market. I focus on a given market in the post-entry spatial competition subgame. However, a similar game is played in each market.

In the second, or location/quality stage, all firms observe the number of competitors they face in the market, n, and the full matrix of cost parameters $\left((k_0, c_0)', ..., (k_{n-1}, c_{n-1})'\right)$. With this common information,

[11] This could be the outcome of a signalling game in which firms offer warranties.

[12] These are standard assumptions in the literature on entry and exit, variants of which are used in Hopenhayn (1992) and Melitz (2003)

[13] This is a standard assumption in the literature. Salop (1979) makes this assumption so that firms earn exactly zero profit with free entry.

the firms simultaneously locate on the circumference of the circle and choose the qualities of output they will produce. In the third, or price stage, all firms observe the locations, qualities, and cost parameters of their market competitors. Firms then simultaneously choose the distributions of prices over which they randomize to determine at which price they sell their varieties. The assumption that firms choose their physical locations and qualities before their prices is realistic. A producer must build a factory and a retailer must build a store. These location investments are fixed at the time that firms make their pricing decisions.

As discussed above, d'Aspremont, Gabszewicz, and Thisse (1979) prove that no pure-strategy SPNE exists to the price-stage subgame because profit functions are globally neither continuous nor quasi-concave. In order to highlight this issue, and to focus on the profit maximization problem more generally, consider the price stage. In particular, suppose that firm i neighbors firm $i+1$ at a distance $d_{i,i+1}$. If

$$p_i - p_{i+1} \in \left(-td_{i,i+1} - q_{i+1}^{\gamma} + q_i^{\gamma}, td_{i,i+1} - q_{i+1}^{\gamma} + q_i^{\gamma}\right)$$

then there exists a consumer located between firm i and firm $i+1$ that is indifferent between buying from the two firms. Let $x_{i,i+1}$ denote the distance from firm i of the consumer located between firms i and $i+1$ that is indifferent between buying from these two firms. The consumer preferences specified in equation (1) imply that

$$x_{i,i+1} = \frac{1}{2t}\left(p_{i+1} - p_i + td_{i,i+1} - q_{i+1}^{\gamma} + q_i^{\gamma}\right). \tag{3}$$

Suppose that firm i also is neighbored on the opposite side by firm $i-1$ at a distance $d_{i-1,i}$. Similarly, let $x_{i,i-1}$ denote the distance from firm i of the consumer located between firms i and $i-1$ that is indifferent between buying from these two firms. If

$$p_i - p_{i-1} \in \left(-td_{i-1,i} - q_{i-1}^{\gamma} + q_i^{\gamma}, td_{i-1,i} - q_{i-1}^{\gamma} + q_i^{\gamma}\right)$$

then

$$x_{i,i-1} = \frac{1}{2t}\left(p_{i-1} - p_i + td_{i-1,i} - q_{i-1}^{\gamma} + q_i^{\gamma}\right). \tag{4}$$

Under the condition that an indifferent consumer exists between *all* neighboring firms, firm i's market share, i.e. the arc-length of the circle over which firm i supplies, denoted by $x_i \equiv x_{i,i-1} + x_{i,i+1}$ when

$x_{i,i-1}, x_{i,i+1} \geq 0$, is

$$x_i = \frac{1}{2t}\left(p_{i-1} + p_{i+1} - q_{i-1}^{\gamma} - q_{i+1}^{\gamma} + t\left(d_{i-1,i} + d_{i,i+1}\right)\right) + \frac{1}{t}\left(q_i^{\gamma} - p_i\right). \tag{5}$$

Firm i's average cost of supplying consumers between itself and firm $i+1$ is $k_i + c_i q_i + \frac{2\tau}{x_{i,i+1}} \int_0^{x_{i,i+1}} \eta d\eta$, which equals $k_i + c_i q_i + \tau x_{i,i+1}$. Similarly, the average cost of supplying consumers located between itself and firm $i-1$ is $k_i + c_i q_i + \tau x_{i,i-1}$. Hence, *under the condition that an indifferent consumer exists between all neighboring firms*, firm i's variable profit is

$$\pi_i = L\left(x_i\left(p_i - \left(k_i + c_i q_i\right)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right) \tag{6}$$

(recall that L is the mass of consumers in the market).

However, if firm $i+1$ (or firm $i-1$) charges a sufficiently low price such that $x_{i,i+1} < 0$ (or $x_{i,i-1} < 0$ respectively), then firm i supplies no consumers and earns zero profit. All consumers between firm $i-1$ and firm i make the same decision when choosing between buying from firm i and firm $i+1$. If firm $i+1$ were to charge a sufficiently low price such that any consumer between firm $i-1$ and firm i preferred buying from firm $i+1$ rather than from firm i, then all consumers would prefer buying from firm $i+1$ rather than from firm i. Hence, if firm $i+1$ (or firm $i-1$) were to charge a price at which $x_{i,i+1} < 0$ (or $x_{i,i-1}$ respectively), then firm i would earn zero profit.

Suppose that the profile of prices is such that $x_{i,i-1} > 0$; that is, a positive mass of consumers prefers buying from firm i rather than firm $i-1$. If firm $i+1$ charges the price $p_{i+1} = p_i - td_{i,i+1} - q_i^{\gamma} + q_{i+1}^{\gamma}$, then $x_{i,i+1} = 0$ implying that all consumers between firms i and $i+1$ prefer buying from firm $i+1$ rather than firm i. Any infinitesimal reduction in the price of firm $i+1$ will induce all consumers between firm $i-1$ and firm i to prefer buying from firm $i+1$ rather than firm i. For a given profile of prices, an infinitesimal reduction in firm $i+1$'s price induces a discrete mass of consumers to switch their brand loyalties. This implies that profit functions are globally neither continuous nor quasi-concave in prices.

Suppose that firm 0 and firm 1 are the only two firms in a market and let $d_{0,1}$ denote the shorter arc-length separating the two firms, where $d_{0,1} < 1/2$. If firm 0 charges any price strictly greater than $p_0 = p_1 + td_{0,1} - q_1^{\gamma} + q_0^{\gamma}$, then it supplies no consumers because all consumers prefer buying from firm 1. If firm 0 charges any price strictly less than $p_0 = p_1 - td_{0,1} - q_1^{\gamma} - q_0^{\gamma}$, then it supplies all consumers in the market.

Figure 2 graphs firm 0's profit as a function of its price:

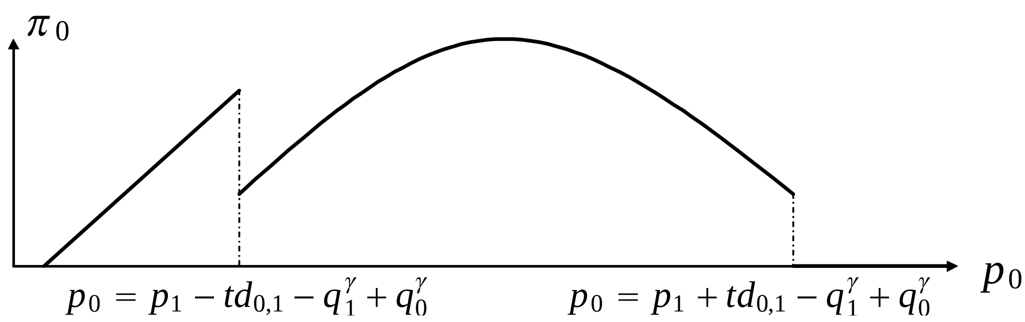


Fig. 2: Discontinuities in profit: firm 0's profit as a function of its price

Because profit functions are globally neither continuous nor quasi-concave in prices, there exist many price-stage subgames for which there is no pure-strategy equilibrium. In a deviation from previous work, I do not add additional assumptions to make profit functions quasi-concave. Instead, I allow firms to pursue mixed strategies in their pricing.

3 Location/quality and price stages

Before describing the equilibrium, I review the structure of the post-entry subgame, provide two definitions, and explain notation. This is a two-stage, simultaneous move game of complete information. The set of players is the set of firms, N with $|N| = n$, and each firm $i \in N$ maximizes its profit. Two-dimensional firm types $(k, c) \in [k_{\min}, k_{\max}] \times [c_{\min}, c_{\max}]$ are common knowledge at the beginning of the location/quality stage. In this stage all firms simultaneously choose their locations along the circumference of the circle and the qualities that they produce. All locations, qualities, and firm types are common knowledge at the beginning of the price stage. In this stage all firms simultaneously choose their prices (or more precisely the distribution of prices over which they randomize). Denote the strategy space by Ω and an outcome in strategy space by $\omega \in \Omega$. Let $\Omega^n \equiv \Omega_1 \times ... \times \Omega_n$ and denote $\vec{\omega} \in \Omega^n$ by a *strategy vector.*

Definition 1 *An equivalence set $O \subseteq \Omega^n$ is a set of strategy vectors such that for any $\vec{\omega}, \vec{\omega}' \in O$, all firm characteristics (price, quality, market share, and profit) except location are identical across $\vec{\omega}$ and $\vec{\omega}'$.*

There may be multiple strategy vectors that yield identical outcomes in terms of price, quality, market share, and profit. I denote the set of all such strategy vectors an equivalence set.

Definition 2 *An equilibrium is strict if in any stage firm i's best response is unique, for all $i = 0, ..., n-1$, holding fixed the strategies of all firms $j \neq i$ in that stage.*

Finally, I provide some notation. Define

$$\chi_i \equiv (1-\gamma)\left(\frac{c_i}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} - k_i, \tag{7}$$

where χ_i is a measure of firm i's productivity (henceforth firm i's productivity); χ_i increases as both or either cost parameter decreases. Let $\bar{\chi} \equiv \frac{1}{n}\sum_{i=0}^{n-1}\chi_i$ denote the average productivity in the market. Define

$$\lambda \equiv \frac{1}{n} - \frac{2}{3t+2\tau}\bar{\chi}_i, \tag{8}$$

where λ measures market softness. The greater is λ, the softer is competition in the market. A market is softer if fewer firms compete, holding average productivity fixed. A market is softer if average productivity decreases, holding the number of competing firms fixed.

Proposition 1 *Suppose $\tau > 0$. For any (k, c) there exists a pair $\varepsilon_k, \varepsilon_c > 0$ such that if $(k_i, c_i) \in [k, k+\varepsilon_k] \times [c, c+\varepsilon_c]$ for all i, then there is a unique non-empty equivalence set O^* such that an outcome is a strict SPNE if and only if $\vec{\omega} \in O^*$.*

Given an order in which firms locate, label any firm 0 and label subsequent firms in a clockwise direction (to firm $n-1$). For any outcome in O^* the distance between any two neighbors, firms i and $i+1$, is

$$d^*_{i,i+1} = \lambda + \frac{1}{3t+2\tau}\left(\chi_i + \chi_{i+1}\right), \tag{9}$$

firm i's quality, price, market share, and profit are

$$q^*_i = \left(\frac{c_i}{\gamma}\right)^{\frac{1}{\gamma-1}}, \tag{10}$$

$$p^*_i = (t+\tau)\lambda + \frac{1}{3t+2\tau}\left(\left[2(t+\tau) + t\gamma\right]\left(\frac{c_i}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} + tk_i\right), \tag{11}$$

$$x^*_i = \lambda + \frac{2}{3t+2\tau}\chi_i, \tag{12}$$

and

$$\pi_i = Lt\left(\lambda + \frac{2}{3t+2\tau}\chi_i\right)^2. \tag{13}$$

Moreover, if $\tau = 0$ then although any strategy vector $\vec{\omega} \in O^*$ is a SPNE, no strategy vector $\vec{\omega} \in O^*$ is a strict SPNE.[14]

In equilibrium all relevant outcomes–quality, price, market share, and profit–depend on other producers' productivities only through an aggregate measure of market softness, λ. This might seem surprising given that in most subgames a firm's profit is clearly affected more by its neighbors' productivities than by the productivities of other firms in the market. In order to highlight the intuition behind this result, abstract from price-stage subgames in which firms mix over prices. Suppose instead that all n firms choose locations and qualities such that a pure-strategy price-stage equilibrium exists in which there is an indifferent consumer between each pair of neighbors. In this case, firm i's profit is an increasing function of its isolation, $d_{i-1,i} + d_{i,i+1}$, as well as the sum of the prices of its two neighbors, $p_{i-1} + p_{i+1}$:

$$\pi_i = \pi\left(d_{i-1,i} + d_{i,i+1}, p_{i-1} + p_{i+1}\right),$$

where p and d are defined $\bmod\,(n)$.[15]

A firm has two clear preferences in the location/quality stage. First, all else equal, a firm would prefer to be as isolated as possible from its neighbors; this increases its market share and its optimal price. Second, a firm would prefer to locate between relatively unproductive neighbors; these neighbors will charge higher prices, increasing the firm's market share and its own optimal price. Hence, in order for a given firm to earn the same profit locating between more productive firms as it would if locating between less productive firms, the firm would have to be more isolated. In equilibrium, a firm is indifferent between locating next to a productive or unproductive firm under the condition that it is compensated for locating next to a productive firm with greater isolation. Because the distance between direct competitors adjusts such that a firm is indifferent to the productivity of its neighbors, the productivity and number of competitors in a market can only influence a firm's market share and post-entry profit via an aggregate measure of market softness.

For concreteness, suppose that there are four firms. Two of these firms have high productivity, χ and two have lower productivity, χ'. In this simplified case, there are two possible arrangements of the firms

[14] All proofs are relegated to the Appendix.

[15] For p this is standard notation. Let the vector $p = (p_0, ..., p_{n-1})$. Then p is $\bmod\,(n)$ implies that $p_j = p_{j-n}$ for all j. For d this is more unusual. Let the vector $d = (d_{0,1}, d_{1,2}, ..., , d_{n-2,n-1}, d_{n-1,0})$. Then d is $\bmod\,(n)$ implies that $d_{j,j+1} = d_{j+n,j+1+n}$ for all j.

around the circle. In one, the two productive firms neighbor one another. In the other, the more productive firms are separated by the less productive firms. Each of these arrangements corresponds to an equilibrium strategy in the equivalence set O^*. The two possible orders are shown in Figure 3:

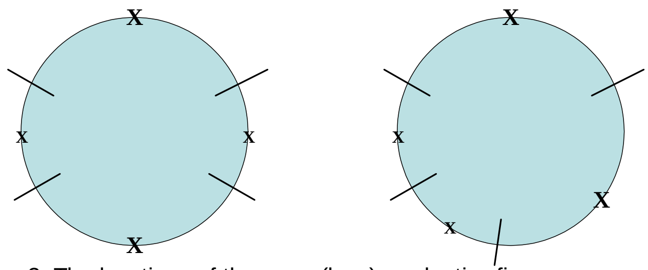


Fig. 3: The locations of the more (less) productive firms are represented by the large X's (small x's). The figure includes graphical representations of firm market shares.

If $\tau > 0$ then both of these equilibria are strict SPNE.

Why does $\tau > 0$ serve as an equilibrium refinement? When firms play $\vec{\omega} \in O^*$, each firm is located at the midpoint of the mass of consumers that it supplies (as in Figure 3). That is, $x_{i,i-1} = x_{i,i+1}$ for all i. If firm i were to make a small, unilateral deviation in location, then $x_{i,i-1} \neq x_{i,i+1}$. If $\tau > 0$ then this deviation serves to increase firm i's shipping cost without affecting its revenue or production cost. On the other hand, if $\tau = 0$ then this deviation does not affect revenue, production cost, or shipping cost. Hence, sufficiently small deviations in location are strictly dominated if $\tau > 0$ but are only weakly dominated if $\tau = 0$.

3.1 Structure of the proof

The proof of Proposition (1) is located in the Appendix. In this section I provide an outline of that proof. The proof of Proposition (1) is in four sections. I prove that any strategy vector $\vec{\omega} \in O^*$ is a strict SPNE if $\tau > 0$ (and is a SPNE if $\tau = 0$) in the first three sections. In the fourth section, I prove that there exists no strict SPNE $\vec{\omega}' \in \Omega^n$, such that $\vec{\omega}' \notin O^*$.

A standard proof of subgame perfection in a two-stage game has two sections. First it is shown that second-stage strategies are dominant. Then it is shown that first-stage strategies are dominant given second-stage strategies. However, my proof differs slightly. I begin with a section that simplifies what follows.

A firm is *overtaken* if it makes no sales in equilibrium. In the first section I prove that any unilateral deviation that leads firm i to overtake another firm with certainty[16] is strictly dominated. I also prove that

[16] Uncertainty can arise in a subgame in which firms mix over prices. When firms mix, a firm i overtakes another firm j with certainty if firm j makes no sales at any price over which it mixes with positive probability.

if firm i unilaterally deviates by locating between firms j and $j+1$, no firm other than i, j, or $j+1$ would ever overtake another firm with probability one. Thenceforth, I focus exclusively on equilibria in which no firm is overtaken with certainty.

I proceed as usual in the remainder of the proof of subgame perfection. In the second section of the proof I prove that price strategies are strictly dominant. In the third section I prove that location/quality-stage strategies are strictly (weakly) dominant if $\tau > 0$ ($\tau = 0$) given price-stage strategies.

The first two sections are relatively straightforward. Proving that a unilateral deviation from location/quality-stage strategies is strictly dominated if $\tau > 0$ is more complex. The proof is in four steps. I define three terms before describing these steps: "local" deviations, "drastic" deviations, and "benchmark profit." If firm i deviates in location, it can either deviate locally or drastically. Firm i, deviates locally if it locates between the same two firms as in its equilibrium location, but not in the exact location specified by equation (9). Firm i deviates drastically if it locates next to any firm that it does not neighbor under its equilibrium strategy.

The third term that I define is "benchmark profit." Recall that firm i's profit is

$$\pi_i = \begin{cases} L\left(x_i\left(P_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right) & \text{if } x_{i,i-1}, x_{i,i+1} \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

I define a "benchmark profit" function that differs slightly from the real profit function. A firm's benchmark profit is

$$\pi_i^B = \begin{cases} L\left(x_i\left(P_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right) & \text{if } x_{i,i-1} + x_{i,i+1} \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

The difference between actual profit and benchmark profit is that benchmark profit allows negative sales on one side as long as total sales are positive. To fix ideas, consider again the example in which $n = 2$. Firm 0's profit as a function of its price is depicted in Figure 2. I replicate this on the left-hand side of Figure 4.

On the right-hand side I graph firm 0's benchmark profit.

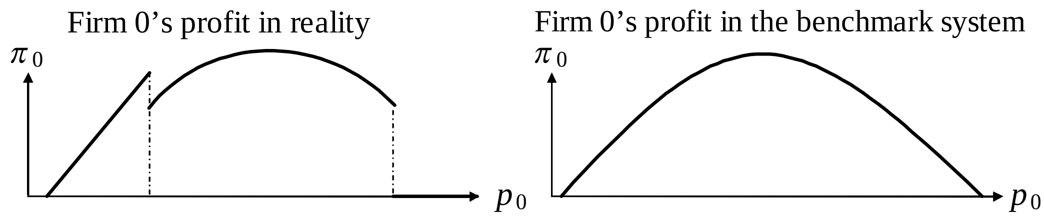


Fig. 4: Firm 0's profit as a function of its price in reality and in the benchmark system

The benchmark system of reaction functions is relevant for two reasons. First, if there exists a price-stage subgame equilibrium in pure strategies in which no firm is overtaken, the equilibrium vector of prices is the solution to the benchmark system of reaction functions. Second, when no pure-strategy price-stage subgame equilibrium exists, benchmark profit serves as a useful comparison to the profit that firms earn when they adopt mixed strategies.

In the first step toward proving that firms have no incentive to deviate in the location/quality stage, I solve the benchmark system of reaction functions for optimal benchmark prices. Second, I prove that for any choice of location, firm i's optimal quality is q_i^* if firms set benchmark prices in the subsequent price stage.

Third, I prove that any location deviation is strictly (weakly) dominated if $\tau > 0$ ($\tau = 0$), no firm is overtaken, and firms price according to the benchmark system in the subsequent price stage. I divide this third step into two substeps. In the first substep I consider only local deviations. Recall that $\pi_i = L\left(x_i\left(p_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right)$. When firms price according to the benchmark system and firm i unilaterally and locally deviates, then firm i's market share and price remain unchanged. This implies that $x_i\left(p_i - (k_i + c_i q_i)\right)$ does not change after a local deviation. Suppose that firm i deviates locally by moving $\delta \in (0, d_{i,i+1})$ units towards firm $i+1$. In this case $\Delta x_{i,i+1} = -\Delta x_{i,i-1}$. Hence, $\pi_i' < \pi_i^*$ if and only if $\tau > 0$.[17] In the second substep I prove that drastic deviations are strictly dominated if firms all choose q_i^*, firms set benchmark prices in the following subgame, and no firm is overtaken.

The fourth and final step in proving that any $\vec{\omega} \in O^*$ is a strict SPNE if $\tau > 0$ (and is a SPNE if $\tau = 0$) involves considering location/quality-stage deviations after which firms must mix over prices in the price subgame. I prove that each firm's profit is bounded above by its benchmark profit for any choice of quality if there exists no pure-strategy equilibrium in the price stage and if no firm is overtaken with certainty. This

[17]It is this distinction between $\tau = 0$ and $\tau > 0$ that implies that all $\omega \in \Omega$ are *strict* SPNE if and only if $\tau > 0$.

completes the proof that any strategy $\vec{\omega} \in O^*$ is a strict SPNE if $\tau > 0$ (and is a SPNE if $\tau = 0$).

In the fourth section, I prove that there exists no strict SPNE $\vec{\omega}' \in \Omega^n$, such that $\vec{\omega}' \notin O^*$. In any strict equilibrium firm prices are the the solution to the benchmark system of reaction functions. This implies that all firms choose their qualities according to equation (10). Moreover, in any strict SPNE each firm must be centered in its zone of supply: $x_{i,i+1} = x_{i,i-1}$ for all $i \in N$. I then prove that there is a unique vector of distances separating neighbors, given the order in which firms locate, such that all firms are centered in their zones of supply. This vector must correspond with the vector characterized by equation (9). This completes the proof of Proposition (1).

3.2 Within market comparative statics

In this subsection I consider comparative statics for firms in the same market. How do firms' choices of locations given in equation (9) differ from those predicted, by Lancaster (1979) and Economides (1989), or assumed, by Salop (1979) and Syverson (2004), in the symmetric-firm model on the circumference of a circle? In the aforementioned symmetric-firm models, each pair of neighbors is separated by a distance of $1/n$. Equation (9) can be expressed as

$$d\left(\chi_i, \chi_{i+1}, \lambda\right) = \frac{1}{n} + \frac{2}{3t + 2\tau}\left(\frac{\chi_i + \chi_{i+1}}{2} - \bar{\chi}\right).$$

This implies that if the average productivity of two neighbors equals the average productivity of all firms in the market, then the neighbors' varieties are separated by $1/n$.[18] With heterogeneous firms, two neighbors, firms i and $i+1$, produce varieties that are separated by more than $1/n$ if and only if the average of χ_i and χ_{i+1} is greater than $\bar{\chi}$. Proposition (1) explains that relative isolation is a function of relative productivity. Intuitively, less productive firms shy away from the harsh competition of highly productive firms. This intuition is an equilibrium argument, however, only if $\tau > 0$.

A firm's marginal cost of production is $\gamma\left(c/\gamma\right)^{\frac{\gamma}{\gamma-1}} + k$. When consumers place no value on quality ($\gamma = 0$), as in the standard Hotelling model, more productive firms have lower marginal costs of production. This relationship is no longer unambiguous when quality is relevant ($\gamma > 0$). A firm that can produce quality more productively, one that draws a lower parameter c, chooses to produce a higher level of quality. All else

[18]If all firms are identical, then there exists a unique strict SPNE to the post-entry subgame of my model (ignoring the order of the firms because they're identical) if $\tau > 0$. In this equilibrium all neighbors are separated by an equal distance. This is a stronger result than found in Economides (1989). Economides proves that there exists an equilibrium in which firms are symmetrically spaced. He does not obtain a uniqueness result. With symmetric firms my approach differs from Economides' in two respects. I assume a linear cost of transportation while he assumes a quadratic cost of transportation. I also include a shipping cost τ.

equal, it will have a higher realized marginal cost.

It is to match this aspect of reality, that more productive firms do not always have lower realized marginal costs, that I introduce the vertical dimension of differentiation. Macy's and Saks Fifth Avenue each sell suits. However, the fact that Macy's tends to sell suits at a lower cost than Saks does not necessarily imply that Macys is more productive than Saks. Instead, it may be that Saks chooses to sell higher quality suits than Macys. Given the observation that Saks sells suits of a higher quality, my model predicts that Saks is more efficient at producing (or selling) high quality suits. Indeed, given a measure of the qualities at which two firms in the same industry sell, firms i and j, one could back out a measure of the relative cost draw c of firm i to firm j.

More productive firms charge lower prices per unit of quality according to equation (11): $\frac{\partial(p^*/q^*)}{\partial c}$, $\frac{\partial(p^*/q^*)}{\partial k} > 0$. However, more productive firms do not pass along to consumers the full benefit of their lower costs through lower prices. A more productive firm is more isolated from its neighbors, all else equal. This implies that it exerts greater market power because, on average, its consumers face a greater cost of substituting to the products of its competitors. This explains why a firm's average variable profit per sale, $p_i^* - (k_i + q_i^* c_i + \tau x_i^*)$, is increasing in its productivity.[19]

The market share of a firm with productivity χ_i can be expressed as

$$x(\chi_i, \lambda) = \frac{1}{n} + \frac{2}{3t + 2\tau}(\chi_i - \bar{\chi}).$$

A firm's market share is greater than average if and only if its productivity is greater than average. There are two forces supporting this relationship. The standard mechanism is that more productive firms charge lower prices per unit of quality. Additionally, there is novel mechanism: all else constant, more productive firms' are more isolated from their neighbors.

A firm with productivity χ_i earns variable profit

$$\pi(\chi_i, \lambda) = Lt(x(\chi_i, \lambda))^2$$

A firm's profit is greater than average if and only if its productivity is greater than average. More productive firms earn higher profits for two reasons: they have larger market shares and they earn higher variable profit

[19] A firm's average variable profit per sale is a linear function of its market share in equilibrium:

$$p_i^* - (k_i + q_i^* c_i + \tau x_i^*) = t x_i^*$$

per sale. In summary,

Proposition 2 *Suppose that each firm i plays a strategy ω_i such that $\vec{\omega} \in O^*$ and that the average productivity in the market is $\bar{\chi}$.*

1. *If firm i neighbors firm $i+1$, then $d\left(\chi_i, \chi_{i+1}, \lambda\right) > \frac{1}{n}$ if and only if $\frac{\chi_i + \chi_{i+1}}{2} > \bar{\chi}$.*
2. *Firm i's market share, variable profit per sale, and variable profit are greater than the market average if and only if $\chi_i > \bar{\chi}$.*

3.3 Empirical implementation

The central prediction of the theory, and the prediction from which the other results derive, is given in Part (1) of Proposition (2). This proposition states that all else equal, two direct competitors are more isolated from one another the greater their average productivity. Empirically testing Part (1) of Proposition (2) requires a measure of physical productivity and a measure of distance.

Physical productivity must be measured directly. Inferring productivity from markups is insufficient because, as the theory predicts, markups themselves depend on remoteness in product or geographic space. If input and output data were available, one could construct a measure of total factor productivity which would be independent of isolation in space.

Clearly, testing a prediction about remoteness in space requires a measure of distance between either firm locations (in a homogeneous good industry in which products are differentiated at most by quality) or products in product characteristic space (in a differentiated good industry). It is arguably easier to measure geographic distance than it is to measure distance in product space. Hence, testing this prediction would be easiest in an industry in which firms sell homogenous products that are differentiated by distance and quality alone. Examples of homogeneous good industries in which firms are differentiated by location and (potentially) quality include ready-mixed concrete (Syverson (2004) and Collard-Wexler (2006)), movie theaters (Davis (2005)), motels (Mazzeo (2002)), video retail (Seim (2001)), and eyeglass retail (Watson (2004)).

4 Entry and exit

There is an unbounded set of ex-ante identical entrepreneurs. In order to enter, an entrepreneur pays a fixed cost, $f_e > 0$. Each entrepreneur who enters draws her cost parameters k and c from the common density

$\tilde{g}(k,c)$. An entrepreneur can choose to exit or remain active. If an entrepreneur chooses to remain active, she pays another fixed cost, $f_p > 0$, that enables her to set up a firm. All entrepreneurs that choose to remain active simultaneously choose the markets in which their firms will operate. All markets are identical ex ante.

Under the two simplifying assumptions–that there are a continuum of markets and that the number of firms within a market need not be an integer–I abstract from aggregate uncertainty, leaving only idiosyncratic uncertainty. With these two assumptions, a standard arbitrage argument implies that firms will allocate themselves across markets such that each market is equally competitive; that is, λ is constant across markets in equilibrium.[20]

I hypothesize the existence of an equilibrium with selection, one in which a positive mass of entrants choose to exit while a positive mass of entrants choose to remain active. An entrant remains active if and only if her firm is sufficiently productive. Under an assumption provided below, I prove that such an equilibrium exists.

In order to prove that there exists a cutoff productivity above which firms remain active, I need to provide a mapping from the space of cost draws (k,c) into the space of productivities, χ. Define the range $[\chi_{\min}, \chi_{\max}]$ by $\chi_{\min} \equiv \chi(k_{\max}, c_{\max})$ and $\chi_{\max} \equiv \chi(k_{\min}, c_{\min})$, where the function χ is defined in equation (7). The joint density of cost draws, $\tilde{g}(k,c)$, induces a density of productivity measures, $g(\chi)$, that is continuous over the interval $[\chi_{\min}, \chi_{\max}]$ inasmuch as $\tilde{g}(k,c)$ is continuous in both of its arguments and χ is a continuous function of both k and c. Moreover, $g(\chi)$ is strictly positive for all $\chi \in [\chi_{\min}, \chi_{\max}]$ because $\tilde{g}(k,c)$ is strictly positive for all $(k,c) \in [k_{\min}, k_{\max}] \times [c_{\min}, c_{\max}]$. Let $G(\chi)$ be the continuous distribution associated with the density $g(\chi)$.

Suppose that there exists a cutoff productivity, $\chi^* \in (\chi_{\min}, \chi_{\max})$, such that an entrant remains active if and only if $\chi > \chi^*$. A firm with productivity χ^* is indifferent between producing and exiting because its variable profit, $\pi(\chi^*, \lambda)$, equals the fixed cost of production, f_p. This relation provides a solution for market

[20] All results would remain qualitatively unchanged under a more realistic set of assumptions. However, the model becomes significantly less tractability under alternative assumptions.

For example, I could assume that there is only one market and impose the restriction that the number of firms must be an integer. I could add the assumption that firms enter sequentially and potential entrants observe the number and productivities of all past entrants to ensure that at least two firms choose to remain active. Under this set of assumptions, there would exist a cutoff market softness, λ^*, such that potential entrant $j+1$ would not enter after entrant j if $\lambda \leq \lambda^*$. Operating firms would earn non-negative post-entry profits while potential entrants would anticipate negative profits.

Alternatively, I could maintain the assumption that there are a continuum of markets but require that the number of firms in each market is an integer. In this case, aggregate uncertainty would go to zero as the mass of consumers in each market, L, went to infinity.

softness as a function of the cutoff productivity

$$\lambda\left(\chi^{*}\right)=\sqrt{f_{p}/tL}-\frac{2}{3t+2\tau}\chi^{*} \tag{14}$$

Lower cutoff productivities are associated with softer markets. Given this solution for λ, a firm's post-entry profit, $\Pi\left(\chi\right)=\pi\left(\chi\right)-f_{p}$, can be expressed as a function of the cutoff productivity measure

$$\Pi\left(\chi,\chi^{*}\right)=\frac{4tL}{3t+2\tau}\left(\chi_{i}-\chi^{*}\right)\left[\sqrt{f_{p}/tL}+\frac{\chi_{i}-\chi^{*}}{3t+2\tau}\right]$$

For a given cutoff productivity, the expected post-entry profit of successful entrants is $\bar{\Pi}\left(\chi^{*}\right)=\frac{1}{1-G(\chi^{*})}\int_{\chi^{*}}^{\chi_{\max}}\Pi\left(\eta,\chi^{*}\right)d\eta$. This can be expressed as:

$$\bar{\Pi}\left(\chi^{*}\right)=k\left(\chi^{*}\right),$$

where

$$k\left(\chi\right)\equiv\frac{1}{1-G\left(\chi\right)}\int_{\chi}^{\chi_{\max}}\frac{4tL\left(\eta-\chi\right)}{3t+2\tau}\left[\sqrt{f_{p}/tL}+\frac{\eta-\chi}{3t+2\tau}\right]g\left(\eta\right)d\eta. \tag{15}$$

Denote $k\left(\chi^{*}\right)$ by the zero cutoff profit (ZCP) curve. The graph of $k\left(\chi^{*}\right)$ in $\left(\chi^{*},\bar{\Pi}\right)$-space represents the expected post-entry profit of successful entrants at which a firm with productivity χ^{*} earns zero post-entry profit.

Firms enter until expected total profit is driven down to zero. The expected total profit of an entrant is given by $\left(1-G\left(\chi^{*}\right)\right)\bar{\Pi}\left(\chi^{*}\right)-f_{e}$. This is the expected post-entry profit of an entrant minus the cost of entry. Expected total profit is driven down to zero if:

$$\bar{\Pi}\left(\chi^{*}\right)=\frac{f_{e}}{1-G\left(\chi^{*}\right)}.$$

Denote $\frac{f_{e}}{1-G(\chi^{*})}$ the free entry (FE) curve. In $\left(\chi^{*},\bar{\Pi}\right)$-space, the FE curve represents the expected post-entry profit of a successful entrant at which the expected total profit of entrants equals zero. In order to maintain the expectation that entrants earn zero profit, the anticipated post-entry profit of a successful entrant must increase with the cutoff productivity. This implies that the FE curve is strictly increasing.[21]

[21] I borrow this notation–the ZCP curve and the FE curve–as well as the general strategy for proving Proposition (3) from Melitz (2003).

Proposition 3 *Define*

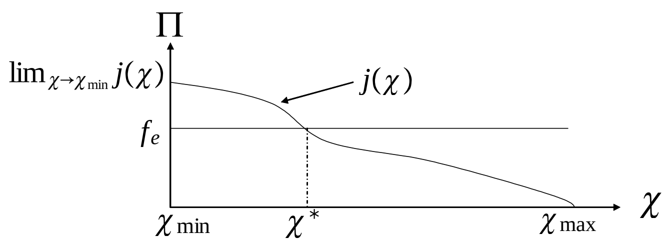
$$j(\chi) \equiv (1 - G(\chi))\, k(\chi) \tag{16}$$

and assume that

$$\lim_{\chi \to \chi_{\min}} j(\chi) > f_e. \qquad \text{(Assumption (SEL))}$$

In $(\chi, \bar{\Pi})$*-space, the FE curve is cut once from above by the ZCP curve at some* $\chi^* \in (\chi_{\min}, \chi_{\max})$.

Proposition (3) implies that when Assumption (SEL) is satisfied, there is an equilibrium with an active selection effect. Figure 5 depicts $j(\chi)$ under Assumption (SEL):



In equilibrium, a firm with productivity χ chooses to remain active if and only if $j(\chi) < f_e$. An entrant remains active if and only if it is sufficiently productive because $j(\chi) < f_e$ if and only if $\chi > \chi^*$. This implies that there is an active selection effect: productive firms choose to remain active and unproductive firms choose to exit. In all that follows I assume that Assumption (SEL) is satisfied in order to focus on equilibria with selection.

Although competition is equally tough across ex-ante identical markets, these markets may differ ex-post in the number of firms that compete and the average productivity of the firms in the market. For a given market in which firms have an average productivity $\bar{\chi}$, the number of firms in the market is found by equating equations (8) and (14):

$$n(\bar{\chi}, \chi^*) = \frac{1}{\sqrt{f_p/tL} + \frac{2}{3t+2\tau}(\bar{\chi} - \chi^*)}$$

The number of firms in a given market is decreasing in the difference between the average productivity in the market and the cutoff productivity. If the productivity of all the firms in a given market equals the cutoff productivity, then there are $n(\chi^*, \chi^*) = \sqrt{tL/f_p}$ firms in the market. Not surprisingly, this is the number of firms that would enter in a model with symmetric firms, only one market, and no fixed cost of entry, f_e, separate from the fixed cost of production, f_p.[22] More generally, a market, A, may have more firms than

[22] This is the number of firms that enter in the Salop (1979) model when there is a "competitive equilibrium." A competitive

another market, B: $n_A > n_B$. However, this implies that the average productivity of firms in market B must be strictly greater than the average productivity of the firms in market A: $\bar{\chi}_B > \bar{\chi}_A$.

4.1 Across market comparative statics

For tractability I have assumed that there are a continuum of ex-ante identical markets between which successful entrants allocate themselves. In order to perform comparative statics, however, markets must differ ex-ante. To maintain tractability, I investigate the effect of changing a parameter, e.g. L, in *all* markets. The results are qualitatively similar to those that would be obtained in the more realistic, but less tractable framework in which there is only one, rather than a continuum of markets, and L changes in this market.

In what follows I focus on the comparative statics of changes in demand density because these are the most empirically relevant. Although I focus on demand density, I also provide comparative statics for both fixed costs.

4.1.1 Demand density

How do changes in demand density affect the cutoff productivity and market softness? I find $d\chi^*/dL$ by differentiating equation (29) with respect to L, yielding

$$\frac{d\chi^*}{dL} = \frac{j(\chi^*)/2L}{-j'(\chi)|_{\chi^*}} > 0.$$

As demand density increases, the cutoff productivity increases. An increase in L shifts the ZCP curve to the right without affecting the FE curve, causing an increase in the cutoff productivity.

Intuitively, holding the mass of entrants and the cutoff productivity constant, an increase in L increases variable profit. This implies that χ^* decreases if the mass of entrants is held constant. However, an increase in L increases entry. As entry increases, variable profits decrease. This implies that χ^* increases. The indirect effect of an increase in L channeled through entry outweighs the direct effect of an increase in L on χ^* because as more firms enter it becomes easier for consumers to substitute from unproductive to productive firms. The result that the cutoff productivity increases with demand density accords well with the empirical findings of Syverson (2004) and Campbell and Hopenhayn (2002).

equilibrium is one in which at least some consumers buying from each firm i would buy from another firm if firm i raised its price.

I find the effect of a change in demand density on market softness by differentiating equation (14) with respect to L:

$$\frac{d\lambda}{dL} = -\left(\frac{1}{2L}\sqrt{f_p/tL} + \frac{2}{3t+2\tau}\frac{d\chi^*}{dL}\right) < 0.$$

An increase in demand density has two effects on market softness. First, the cutoff productivity increases with demand density. This increases the average productivity across markets.[23] Moreover, an increase in L sufficiently increases entry such that the average number of firms per market increases.[24] Both of these effects cause markets to become tougher.

4.1.2 Fixed costs

I find the effect of a change in fixed costs on the cutoff productivity by differentiating equation (29) with respect to each fixed cost. Increasing the fixed cost of entry leaves $j(\chi)$ unchanged. As is evident from Figure 5, this decreases the cutoff productivity:

$$\frac{d\chi^*}{df_e} = \frac{1}{j'(\chi^*)} < 0$$

Increasing the fixed cost of production leaves f_e unchanged and shifts $j(\chi)$ upwards in $\left(\chi, \bar{\Pi}\right)$-space. As is evident from Figure 5, this increases the cutoff productivity:

$$\frac{d\chi^*}{df_p} = \frac{\partial j(\chi^*)/\partial f_p}{-j'(\chi^*)} > 0.$$

I find the effect of a change in fixed costs on market softness by differentiating equation (14). The effects of changes in the fixed costs on market softness are

$$\frac{d\lambda}{df_e} = \frac{-2}{3t+2\tau}\frac{d\chi^*}{df_e} > 0$$

and

$$\frac{d\lambda}{df_p} = \frac{1 - G(\chi^*)}{2tL\int_{\chi}^{\chi_{\max}}\left(\sqrt{f_p/tL} + \frac{2(\eta-\chi)}{3t+2\tau}\right) g(\eta)\, d\eta} > 0.$$

It is not surprising that softer markets are associated with higher fixed costs. Higher fixed costs decrease

[23] There may be some markets that experience no change or even a decrease in the average productivity of firms after an increase in L (these markets will experience an increase in n). However, when averaging across markets, average productivity increases.

[24] Again, some markets may actually experience a decrease in the number of firms after an increase in L (these markets will experience a sufficiently large increase in average productivity). However, on average the number of firms per market increases.

entry, which softens markets.

5 Conclusion

In this paper, I have created a realistic and tractable framework for studying endogenous product or geographic differentiation. The model incorporates firm heterogeneity. Firms' location decisions depend on the strength of their competition. The model also combines horizontal and vertical dimensions of differentiation because most industries are characterized by both.

This paper demonstrates that two direct competitors' products are less substitutable than average if and only their average productivity is greater than the average productivity in the market. This provides a new explanation for the effect of productivity on profit. More productive firms earn higher profit because their competitors offer relatively poor substitutes, all else constant. This is in addition to the standard argument that more productive firms charge larger markups.

Although the model focuses on the case in which demand density is constant within a market, in reality demand varies within markets. The model shows that denser markets are characterized by greater competition, implying that the products of equally productive neighbors are better substitutes in denser markets. This intuition suggests that if demand density varies within a market, the substitutability of direct competitors may vary across different portions of the market. Extending the model to incorporate variable densities is therefore an important task for future research.

References

Anderson, S. P., A. de Palma, and J.-F. Thisse (1992): "Discrete choice theory of product differentiation," *MIT Press.*

Campbell, J. R., and H. A. Hopenhayn (2002): "Market Size Matters," *National Bureau of Economic Research, Inc, NBER Working Papers: 9113.*

Collard-Wexler, A. (2006): "Plant Turnover and Demand Fluctuations in the Ready-Mix Concrete Industry," *Center for Economic Studies, U.S. Census Bureau, Working Papers.*

Dasgupta, P., and E. Maskin (1986a): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

——— (1986b): "The Existence of Equilibrium in Discontinuous Economic Games, II: Applications," *Review of Economic Studies*, 53(1), 27–41.

d'Aspremont, C., J. J. Gabszewicz, and J.-F. Thisse (1979): "On Hotelling's "Stability in Competition"," *Econometrica*, 47(5), 1145–50.

Davis, P. (2005): "Spatial Competition in Retail Markets: Movie Theaters," *CEPR*.

Dixit, A. K., and J. E. Stiglitz (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67(3), 297–308.

Economides, N. (1989): "Symmetric Equilibrium Existence and Optimality in Differentiated Product Markets," *Journal of Economic Theory*, 47(1), 178–94.

Hopenhayn, H. A. (1992): "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–50.

Hotelling, H. (1929): "Stability in Competition," *The Economic Journal*, 39(153), 41–57.

Jovanovic, B. (1982): "Selection and the Evolution of Industry," *Econometrica*, 50(3), 649–70.

Lancaster, K. J. (1979): *Variety, Equity, and Efficiency: Product Variety in an Industrial Society.* Columbia University Press, New York.

Lucas, Robert E., J. (1978): "On the Size Distribution of Business Firms," *Bell Journal of Economics*, 9(2), 508–23.

Mazzeo, M. J. (2002): "Product Choice and Oligopoly Market Structure," *RAND Journal of Economics*, 33(2), 221–242.

Melitz, M. J. (2003): "The Impact of Trade on Intra-industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 71(6), 1695–1725.

Melitz, M. J., and G. I. P. Ottaviano (2005): "Market Size, Trade, and Productivity," *National Bureau of Economic Research, Inc, NBER Working Papers: 11393*.

Rauch, J. E. (1999): "Networks versus Markets in International Trade," *Journal of International Economics*, 48(1), 7–35.

Salop, S. C. (1979): "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, 10(1), 141–56.

Seim, K. (2001): "Spatial Differentiation and Market Structure: The Video Retail Industry," *Dissertation.*

Spence, M. (1976): "Product Selection, Fixed Costs, and Monopolistic Competition," *Review of Economic Studies*, 43(2), 217–35.

Syverson, C. (2004): "Market Structure and Productivity: A Concrete Example," *Journal of Political Economy*, 112(6), 1181–1222.

Viner, J. (1932): "Cost Curves and Supply Curves," *Zeitschrift fur Nationalokonomie*, 3, 23–46.

Watson, R. (2004): "Product Variety and Competition in the Retail Market for Eyeglasses," *Mimeo: University of Texax at Austin.*

A Proof of Proposition (1)

The proof of Proposition (1) is in four sections. I prove that any strategy $\vec{\omega} \in O^*$ is a strict SPNE if $\tau > 0$ in the first three sections. In the fourth section I prove that there exists no strict SPNE $\vec{\omega}' \in \Omega^n$, such that $\vec{\omega}' \notin O^*$.

A.1 Overtaking

A firm is overtaken if it supplies no consumers in equilibrium. In this section I prove that any unilateral deviation that leads firm i to overtake another firm with certainty is strictly dominated. I also prove that if firm i unilaterally deviates by locating between firms j and $j+1$, no firm other than i, j, or $j+1$ would ever overtake another firm with probability one. By doing this, I am able to focus exclusively on equilibria in which no firm is overtaken with certainty in the remainder of the proof.

Suppose that firm i deviates in such a way that it overtakes another firm, j, with probability one. This implies that firm i charges a price no greater than $k_j + c_j q_j + td_{i,j} + q_i^\gamma - q_j^\gamma$. Suppose that all firms are identical. Then $k_j + c_j q_j = k + \gamma \left(\frac{c}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}$. The upper bound on firm i's price is greatest when $d_{i,j} = 0$. Hence, firm i's price must be no less than $p_i' \equiv k - (1-\gamma)\left(\frac{c}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} + q_i^\gamma$.

I prove that $p_i' \leq k + cq_i$ for any k, c, and q_i. $p_i' \leq k + cq_i \iff q_i^\gamma - (1-\gamma)\left(\frac{c}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} \leq cq_i$. The right-hand-side ($cq_i$) is increasing and linear in q_i. The left-hand-side ($q_i^\gamma - (1-\gamma)\left(\frac{c}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}$) is increasing and concave in q_i. If the left-hand-side is ever greater than the right-hand-side, it must be greater at the q_i at which the two slopes are equated (at $q_i = \left(\frac{c}{\gamma}\right)^{\frac{1}{\gamma-1}}$). At this q_i, the left-hand-side equals the right-hand-side. This implies that firm i's profit per sale is bounded above by zero if it overtakes firm j with certainty. When firms are identical, overtaking another firm with probability one is a strictly dominated strategy. This implies that for asymmetries that are not too large, overtaking with probability one remains strictly dominated.

A similar argument can be used to prove that if firm i unilaterally deviates, locating between firms j and $j+1$, no firm other than i, j, or $j+1$ would ever overtake another firm with probability one (for sufficiently small asymmetries).

A.2 Pricing stage deviations

In this section I prove that if all firms follow their equilibrium strategies in the location/quality stage, then all unilateral deviations are strictly dominated in the price stage.

If firm i sets a price such that there is an indifferent consumer located between i and $i+1$ and an indifferent consumer between i and $i-1$ then its first-order condition is given by equation (**??**). Suppose that all firms choose quality according to equation (10) and locate according to equation (9) in the location/quality stage. Suppose that all firms $j \neq i$ choose their price according to equation (11) in the price stage. Substituting into firm i's reaction function from equation (**??**) the distances between all firms using equation (9), the qualities of firm i and its neighbors using equation (10), and the prices of firms $i-1$ and $i+1$ using equation (11) yields equation (11). Hence, when firm i does not overtake its neighbors, any price-stage deviation is strictly dominated.

A.3 Location/quality stage deviations

In this section I prove that all unilateral location/quality-stage deviations are strictly dominated if $\tau > 0$ (and are weakly dominated if $\tau = 0$). I focus on the case in which no firm is overtaken with certainty throughout this section, because I have already considered all cases in which at least one firm supplies no consumers with probability one.

If a firm deviates in the location/quality stage, it may be the case that there exists no pure-strategy equilibrium to the subsequent price stage. I separately consider the cases in which there is and is not a pure-strategy equilibrium to the price stage. I define a new function, which I denote "benchmark profit." Benchmark profit is relevant for two reasons. First, if there exists a price-stage subgame equilibrium in pure strategies, the equilibrium vector of prices is the solution to the benchmark system of reaction functions. Second, when no pure-strategy price-stage subgame equilibrium exists, benchmark profit serves as a useful comparison to the profit that firms earn when they mix.

I begin by defining benchmark profit and solving the resultant "benchmark system" of price-stage reaction functions. In the next step I prove that wherever firm i locates, it will always choose to set its quality equal to q_i^* if firms price according to the benchmark system in the pricing stage.

I then consider location deviations under the assumption that no firm deviates in quality. If firm i deviates in location in the location/quality stage, it can either deviate locally or drastically. Firm, i, deviates locally if it locates between the two firms between which it is supposed to locate, but not in the exact location specified by equation (9). Firm i deviates drastically if it locates between two firms between which it is not supposed to locate.

I prove firm i's benchmark profit is strictly lower if it drastically deviates than if it follows its equilibrium strategy. This implies that any drastic deviation is strictly dominated if firms use pure strategies in the subsequent price stage. I similarly prove that firm i's benchmark profit is strictly (weakly) lower if it locally deviates than if it follows its equilibrium strategy when $\tau > 0$ ($\tau = 0$). This implies that any local deviation is strictly (weakly) dominated if firms use pure strategies in the subsequent price stage and $\tau > 0$ ($\tau = 0$).

Finally I prove that each firm's profit is bounded above by its benchmark profit for any location and quality profile such that the price-stage equilibrium involves mixing. This implies that any location/quality stage deviation is strictly dominated if $\tau > 0$ and is weakly dominated if $\tau = 0$.

A.3.1 Benchmark prices

Recall that firm i's profit is

$$\pi_i = \begin{cases} L\left(x_i\left(P_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right) & \text{if } x_{i,i-1}, x_{i,i+1} \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

I define a "benchmark profit" function that differs slightly from the real profit function. A firm's benchmark profit is

$$\pi_i^B = \begin{cases} L\left(x_i\left(P_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right) & \text{if } x_{i,i-1} + x_{i,i+1} \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The difference between profit and benchmark profit is the condition under which profit is not equal to zero. A firm's real profit is not restricted to zero if $x_{i,i-1} \geq 0$ and $x_{i,i+1} \geq 0$. It is this restriction that causes profit to be discontinuous and not quasi-concave. On the other hand, a firm's benchmark profit is not restricted to zero if $x_{i,i-1} + x_{i,i+1} \geq 0$. The benchmark system of reaction functions is relevant for two reasons. First, if there exists a price-stage subgame equilibrium in pure strategies in which no firm is overtaken, the equilibrium vector of prices is the solution to the benchmark system of reaction functions. Second, when no pure-strategy price-stage subgame equilibrium exists, benchmark profit serves as a useful comparison to the profit that firms earn when they mix.

Let the vector $\vec{P} \equiv (P_0, ..., P_{n-1})'$ denote the unique solution to the benchmark system of equations. The importance of this benchmark system is twofold. First, firm i's benchmark profit is continuous and quasi-concave in its price. Second, whenever a pure-strategy equilibrium exists, the equilibrium coincides with the solution to the benchmark system. The two coincide because there exists no pure strategy equilibrium in which no firm is overtaken with probability one and at least one firm j faces a binding constraint of either $x_{j,j-1} = 0$ or $x_{j,j+1} = 0$. Suppose such an equilibrium existed and $x_{j,j+1} = 0$. Then firm $j+1$ could infinitesimally lower its price and gain a discrete mass of consumers ($x_{j,j-1} > 0$). This implies that no equilibrium of this form can exist.

The vector $\vec{P}$ is the solution to the system $A\vec{P} = \vec{b}$, where A is an $n \times n$ symmetric, circulant matrix in which the first row is

$$\left(\frac{2\,(2t+\tau)}{t+\tau}, -1, 0, ..., 0, -1 \right).$$

Since A is circulant, row $(j+1)$ has the same elements as row j, but the elements are moved one position to the right and wrapped around. The A matrix takes the following form:

$$A = \begin{bmatrix} \frac{2(2t+\tau)}{t+\tau} & -1 & 0 & 0 & -1 \\ -1 & \frac{2(2t+\tau)}{t+\tau} & -1 & 0 & 0 \\ ... & ... & ... & ... & ... \\ -1 & 0 & 0 & -1 & \frac{2(2t+\tau)}{t+\tau} \end{bmatrix}$$

The vector $\vec{P}$ is $\bmod{(n)}$ where

$$(P_0, P_1, ..., P_{n-1})'$$

The vector $\vec{b} \equiv (b_0, ..., b_{n-1})'$ is $\mathrm{mod}\,(n)$ where

$$b_i \equiv t\,(d_{i-1,i} + d_{i,i+1}) - q_{i-1}^{\gamma} - q_{i+1}^{\gamma} + \frac{2t}{t+\tau}\,(k_i + q_i c_i) + 2q_i^{\gamma}$$

and $\vec{d} \equiv (d_{0,1}, d_{1,2}..., d_{n-2,n-1}, d_{n-1,0})'$ is also $\mathrm{mod}\,(n)$.

Lemma 1 *Suppose firms set benchmark prices according to $A\vec{P} = \vec{b}$ in the pricing stage.*

1. *If n is even then*

$$\begin{aligned} P_i &= t\left(\beta_1\,(d_{i-1,i} + d_{i,i+1}) + ... + \beta_{\frac{n}{2}}\left(d_{i+1-\frac{n}{2},i-\frac{n}{2}} + d_{i-1+\frac{n}{2},i+\frac{n}{2}}\right)\right) \\ &\quad + \left(\psi_0\,(q_i^{\gamma}) + \psi_1\,(q_{i-1}^{\gamma} + q_{i+1}^{\gamma}) + ... + \psi_{\frac{n}{2}}\left(q_{i-\frac{n}{2}}^{\gamma}\right)\right) \\ &\quad + \frac{2t}{t+\tau}\left(\delta_0\,(k_i + q_i c_i) + \delta_1\,(k_{i-1} + q_{i-1}c_{i-1} + k_{i+1} + q_{i+1}c_{i+1}) + ... + \delta_{\frac{n}{2}}\left(k_{i-\frac{n}{2}} + q_{i-\frac{n}{2}}c_{i-\frac{n}{2}}\right)\right) \end{aligned}$$

   for all $i = 0, ..., n-1$. If n is odd then

$$\begin{aligned} P_i &= t\left(\beta_1\,(d_{i-1,i} + d_{i,i+1}) + ... + \beta_{\frac{n+1}{2}}\left(d_{i-\frac{n-1}{2},i-\frac{n}{2}}\right)\right) \\ &\quad + \left(\psi_0\,(q_i^{\gamma}) + \psi_1\,(q_{i-1}^{\gamma} + q_{i+1}^{\gamma}) + ... + \psi_{\frac{n-1}{2}}\left(q_{i-\frac{n-1}{2}}^{\gamma} + q_{i+\frac{n-1}{2}}^{\gamma}\right)\right) \\ &\quad + \frac{2t}{t+\tau}\left(\delta_0\,(k_i + q_i c_i) + ... + \delta_{\frac{n-1}{2}}\left(k_{i-\frac{n-1}{2}}c_{i-\frac{n-1}{2}} + q_{i-\frac{n-1}{2}}c_{i-\frac{n-1}{2}} + k_{i+\frac{n-1}{2}}q_{i+\frac{n-1}{2}}c_{i+\frac{n-1}{2}}\right)\right) \end{aligned}$$

   for all $i = 0, ..., n-1$.

2. $\beta_1 > \beta_2 > ...$

3. $\frac{2t}{t+\tau}\,(\delta_0 - \delta_1) = \psi_1 - \psi_0 + 1$

Proof. *The solution to this system is found by pre-multiplying both sides by A^{-1}. Define the matrix H by $H \equiv A^{-1}$. The solution is given by $\vec{P} = H\vec{b}$. The inverse of a symmetric matrix is symmetric. The inverse of a circulant matrix is circulant. Therefore, H is symmetric and circulant. Let $h_{i,k}$ denote the element of H in the $(i+1)$'st row and $(k+1)$'st column (where $h_{i,k}$ is $\mathrm{mod}\,(n)$). The row in which $h_{i,k}$ is located can be suppressed because H is circulant. Instead, let $h\,(k)$ denote the $(k+1)$'st column in the first row of H. With this notation $h_{i,k} = h\,(k-i)$. Firm i's price is $P_i = \sum\limits_{k=0}^{n-1} h\,(k-i)\,b_k$.*

I now prove Part (1) of Lemma (1). I do not focus on proving the distinction between the cases in which n is odd and even. Instead, I prove the general form of P_i. I begin by focusing on the distance terms in

P_i. Using the symmetry of H I prove that the coefficients on $d_{i+k,i+k+1}$ and $d_{i-k,i-k-1}$in the solution to P_i are identical. The term $d_{i+k,i+k+1}$ appears in b_{i+k} and b_{i+k+1} and nowhere else in $\vec{b}$. In the solution to P_i, b_{i+k} is multiplied by $h(k)$ while b_{i+k+1} is multiplied by $h(k+1)$. Hence, in the solution to P_i the coefficient multiplying $d_{i+k,i+k+1}$ is $(h(k)+h(k+1))$. Similarly, the term $d_{i-k,i-k-1}$ appears in b_{i-k} and b_{i-k-1} and nowhere else in $\vec{b}$. In the solution to P_i, b_{i-k} is multiplied by $h(-k)$ while b_{i-k-1} is multiplied by $h(-k-1)$. Hence, in the solution to P_i, the coefficient multiplying $d_{i-k,i-k-1}$ is $(h(-k)+h(-k-1))$. Because H is symmetric, $h(k)=h(-k)$ and $h(k+1)=h(-k-1)$. This implies that the distance terms enter the solution of P_i in the form

$$\beta_1\left(d_{i-1,i}+d_{i,i+1}\right)+\beta_2\left(d_{i-2,i-1}+d_{i+1,i+2}\right)+...$$

where

$$\beta_k=h(k-1)+h(k)$$

I have yet to consider the q^γ and $\frac{2t}{t+\tau}(k+qc)$ terms. I will focus on the q^γ terms first. The term $-q_{i-j}^\gamma$ appears in both b_{i-j-1} and b_{i-j+1} while the term $2q_{i-j}^\gamma$ appears in b_{i-j}. In the solution to P_i, b_{i-j-1} is multiplied by $h(-j-1)$; b_{i-j+1} is multiplied by $h(-j+1)$; and b_{i-j} is multiplied by $h(-j)$. Hence, in the solution to P_i, the coefficient multiplying q_{i-j}^γ is $(-h(-j-1)-h(-j+1)+2h(-j))$. Similarly, in the solution to P_i the coefficient multiplying q_{i+j}^γ is $(-h(j-1)-h(j+1)+2h(j))$. Because H is symmetric, $h(-j-1)=h(j+1)$; $h(-j+1)=h(j-1)$; and $h(-j)=h(j)$. Hence, the q^γ terms enter the solution of P_i in the form

$$\psi_0\left(q_i^\gamma\right)+\psi_1\left(q_{i-1}^\gamma+q_{i+1}^\gamma\right)+...$$

where

$$\psi_j=-h(j-1)-h(j+1)+2h(j).$$

To complete the proof of Part (1) of Lemma (1) I consider the $\frac{2t}{t+\tau}(k+qc)$ terms. The term $\frac{2t}{t+\tau}\times(k_{i-j}+q_{i-j}c_{i-j})$ only appears in b_{i-j} and $\frac{2t}{t+\tau}\times(k_{i+j}+q_{i+j}c_{i+j})$ only appears in b_{i+j}. In the solution to P_i, b_{i-j} is multiplied by $h(-j)$ while b_{i+j} is multiplied by $h(j)$. Moreover, $h(-j)=h(j)$ because H is symmetric. Hence, the $\frac{2t}{t+\tau}(k+qc)$ terms enter the solution of P_i in the form

$$\frac{2t}{t+\tau}\left(\delta_0\left(k_i+q_ic_i\right)+\delta_1\left(k_{i-1}+q_{i-1}c_{i-1}+k_{i+1}+q_{i+1}c_{i+1}\right)+...\right)$$

where

$$\delta_j = h(j).$$

This completes the proof of Part (1) of Lemma (1).

Next I prove Part (2) of Lemma (1) focusing on the case in which n *is odd. I do not work out the case in which* n *is even, but the proof follows the same logic. In the case in which* n *is odd, Part (2) of Lemma (1) states that* $\beta_k > \beta_{k+1}$ *for all* $k \leq \frac{n-1}{2}$*. This is equivalent to* $h(k-1) > h(k+1)$ *for all* $k \leq \frac{n-1}{2}$*.*

In order to prove Part (2) of Lemma (1) I introduce one piece of new notation. Fix $h(j)$ *and denote by* j' *the minimal number of elements separating* $h(j)$ *from* $h(0)$*:* $j' = \begin{cases} j & \text{if } j \leq \frac{n-1}{2} \\ n-j & \text{if } j > \frac{n-1}{2} \end{cases}$*. I prove Part (2) of Lemma (1) by proving that* $h(j-i) > h(k-i)$ *if and only if* $j' < k'$*.*

$AH = I$ *because* $H \equiv A^{-1}$*. Considering only the first row of the identity matrix yields a system of equations*

$$\begin{gathered}
\tfrac{2(2t+\tau)}{t+\tau} h(0) - h(1) - h(1) = 1 \\
\tfrac{2(2t+\tau)}{t+\tau} h(1) - h(0) - h(2) = 0 \\
\ldots \\
\tfrac{2(2t+\tau)}{t+\tau} h(j-1) - h(j-2) - h(j) = 0 \\
\ldots \\
\tfrac{2(2t+\tau)}{t+\tau} h(n-1) - h(n-2) - h(0) = 0
\end{gathered}$$

which has a general solution

$$h(j) = \zeta(j)\, h(0) - \nu(j) \tag{17}$$

and the boundary condition

$$h\left(\frac{n-1}{2}\right) = h\left(\frac{n+1}{2}\right) \tag{18}$$

since H *is symmetric.* $h(j) = \left(\frac{2(2t+\tau)}{t+\tau}\zeta(j-1) - \zeta(j-2)\right) h(0) - \left(\frac{2(2t+\tau)}{t+\tau}\nu(j-1) - \nu(j-2)\right)$*, implying that*

$$\begin{aligned}
\zeta(j) &= \frac{2(2t+\tau)}{t+\tau}\zeta(j-1) - \zeta(j-2) \qquad (19) \\
\nu(j) &= \frac{2(2t+\tau)}{t+\tau}\nu(j-1) - \nu(j-2)
\end{aligned}$$

for all $j \geq 2$. Moreover,

$$\zeta(0) = 1,\ \zeta(1) = \frac{2t+\tau}{t+\tau} \text{ and } \nu(0) = 0,\ \nu(1) = \frac{1}{2}.$$

Using the boundary condition and equation (19) to solve for $h(0)$ as a function of $\nu\left(\frac{n-1}{2}\right), \nu\left(\frac{n-3}{2}\right), \zeta\left(\frac{n-1}{2}\right)$, and $\zeta\left(\frac{n-3}{2}\right)$ yields

$$h(0) = \frac{\frac{3t+\tau}{t+\tau}\nu\left(\frac{n-1}{2}\right) - \nu\left(\frac{n-3}{2}\right)}{\frac{3t+\tau}{t+\tau}\zeta\left(\frac{n-1}{2}\right) - \zeta\left(\frac{n-3}{2}\right)} \tag{20}$$

It remains to solve the differential equation given in equation (19). This is equivalent to solving the system

$$x(y) - \frac{2(2t+\tau)}{t+\tau}x(y-1) + x(y-2) = 0$$

given two different sets of initial conditions

1. *$x(0) = 1$ and $x(1) = \frac{2t+\tau}{t+\tau}$ (for $\zeta(y)$)*
2. *$x(0) = 0$ and $x(1) = 1/2$ (for $\nu(y)$)*

The general solution to such a system is $x(y) = kr_1^y + jr_2^y$ where r_1 and r_2 are the roots of $x^2 - \frac{2(2t+\tau)}{t+\tau}x + 1 = 0$. Let

$$z \equiv \frac{2(2t+\tau)}{t+\tau} \in (2, 4].$$

Then

$$r_1 = \frac{1}{2}\left(z - \sqrt{z^2-4}\right) \in (0, 1)$$

and

$$r_2 = \frac{1}{2}\left(z + \sqrt{z^2-4}\right) > 1,$$

implying that $x(y) = k\left(\frac{z-\sqrt{z^2-4}}{2}\right)^y + j\left(\frac{z+\sqrt{z^2-4}}{2}\right)^y$. Solving for $\zeta(y)$ using the first set of initial conditions yields:

$$\zeta(y) = \frac{1}{2}\left(\left(\frac{z+\sqrt{z^2-4}}{2}\right)^y + \left(\frac{z-\sqrt{z^2-4}}{2}\right)^y\right). \tag{21}$$

Solving for $\nu(y)$ using the second set of initial conditions yields:

$$\nu(y) = \frac{1}{2\sqrt{z^2-4}}\left(\left(\frac{z+\sqrt{z^2-4}}{2}\right)^y - \left(\frac{z-\sqrt{z^2-4}}{2}\right)^y\right). \tag{22}$$

Substituting equations (21) and (22) into equation (17) yields a solution for $h(y)$ as a function of $h(0)$:

$$h(y) = \frac{1}{2\sqrt{z^2-4}}\left[(r_2)^y\left(h(0)\sqrt{z^2-4}-1\right)+(r_1)^y\left(h(0)\sqrt{z^2-4}+1\right)\right]$$

Solving for $h(0)$ from equation (20) using equations (21) and (22) yields an explicit solution for $h(0)$:

$$h(0) = \frac{(r_2)^{\frac{n-3}{2}}\left((z-1)r_2-1\right)-(r_1)^{\frac{n-3}{2}}\left((z-1)r_1-1\right)}{\sqrt{z^2-4}\left[(r_2)^{\frac{n-3}{2}}\left((z-1)r_2-1\right)+(r_1)^{\frac{n-3}{2}}\left((z-1)r_1-1\right)\right]} \tag{23}$$

Using the solution for $h(0)$ given in equation (23) yields an explicit solution for $h(y)$:

$$h(y) = \frac{(r_1)^y(r_2)^{\frac{n-3}{2}}\left((z-1)r_2-1\right)-(r_2)^y(r_1)^{\frac{n-3}{2}}\left((z-1)r_1-1\right)}{\sqrt{z^2-4}\left((r_2)^{\frac{n-3}{2}}\left((z-1)r_2-1\right)+(r_1)^{\frac{n-3}{2}}\left((z-1)r_1-1\right)\right)} \tag{24}$$

Finally, I prove that $h(y)$ is strictly decreasing in y if and only if $y<\frac{n}{2}$. Differentiating equation (24) with respect to y yields

$$h'(y)<0 \iff y<\frac{n-3}{2}+T(z),$$

where

$$T(z) \equiv \frac{\ln\left(\left(\frac{(z-1)\frac{1}{2}\left(z+\sqrt{z^2-4}\right)-1}{(z-1)\frac{1}{2}\left(z-\sqrt{z^2-4}\right)-1}\right)\left(\frac{\ln\frac{1}{2}\left(z-\sqrt{z^2-4}\right)}{\ln\frac{1}{2}\left(z+\sqrt{z^2-4}\right)}\right)\right)}{\ln\left(\frac{\frac{1}{2}\left(z+\sqrt{z^2-4}\right)}{\frac{1}{2}\left(z-\sqrt{z^2-4}\right)}\right)} \tag{25}$$

$T(z)$ can be expressed as function of r_1 and r_2, where r_1 and r_2 are the roots of the quadratic x^2-zx+1:

$$T(z) = \frac{\ln\left(\left(\frac{(z-1)r_2-1}{(z-1)r_1-1}\right)\left(\frac{\ln r_1}{\ln r_2}\right)\right)}{\ln\left(\frac{r_2}{r_1}\right)}$$

Substituting out r_1 using $r_1 = 1/r_2$ yields

$$T(z) = \frac{\ln\left(r_2\left(\frac{r_2-zr_2+1}{r_2-z+1}\right)\left(\frac{\ln r_2^{-1}}{\ln r_2}\right)\right)}{\ln r_2^2}$$

If $z>2$ (which it is) this is equivalent to

$$T(z) = \frac{\ln r_2^3}{\ln r_2^2}$$

because $\frac{\ln r_2^{-1}}{\ln r_2} = -1$ and $\frac{r_2-zr_2+1}{r_2-z+1} = -r_2^2$. This implies that $T(z) = 3/2$ for all $z>2$, proving that $h'(y)<0$

if and only if $y < n/2$. *This completes the proof of Part (2) of Lemma (1).*

Finally, I prove Part (3) of Lemma (1). Recall that $\psi_k = 2h(k) - h(k-1) - h(k+1)$ *and* $\delta_k = h(k)$. *Therefore*

$$\begin{gathered}\frac{2t}{t+\tau}(\delta_0 - \delta_1) = \psi_1 - \psi_0 + 1 \iff \\ \frac{2t}{t+\tau}(h(0) - h(1)) = 1 - h(0) - h(2) + 2h(1) - (-h(n-1) - h(1) + 2h(0)) \iff \\ \frac{2t}{t+\tau}(h(0) - h(1)) = 1 - 3h(0) + 3h(1) - h(2) + h(n-1) \iff \\ \Phi(n) \equiv \frac{5t+3\tau}{t+\tau}(h(0) - h(1)) + h(2) - h(n-1) = 1.\end{gathered}$$

$\Phi(n=3) = 1$ *because* $z + 1 = \frac{5t+3\tau}{t+\tau}$. *Therefore, the proof of Part (3) of Lemma (1) is complete if* $\Phi'(n) = 0$. *Define* $\eta \equiv \frac{5t+3\tau}{t+\tau} - \frac{2(3t+2\tau)}{t+\tau} r_1 + r_1^2$ *and* $\kappa \equiv \frac{5t+3\tau}{t+\tau} - \frac{2(3t+2\tau)}{t+\tau} r_2 + r_2^2$, *where* $\eta + \kappa = 0$. *Then*

$$\Phi(n) = \frac{(r_2)^{\frac{n-3}{2}}((z-1)r_2 - 1)\eta - (r_1)^{\frac{n-3}{2}}((z-1)r_1 - 1)\kappa}{\sqrt{z^2-4}\left((r_2)^{\frac{n-3}{2}}((z-1)r_2 - 1) + (r_1)^{\frac{n-3}{2}}((z-1)r_1 - 1)\right)}.$$

Differentiating $\Phi(n)$ *with respect to* n *yields*

$$\begin{gathered}\Phi'(n) = 0 \iff \\ \ln(r_2)(\eta + \kappa) = \ln(r_1)(\eta + \kappa) \iff \\ \eta + \kappa = 0\end{gathered}$$

This implies that $\Phi(n) = 1$ *for all* n, *proving Part (3) of Lemma (1).* ■

A.3.2 Optimal quality

Suppose that firm i unilaterally deviates in the location/quality stage. For any location choice, I find firm i's optimal quality under the assumption that firms set benchmark prices and the assumption that each firm supplies to a positive mass of consumers. Recall that benchmark profit is

$$\pi_i^B = L\left(x_i^2(t+\tau) + \tau\left((x_{i,i+1})^2 + (x_{i,i-1})^2\right)\right).$$

Differentiating benchmark profit with respect to q_i yields:

$$\frac{d\pi_i^B}{dq_i} = L\left(2x_i\left(t+\tau\right)\frac{\partial x_i}{\partial q_i} + 2\tau\left(x_{i,i+1}\frac{\partial x_{i,i+1}}{\partial q_i} + x_{i,i-1}\frac{\partial x_{i,i-1}}{\partial q_i}\right)\right)$$

where

$$\frac{\partial x_{i,i+1}}{\partial q_i} = \frac{1}{2t}\left(\frac{\partial p_{i+1}}{\partial q_i} - \frac{\partial p_i}{\partial q_i} + \gamma q_i^{\gamma-1}\right),$$

$$\frac{\partial x_{i,i-1}}{\partial q_i} = \frac{1}{2t}\left(\frac{\partial p_{i-1}}{\partial q_i} - \frac{\partial p_i}{\partial q_i} + \gamma q_i^{\gamma-1}\right),$$

and

$$\frac{\partial x_i}{\partial q_i} = \frac{1}{2t}\left(\frac{\partial p_{i+1}}{\partial q_i} + \frac{\partial p_{i-1}}{\partial q_i} - 2\frac{\partial p_i}{\partial q_i} + 2\gamma q_i^{\gamma-1}\right).$$

According to Part (3) of Lemma (1) $\frac{\partial p_{i+1}}{\partial q_i} = \frac{\partial p_{i-1}}{\partial q_i} = \psi_1\gamma q_i^{\gamma-1} + \delta_1\frac{2t}{t+\tau}c_i$ while $\frac{\partial p_i}{\partial q_i} = \psi_0\gamma q_i^{\gamma-1} + \delta_0\frac{2t}{t+\tau}c_i$. Therefore,

$$\begin{aligned}\frac{\partial x_{i,i+1}}{\partial q_i} &= \frac{\partial x_{i,i-1}}{\partial q_i} = \frac{1}{2t}\left(\psi_1\gamma q_i^{\gamma-1} + \delta_1\frac{2t}{t+\tau}c_i - \left(\psi_0\gamma q_i^{\gamma-1} + \delta_0\frac{2t}{t+\tau}c_i\right) + \gamma q_i^{\gamma-1}\right)\\ &= \frac{1}{2t}\left(\gamma q_i^{\gamma-1}\left(\psi_1-\psi_0+1\right) + \frac{2t}{t+\tau}c_i\left(\delta_1-\delta_0\right)\right)\end{aligned}$$

and

$$\frac{\partial x_i}{\partial q_i} = 2\frac{\partial x_{i,i+1}}{\partial q_i} = \frac{1}{t}\left(\gamma q_i^{\gamma-1}\left(\psi_1-\psi_0+1\right) + \frac{2t}{t+\tau}c_i\left(\delta_1-\delta_0\right)\right)$$

This implies that

$$\begin{gathered}\frac{d\pi_i^B}{dq_i} = 2Lx_i\left(2t+3\tau\right)\frac{\partial x_{i,i+1}}{\partial q_i} = 0 \iff\\ \gamma q_i^{\gamma-1}\left(\psi_1-\psi_0+1\right) + \frac{2t}{t+\tau}c_i\left(\delta_1-\delta_0\right) = 0 \iff\\ q_i = \left(\frac{\frac{2t}{t+\tau}\left(\delta_0-\delta_1\right)}{\psi_1-\psi_0+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{c_i}{\gamma}\right)^{\frac{1}{\gamma-1}}\end{gathered}$$

According to Part (3) of Lemma (1), this is equivalent to equation (10). Therefore, wherever firm i locates it will always choose to set q_i equal to q_i^* if firms set benchmark prices in the subsequent price stage.

A.3.3 Optimal location in the benchmark system

Local deviations

Lemma 2 *If firm i deviates locally in the location stage, then its benchmark profit is strictly (weakly) less than its equilibrium profit if $\tau > 0$ ($\tau = 0$), $q_i = q_i^*$, and no firm is overtaken.*

Proof. *Recall that benchmark profit is:*

$$\pi_i = L\left(x_i\left(P_i - (k_i + c_i q_i)\right) - \tau\left(x_{i,i-1}^2 + x_{i,i+1}^2\right)\right).$$

If firm i follows its equilibrium strategies denote its variables by an asterisk, *. If it deviates, denote variables by an apostrophe, $'$. Define $\Delta\pi_i \equiv \pi_i' - \pi_i^*$:

$$\frac{1}{L}\Delta\pi_i = x_i'\left(P_i' - (k_i + c_i q_i)\right) - x_i^*\left(P_i^* - (k_i + c_i q_i)\right) - \tau\left(\left(x_{i,i-1}'\right)^2 + \left(x_{i,i+1}'\right)^2 - \left(x_i^*\right)^2\right).$$

I first show that $x_i' = x_i^*$ and $P_i' = P_i^*$. Then I prove that $\left(x_{i,i-1}'\right)^2 + \left(x_{i,i+1}'\right)^2 > (x^*)^2$.

According to Part (1) of Lemma (1) firm i's benchmark price is

$$P_i = \beta_1 t\left(d_{i-1,i} + d_{i,i+1}\right) + \beta_2 t\left(d_{i-2,i-1} + d_{i+1,i+2}\right) + ...$$

The term $(d_{i-1,i} + d_{i,i+1})$ is constant for any choice of $d_{i-1,i}$ given that firm i locates between firm $i-1$ and firm $i+1$ because the locations of firm $i-1$ and firm $i+1$ are fixed. This implies that P_i is constant for any choice of $d_{i-1,i}$.

Moreover, x_i is also constant for any choice of $d_{i-1,i}$. Fix any location of firm i strictly between firm $i-1$ and firm $i+1$. Choose any distance $\delta \in (0, d_{i-1,i})$. If firm i moves δ units towards firm $i-1$ then firm $(i-1)$'s price decreases by $\delta t(\beta_1 - \beta_2)$ while firm $(i+1)$'s price increases by $\delta t(\beta_1 - \beta_2)$. Hence, $(P_{i-1} + P_{i+1})$ remains constant. x_i is a function of $(P_{i-1} + P_{i+1})$, P_i, and $(d_{i-1,i} + d_{i,i+1})$. Because each of these terms remains constant for any choice of $d_{i-1,i}$, x_i' is independent of $d_{i-1,i}$.

Since $x_i'\left(P_i' - (k_i + c_i q_i)\right) = x_i^*\left(P_i^* - (k_i + c_i q_i)\right)$, it is clear that $\Delta\pi_i < 0$ if and only if $\left(x_{i,i-1}'\right)^2 + \left(x_{i,i+1}'\right)^2 > (x^*)^2$ and $\tau > 0$. If firm i locally deviates by moving $\delta \in (0, d_{i-1,i})$ units towards firm $i-1$, then $x_{i,i-1}' = x_{i,i-1}^* - \delta/2$ and $x_{i,i+1}' = x_{i,i-1}^* + \delta/2$. This implies that $\left(x_{i,i-1}'\right)^2 + \left(x_{i,i+1}'\right)^2 > (x^*)^2$. Therefore, if firm i deviates locally in the location stage and $\tau > 0$ ($\tau = 0$), then its benchmark profit is strictly less than (equal to) its equilibrium profit. ■

Drastic deviations Consider the case in which all firms are identical. In equilibrium with symmetric firms, firms locate symmetrically along the circumference of the circle: $d_{j,j+1} = \frac{1}{n}$ for all neighbors j, $j+1$.

I prove that firm i's benchmark profit after a drastic deviation is strictly less than its equilibrium profit if no firm is overtaken.

If firms set benchmark prices then firm i chooses $q_i = q_i^*$, as proven above. According to Lemma (1) each firm charges $P^* = \beta_1 \frac{2}{n} + \beta_2 \frac{2}{n} + ...$ Moreover, $x_i^* = \frac{1}{n}$.

Suppose that firm i drastically deviates by locating between firm $i-k$ and firm $i-k-1$ for any $1 \leq k \leq \frac{n}{2}$ at a distance $d \in (0, 1/n)$ from firm $(i-k)$ when either n is even or n is odd and $k \neq \frac{n-1}{2}$.[25] According to Lemma (1) firm i's benchmark price is

$$p_i = P^* - \frac{t}{n}\left(\beta_1 - \beta_{k+1}\right) < P^*.$$

Therefore, firm i charges a strictly lower price after drastically deviating. Firm $(i-k)$'s benchmark price is

$$p_{i-k} = P^* - \left(\frac{1}{n} - d\right) t\beta_1 - dt\beta_2 + \frac{1}{n} t\beta_k$$

and firm $(i-k-1)$'s benchmark price is

$$p_{i-k-1} = P^* - dt\beta_1 - \left(\frac{1}{n} - d\right) t\beta_2 + \frac{1}{n} t\beta_{k+2}.$$

At these benchmark prices, firm i's market share is

$$x_i' = \frac{1}{2n}\left(1 + \beta_1 - \beta_2 + \beta_k - 2\beta_{k+1} + \beta_{k+2}\right).$$

A sufficient condition under which firm i's benchmark profit is strictly less than its equilibrium profit is $x_i' \leq x^* = \frac{1}{n}$. This is true because firm i's equilibrium price is strictly greater than its price after deviating and because firm i is centered in its zone of supply in equilibrium. $x_i' \leq \frac{1}{n}$ if and only if $\Phi(k) \leq 1$, where $\Phi(k) \equiv \beta_1 - \beta_2 + \beta_k - 2\beta_{k+1} + \beta_{k+2}$. A sufficient condition for $\Phi(k) \leq 1$ is $\beta_1 - \beta_2 + \beta_k \leq 1$. The condition $\beta_1 - \beta_2 + \beta_k \leq 1$ is true for all k if it is true for $k = 1$ because $\beta_1 > \beta_j$ for all $j > 1$. Therefore, $2\beta_1 - \beta_2 \leq 1$ is a sufficient condition for $\Phi(k) \leq 1$. Recall that $\beta_1 = h(0) + h(1)$

$$\beta_1 = \frac{(1+r_1)(r_2)^{\frac{n-3}{2}}((z-1)r_2 - 1) - (1+r_2)(r_1)^{\frac{n-3}{2}}((z-1)r_1 - 1)}{\sqrt{z^2-4}\left[(r_2)^{\frac{n-3}{2}}((z-1)r_2 - 1) + (r_1)^{\frac{n-3}{2}}((z-1)r_1 - 1)\right]}$$

[25] The proof when n is odd and $k = \frac{n-1}{2}$ is very similar. The difference is that in this case both of firm i's neighbors are equidistant from the arc between firms $i-1$ and $i+1$.

and $\beta_2 = h(1) + h(2)$

$$\beta_2 = \frac{\left(r_1 + r_1^2\right)(r_2)^{\frac{n-3}{2}}\left((z-1)r_2 - 1\right) - \left(r_2 + r_2^2\right)(r_1)^{\frac{n-3}{2}}\left((z-1)r_1 - 1\right)}{\sqrt{z^2-4}\left[(r_2)^{\frac{n-3}{2}}\left((z-1)r_2 - 1\right) + (r_1)^{\frac{n-3}{2}}\left((z-1)r_1 - 1\right)\right]}.$$

This implies that

$$2\beta_1 - \beta_2 = \frac{(r_2)^{\frac{n-3}{2}}\left((z-1)r_2 - 1\right)\left(2 + r_1 - r_1^2\right) + (r_1)^{\frac{n-3}{2}}\left((z-1)r_1 - 1\right)\left(r_2^2 - 2 - r_2\right)}{\sqrt{z^2-4}\left((r_2)^{\frac{n-3}{2}}\left((z-1)r_2 - 1\right) + (r_1)^{\frac{n-3}{2}}\left((z-1)r_1 - 1\right)\right)}$$

A sufficient condition for $2\beta_1 - \beta_2 \leq 1$ is $2\beta_1(n=3) - \beta_2(n=3) \leq 1$ because $2\beta_1 - \beta_2$ is maximized at $n = 3$ over the range $n \geq 3$ (which is the range over which drastic deviations are possible).

$$2\beta_1(n=3) - \beta_2(n=3) = \frac{2(z-1)}{(z-1)z-2},$$

which is strictly less than one for all $z > 3$; and $\tau < t$ implies that $z \in (3,4]$. Therefore, the assumption that $\tau < t$ is sufficient to ensure that all drastic deviations are strictly dominated when firms are identical. This implies that all drastic deviations remain strictly dominated for asymmetries that are not too large.

A.3.4 Mixed strategies

In this section I prove that for any profile of locations and qualities such that there exists no pure-strategy price-stage equilibrium and no firm is overtaken with certainty, each firm's profit is bounded above by its benchmark profit.

Suppose that there are $n \geq 2$ firms and that the profile of locations and qualities is such that there exists no pure-strategy price-stage equilibrium. According to Dasgupta and Maskin (1986b), there exists an equilibrium in mixed strategies in the pricing stage for all possible price-stage subgames. Denote by $[m_i, M_i']$ the support over which firm i mixes for all $i = 0, ..., n-1$, where m_i (M_i') represents the lowest (highest) price over which firm i mixes with positive probability, if such a price exists. If the range over which firm i mixes is unbounded, then m_i or M_i' may equal $-\infty$ or ∞.

Define $M_i \equiv \begin{cases} M_i' & \text{if } M_i' \leq v \\ v & \text{if } M_i' > v \end{cases}$. In what follows I focus on an equilibrium in which each firm i mixes over the support $[m_i, M_i]$, where M_i is bounded from above by v. However, this is without loss of generality. Firm i is indifferent between mixing over the range $[m_i, M_i]$ and the range $[m_i, M_i']$ because any price $p \geq M_i$

yields zero variable profit independent of the prices other firms charge. Moreover, firm i's best response is identical for any price $p_j \geq M_j$ that any competitor $j \neq i$ charges.

Consider any profile of locations and qualities such that firms must mix in the price stage. Each firm supplies a positive mass of consumers with positive probability only if

$$M_i < \min\left\{M_{i-1} + td_{i-1,i} - q_{i-1}^{\gamma} + q_i^{\gamma},\ M_{i+1} + td_{i,i+1} - q_{i+1}^{\gamma} + q_i^{\gamma}\right\} \tag{26}$$

for all i.

Lemma 3 *Suppose that all firms satisfy the condition in equation (26). Denote by p_i^* firm i's best response to $p_k = M_k$ for all $k \neq i$ given that firm i overtakes no firms. Then $p_i^* \geq M_i$.*

Proof. *Suppose $p_k = M_k$ for all $k \neq i$ and, to obtain a contradiction, that $p_i^* < M_i$. When $p_k = M_k$ for all $k \neq i$, firm i's best response is unique over the range of prices at which it does not overtake another firm because its profit function is continuous and strictly concave in this range. This implies that p_i^* is firm i's unique best response given that it does not overtake another firm. Hence, if firm i charges the price p_i^* it earns strictly more profit than if it charges the price M_i when $p_k = M_k$ for all $k \neq i$.*

The price $p_i = p_i^*$ yields firm i profit that is never less than than that yielded by the price $p_i = M_i$ for any profile of prices $p_k \leq M_k$ for all $k \neq i$ because prices are strategic complements. The fact that p_i^* strictly dominates M_i implies that firm i does not mix over M_i in equilibrium, contradicting the assumption that $p_i^* < M_i$. ■

Lemma 4 *Suppose that all firms satisfy the condition in equation (26). Suppose that there is no equilibrium in which firms use pure strategies along the equilibrium path. Then*

$$M_i \leq P_i$$

for all $i = 0, ..., n-1$.

Proof. *To obtain a contradiction, suppose that there exists a non-empty set of firms $J \equiv \{k \in N \mid M_k > P_k\}$. Denote by J^c the complement of J in N.*

Suppose all firms charge the upper bound over which they mix; that is each firm $j^c \in J^c$ (if any such firms exist) charges price M_{j^c} and each firm $j \in J$ (by assumption at least one such firm exists) charges price M_j. No firm overtakes another firm because all satisfy equation (26) by assumption.

Benchmark prices are the unique solution to the system of reaction functions when no firm overtakes *another firm. Benchmark prices are strategic complements. Thus, when all firm $j^c \in J^c$, if any such firms exist, charge $M_{j^c} \le P_{j^c}$ then the unique solution to the system of reaction functions for the remaining firms $j \in J$ must yield prices that are bounded above by P_j; in the case in which J^c is empty, the solution is P_j for all j. This implies that $p_j^* < M_j$ for at least one firm $j \in J$, contradicting Lemma (??). Thus, the set J must be empty.* ■

Now I prove that if firm i deviates in the location/quality stage in such a way that all firms satisfy the condition in equation (26) and there exists no equilibrium in which firms use pure strategies along the equilibrium path, then each firm's expected profit is no larger than its benchmark profit.

Lemma 5 *Suppose that the profile of locations and qualities is such that there exists no price-stage equilibrium in which firms use pure strategies and that all firms satisfy the condition in equation (26). Then each firm's expected profit in the mixing equilibrium is bounded above by its benchmark profit.*

Proof. *When firm i sets the price $p_i = M_i$ it is unable to overtake either of its two neighbors by assumption. Given that $p_i = M_i$, the upper bound of firm i's profit is increasing in p_{i-1} and p_{i+1}. Hence, firm i's profit is maximized when $p_j = M_j$ for all $j \neq i$. An upper bound on firm i's profit when it sets $p_i = M_i$ is given by*

$$\pi_i\left(p_j = M_j \forall j\right) = L \left(\begin{array}{c} x_i\left(p_j = M_j \forall j\right)\left(P_i - \left(k_i + c_i q_i\right)\right) \\ -\tau\left(\left(x_{i,i-1}\left(p_j = M_j \forall j\right)\right)^2 + \left(x_{i,i+1}\left(p_j = M_j \forall j\right)\right)^2\right) \end{array} \right)$$

where $\pi_i\left(p_j = M_j \forall j\right)$ is firm i's profit given that each firm charges the price that is the upper bound over which it mixes and $x_i\left(p_j = M_j \forall j\right)_i$, $x_{i,i-1}\left(p_j = M_j \forall j\right)$, and $x_{i,i+1}\left(p_j = M_j \forall j\right)$ are defined similarly. Let $\pi_i\left(p_j = P_j \forall j\right)$ be firm i's profit given that each firm charges its benchmark price. Clearly, $\pi_i\left(p_j = M_j \forall j\right) \le \pi_i\left(p_j = P_j \forall j\right)$.

$E\left[\pi_i\left(p_i = M_i\right)\right] \le \pi_i\left(p_j = M_j \forall j\right)$ because $\pi_i\left(p_j = M_j \forall j\right)$ is the upper bound on profits when i sets its price at the maximum price over which it mixes. Moreover, $E\left[\pi_i\left(p_i = M_i\right)\right]$ equals firm i's expected profit for any price over which it mixes with positive probability. This implies that firm i's expected profit when it mixes is bounded above by $\pi_i\left(p_j = M_j \forall j\right)$. ■

By transitivity, when all firms satisfy equation (26) and firm i locates such that there exists no equilibrium in which firms use pure strategies along the equilibrium path, firm i's expected profit in the mixing equilibrium is strictly less (if $\tau > 0$) or is no greater (if $\tau = 0$) than its equilibrium profit.

If equation (26) is violated then at least one firm never supplies a positive mass of consumers. If this is the case, firm i's profit is necessarily less than its equilibrium profit, as proven in the overtaking section.

A.4 Uniqueness

Fix $\tau > 0$. I have proven that if $\vec{\omega} \in O^*$, then $\vec{\omega}$ is a strict SPNE. It remains to prove for asymmetries that are not too large that a SPNE $\vec{\omega}$ is strict only if $\vec{\omega} \in O^*$.

To obtain a contradiction, suppose that there exists a strict SPNE $\vec{\omega}$ such that $\vec{\omega} \notin O^*$. Because asymmetries are not too large, there exists no SPNE in which a firm supplies no consumers with certainty. Along the equilibrium path firms locate such that they do not mix in the price-stage. This implies that firms set benchmark prices. When firms set benchmark prices each firm i chooses $q_i = q_i^*$.

Because $\vec{\omega}$ is a strict SPNE, each firm's price-stage best response must be unique. This implies that each firm is located such that any firm could move a positive distance in either direction and its best response would remain unique. If not, then at least one firm would have to be indifferent between mixing and using a pure strategy. If this were the case, then the equilibrium would not be strict.

Moreover, in every strict SPNE each firm must be centered in its zone of supply; that is, $x_{i,i-1} = x_{i,i+1}$ for all i. To obtain a contradiction suppose that there exists a strict SPNE in which $x_{i,i-1} \neq x_{i,i+1}$ for some i. If $x_{i,i-1} \neq x_{i,i+1}$ and best responses are unique, then firm i would increase its profit by moving towards the center of its zone of supply (under the condition that after moving, firms continue to use pure strategies in the subsequent price stage). Such a move is always possible from a strict equilibrium. This implies that a SPNE is strict only if $x_{i,i-1} = x_{i,i+1}$ for all i.

Each firm i is centered in its zone of supply if and only if

$$p_{i-1} + td_{i-1,i} - q_{i-1}^{\gamma} = p_{i+1} + td_{i,i+1} - q_{i+1}^{\gamma}. \tag{27}$$

Let $\Phi_i \equiv p_i + td_{i,i+1} - q_i^{\gamma}$. When firms set benchmark prices

$$\begin{aligned}
\Phi_i \quad = \quad & t\left(\beta_1\left(d_{i-1,i} + (1 + 1/\beta_1)\, d_{i,i+1}\right) + \beta_2\left(d_{i-2,i-1} + d_{i+1,i+2}\right) + ...\right) \\
& + \left(\left(\psi_0 + \frac{2t\gamma}{t+\tau}\delta_0 - 1\right)(q_i^{\gamma}) + \left(\psi_1 + \frac{2t\gamma}{t+\tau}\delta_1\right)\left(q_{i-1}^{\gamma} + q_{i+1}^{\gamma}\right) + ...\right) \\
& + \left(\frac{2t}{t+\tau}\left(\delta_0 k_i + \delta_1\left(k_{i-1} + k_{i+1}\right) + ...\right)\right).
\end{aligned}$$

Equation (27) can be expressed as $\Phi_{i-1} = \Phi_{i+1}$, which can itself be expressed as

$$\vec{a}_{i-1,i+1}'\vec{d} = b_{i-1,i+1} \tag{28}$$

for the appropriately defined vector $\vec{a}'_{i-1,i+1}$ and scalar $b_{i-1,i+1}$, each of which is constant for a given order in which the firms locate given that all firms choose quality according to equation (10).

The vector $\vec{d}^*$ satisfies equation (28) for all i. Either $\vec{d}^*$ is the unique solution to equation (28) for a given i or any vector $\vec{d}$ satisfies equation (28) for a given i. I have already proven that if an arbitrary firm i locally deviates from an equilibrium in O^* then it is not centered in its zone of supply. The fact that this arbitrary $\vec{d}$ does not satisfy equation (28) implies that $\vec{d}^*$ must be the unique solution to equation (28).

Therefore, given an order in which firms locate, a strict SPNE is characterized by the unique vector of distances given by equation (9). This contradicts the assumption that there exists a strict SPNE characterized by a vector of distances such that at least one element $d_{i,i+1}$ does not satisfy equation (9) and proves that a SPNE, $\vec{\omega}$, is strict only if $\vec{\omega} \in O^*$.

B Proof of Proposition (3)

Proof. If it exists, χ^* is defined by the equation

$$j(\chi^*) = f_e. \tag{29}$$

I prove that $j(\chi)$ is monotonically decreasing to zero. The derivative of $j(\chi)$ is

$$j'(\chi) = (1 - G(\chi))\, k'(\chi) - g(\chi)\, k(\chi)$$

where

$$k'(\chi) = \frac{g(\chi)}{1 - G(\chi)} k(\chi) - \frac{4tL}{3t + 2\tau} \frac{1}{1 - G(\chi)} \int_{\chi}^{\chi_{\max}} \left(\sqrt{f_p/tL} + \frac{2(\eta - \chi)}{3t + 2\tau} \right) g(\eta)\, d\eta.$$

Hence

$$j'(\chi) = \frac{-4tL}{3t + 2\tau} \int_{\chi}^{\chi_{\max}} \left(\sqrt{f_p/tL} + \frac{2(\eta - \chi)}{3t + 2\tau} \right) g(\eta)\, d\eta < 0$$

Moreover, $\lim_{\chi \to \chi_{\max}} j(\chi) = 0$. This implies that $j(\chi)$ is either always less than f_e or there is a cutpoint χ^* such that $j(\chi) > f_e$ for all $\chi < \chi^*$ and $j(\chi) > f_e$ for all $\chi > \chi^*$. The assumption that $\lim_{\chi \to \chi_{\min}} j(\chi) > f_e$ implies that there exists a unique $\chi^* \in (\chi_{\min}, \chi_{\max})$ at which the ZCP curve intersects the FE curve from above. ■