3. DIFFERENCE EQUATIONS AND LINEAR DYNAMIC MODELS

So gradually it came to pass that, from looking back together, they took also to looking forward together.

– Christina Rossetti (1830–1894)

(a) Rational Expectations and Rules of Forecasting

Many behavioural relationships in economics seem likely to depend on expectations of future values of exogenous variables. For example, inventory investment by a firm may increase if a strike is expected, stock market activity may be affected by an expected electoral decision, consumption this year may depend on expected income next year, investment may depend on expected future sales, and portfolio investment may depend on expected future returns or interest rates. Almost always these expectations are unobservable, so a model must contain some auxiliary assumption about how expectations are formed. In this section we'll study rational expectations in linear models. Then we'll apply these tools to study speculative bubbles, hyperinflation, and exchange-rate overshooting.

A crucial element in describing expectations concerns the information set attributed to agents. We'll deal with rational expectations, which sometimes can be described by saying that the entire structure of the economy is in the information set. More generally, we represent rational expectations as optimal forecasts, and what is optimal depends on what information is available *e.g.* $\mathrm{E}_1 y_2 \neq \mathrm{E} y_2$, in general. For simplicity, we also shall assume that everyone has the same expectations or that information is homogeneous, even though this ignores one of the main motives for trade.

Section 2 discussed modelling expectations in two-period models. Two-period models are often simple to solve and provide very helpful guides. But they obviously are not enough if we hope to discuss cycles, policy changes, and growth. So let us suppose that we have a ten-period model or a model with an infinite number of periods. If we hope to model uncertainty must we write down a joint density for the exogenous variables at all dates? *i.e.* In a ten-period endowment economy must we assume a joint density for $\{y_1, y_2, y_3, \ldots, y_{10}\}$?

Not at all. Instead we start with an initial value, say y_1. Then we write down a rule which gives the density of y_t given y_{t-1} for any t running from 1 to 10. This is simply the conditional density. Usually we assume that this density is the same for all values of t – that is sometimes referred to as stationarity. Then we can use it to construct a probability tree branching into future time periods.

To make all this clearer let us turn to some examples. First, suppose that y equals either 1 or 2 and that the transition rule is: stay with probability 0.6 and switch with probability 0.4. You can see that this is the natural extension of our earlier example.

Another way to describe a conditional probability model is to list the values and probabilities of the innovations or shocks to the process. For example, suppose that the rule for conditional probabilities is

$$\begin{aligned} y_t &= \rho y_{t-1} + \epsilon_t, \\ \epsilon_t &= \overline{\epsilon} \ \ w.p.\ \ \pi, \\ &= \underline{\epsilon} \ \ w.p.\ \ 1-\pi \end{aligned}$$

and (this is very important) ϵ is independently and identically distributed over time. We'll also assume that $E(\epsilon) = \overline{\epsilon}\pi + \underline{\epsilon}(1-\pi) = 0$, and thus define this shock as something which is unpredictable. Thus $E_{t-1}y_t = \rho y_{t-1}$.

To use this in modelling the multi-period expectations of agents one simply applies it repeatedly as follows:

$$\mathrm{E}_1 y_2 = \mathrm{E}(y_2|y_1) = \mathrm{E}(\rho y_1 + \epsilon_2|y_1) = \rho y_1 + E(\epsilon) = \rho y_1$$

That first step is obvious, but let us now find $\mathrm{E}_1 y_3$:

$$\mathrm{E}_1 y_3 = \mathrm{E}(y_3|y_1) = \mathrm{E}(\rho y_2 + \epsilon_3|y_1) = \rho \mathrm{E}(y_2|y_1) + E(\epsilon).$$

Notice that this expectation depends on the one we already have calculated. So substituting our first step gives:

$$\mathrm{E}_1 y_3 = \rho^2 y_1.$$

This is called the *chain rule of forecasting*. We can use it to build up distant forecasts by induction.

In some cases it makes sense to allow the shock and hence the variable itself, to be a continuous random variable. That modification is straightforward. We simply write:

$$y_t = \rho y_{t-1} + \epsilon_t; \quad \epsilon_t \sim \mathrm{iid}(0, \sigma^2),$$

for example. The conditional expectations are unchanged but now the variable y can take on more values.

When you regard these conditional probability models you may observe that they look like regression models from econometrics. That analogy is exact. If an econometrician believed that the value of y_1 were related systematically to its previous value then a good way to form forecasts would be to regress y_t on y_{t-1}, in historical data. The one-step forecast then would be $\hat{\rho} y_{t-1}$.

But that is exactly the forecast that we attribute to agents here, except that we usually ignore sampling variability in $\hat{\rho}$ and assume in theoretical models that agents know the true ρ. Expectations modelled in this way are called rational for two reasons. One is that the agent is assumed to forecast just as a statistician would. A second is that the agent also often is assumed to know the true ρ. But agents with rational expectations need not have superhuman knowledge. Recall from section 2 that conditional expectations depend on the information set which we attribute to agents (though economic theory may not tell us much about that). If they have little information in period $t-1$ then their expectations for period t, while efficiently constructed and not systematically wrong, may not be very accurate period by period.

The analogy between rational expectations and regression also suggests some properties that rational expectations should have. For example, forecast errors should have zero mean and one-step forecast errors should be serially uncorrelated and thus unpredictable. An example of a forecast error is

$$y_2 - \mathrm{E}_1 y_2 = y_2 - \rho y_1 = \rho y_1 + \epsilon_2 - \rho y_1 = \epsilon_2,$$

which has these properties. Also $\mathrm{E}(\epsilon_t \cdot y_{t-1}) = 0$ which again is familiar from ordinary least squares.

We also may use these ideas to show how forecasts might be revised. Notice that

$$\mathrm{E}_2 y_3 - \mathrm{E}_1 y_3 = \rho y_2 - \rho^2 y_1 = \rho(\rho y_1 + \epsilon_2) - \rho^2 y_1 = \rho\epsilon_2,$$

and thus

$$\mathrm{E}_2 y_3 - \mathrm{E}_1 y_3 = \rho(y_2 - \mathrm{E}_1 y_2).$$

This expression might remind some readers of adaptive expectations. The difference here is that in attributing these expectations to agents we choose ρ to match the actual regression coefficient rather than simply positing a number.

The final tool we shall need describes how to forecast forecasts. The rule for that is very simple:

$$\mathrm{E}_1[\mathrm{E}_2 y_3] = \mathrm{E}_1 y_3.$$

If this were not the case then there would be predictable changes in our forecast, which wouldn't make much sense. This rule is called the *law of iterated expectations.* We shall see a concrete example below.

In the conditional probability rules so far we have assumed that $\mathrm{Prob}(y_t|y_{t-1}) = \mathrm{Prob}(y_t|y_{t-1}, y_{t-2}, \ldots)$ so that further lags in information do not provide any additional information. But we could generalize what we have done to allow rules such as:

$$y_t = \rho_0 + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t,$$

just as one could add lags to a forecasting model. Here

$$\mathrm{E}_{t-1} y_t = \rho_0 + \rho_1 y_{t-1} + \rho_2 y_{t-2}.$$

In other cases the information set will include variables other than past values of y itself. The value for the conditional expectation depends on what we include in z, the information set. For that reason it is usually best to write $\mathrm{E}(y_t|z_{t-1})$, say, where z_{t-1} is the information set of known variables, rather than $\mathrm{E}_{t-1} y_t$, which is less clear.

(b) Expectations of Exogenous Variables

Next we begin to use these expectations models to describe behaviour. Suppose that y is exogenous to an agent who determines a price p_t, based on expectations of future y:

$$p_t = f(\mathrm{E}_t y_{t+1}),$$

where f perhaps is derived from an optimization problem. As we saw at the conclusion of section 2, models with nonlinear functions of random variables can be tricky. So for simplicity here in section 3 we shall work with models in which functions such as f are linear:

$$p_t = \lambda \mathrm{E}_t y_{t+1}.$$

That makes it easy to work out the properties of p_t.

Now we make the hypothesis of rational expectations a practical tool by assuming that agents use the forecast of y_{t+1} given some information set (simply because alternatives often will be systematically wrong). If we assume that the true conditional probability rule is linear as in our first examples,

$$y_t = \rho y_{t-1} + \epsilon_t,$$

then

$$p_t = \lambda \rho y_t.$$

When we replace the unobservable expectations with a forecast we find a relationship between two variables which we can observe. We get this relationship between currently observable variables from the behavioural rule and the auxiliary model of expectations.

We could test this statistically by estimating $y_t = \rho y_{t-1}$ at the same time. Note that

(a) p_t is related to y_t here statistically even though behaviour is forward-looking, and

(b) the coefficient in this relation is an amalgam of the structural parameter λ and of the parameter in the law of motion of the y's. Thus λ and ρ cannot be separately identified from this one equation, though they will be identified if we include the forecasting model itself. The fact that ρ appears in both equations is an example of a rational expectations cross-equation restriction. Finally,

(c) even if agents' behaviour (λ) is unchanging, the value of this composite coefficient will change if the behaviour of y changes (*e.g.* permanently and unexpectedly). This is a simple example of the Lucas critique.

Before going on, see if you can apply this method to the expectations hypothesis of the term structure. Suppose that the one-month interest rate evolves this way:

$$r_{1t} = \rho_0 + \rho_1 r_{1t-1} + \epsilon_t,$$

with ϵ_t unpredictable. The economic hypothesis concerns the two-month rate:

$$r_{2t} = 0.5(r_{1t} + E_t r_{1t+1}).$$

Solve the model and describe how you would estimate and test it.

(c) Expectations of Endogenous Variables

So far this seems easy; we simply replace expectations by statistical forecasts. But this method applies only to exogenous variables. At a deeper level, outcomes in economic models depend on the expectations of future endogenous variables. An example is the expected rate of inflation, which influences today's interest rates. But we cannot simply estimate a time series model for inflation, forecast with it, and use those forecasts in a model, because inflation is endogenous in any large macroeconomic model. Somehow we must ensure that the model is internally consistent, so that the law of motion for inflation (used to make forecasts) is the one produced endogenously by the model itself. When we have this consistency we have a rational expectations equilibrium (REE).

Suppose that the structural equation in a model is:

$$p_t = \beta \mathrm{E}_t p_{t+1} + \lambda y_t$$

with $0 < \beta < 1$ and $\lambda > 0$. Again p is endogenous and y is exogenous. Here the price depends on next period's price, a common feature in asset markets. *e.g.* I'll hold this asset if it earns a high rate of return, but that involves what I'll be able to sell it for next period. An example might be as follows: p_t is the price of an house, which depends on the expected resale value $\mathrm{E}_t p_{t+1}$ as well as on fundamentals – location, size, view, taxes – listed in y_t.

What is the reduced-form price equation? We want to know how the price is related to the exogenous variable y. Solving means isolating p_t on the left-hand side.

Now let me suggest another analogy which may be helpful. The relationship above is a standard difference equation, with two modifications. First, the exogenous component is not zero or a constant but a variable, y_t. Second, the equation contains $\mathrm{E}_t p_{t+1}$ rather than p_{t+1}.

I shall outline two simple ways to solve these equations, without going into general methods.

Method A: Undetermined Coefficients.

In this method we guess a form for the solution and then solve for its coefficients. Suppose we guess that the answer is

$$p_t = k y_t,$$

where k is some number we don't yet know. Then if our guess is correct $p_{t+1} = k y_{t+1}$, so

$$\mathrm{E}_t p_{t+1} = E_t k y_{t+1} = k \rho y_t,$$

using our probability law for y, which is part of the model. Once we have an expression in expectations of exogenous variables we can simply replace them by forecasts.

Now we replace terms in p in the original difference equation:

$$ky_t = \beta k\rho y_t + \lambda y_t = (\beta k\rho + \lambda)y_t.$$

If the solution is to satisfy the original equation then the two sides must be equal, so

$$k = \beta k\rho + \lambda,$$

or

$$k = \frac{\lambda}{1 - \beta\rho}.$$

The solution is:

$$p_t = \frac{\lambda y_t}{1 - \beta\rho}.$$

This seems simple, and it is. The only worrying part is: How do we guess the form of the solution in the first place? Because everything is linear in this section the solution will be linear so that makes choosing a functional form easy. But what variables should be included? Here some practice may help. Also if you are in doubt then include more variables. In this example, try guessing that $p_t = ky_t + cy_{t-1}$, for example. If you carry through the method you will find that $c = 0$.

A final word of advice on guessing: Make the lag length in the guess one less than in the law of motion for the exogenous variable. In the example, the law of motion was $y_t = \rho y_{t-1} + \epsilon_t$ and we found that we needed to include only y_t in the guess.

Method B: Repeated Substitution

A second method may provide some additional economic intuition. Notice that in the original model we are told that $|\beta| < 1$. That parameter can be thought of as a root of this first-order, stochastic, difference equation. The second method is equally simple: substitute for the endogenous variable repeatedly in the direction of the stable root. Here the stable root is forwards.

There are three steps: (1) lead; (2) forecast; (3) substitute in the original equation. To see this, begin with:

$$p_t = \beta \mathrm{E}_t p_{t+1} + \lambda y_t,$$

so that, in step 1,

$$p_{t+1} = \beta \mathrm{E}_{t+1} p_{t+2} + \lambda y_{t+1}.$$

That means that, in step 2,

$$\mathrm{E}_t p_{t+1} = \mathrm{E}_t \beta \mathrm{E}_{t+1} p_{t+2} + \mathrm{E}_t \lambda y_{t+1} = \mathrm{E}_t \beta p_{t+2} + \mathrm{E}_t \lambda y_{t+1},$$

by the law of iterated expectations. That law holds that $E_tE_{t+1}p_{t+2} = E_tp_{t+2}$. If one expected to change one's expectations then one's expectations would not be rational. Substituting in the original equation, step 3, gives:

$$p_t = \lambda y_t + \beta E_t \lambda y_{t+1} + \beta^2 E_t p_{t+2}.$$

If we continue in this vein we find:

$$p_t = E_t \sum_{i=0}^{\infty} \beta^i \lambda y_{t+i} + \lim_{i\to\infty} E_t \beta^i p_{t+i}.$$

But here we use our knowledge of β's being a fraction. As long as the price is not growing explosively the last term will equal zero, simply because it includes a fraction taken to higher and higher powers. In general there are many solutions to the difference equation but we shall focus on ones in which this term is zero; that is sometimes called a transversality condition. In a moment we'll discuss its relation to bubbles and Ponzi schemes.

Thus, we have

$$p_t = E_t \sum_{i=0}^{\infty} \beta^i \lambda y_{t+i}$$

which provides some economic intuition. If we interpret the original equation as describing an asset price p with dividend y then this result tells us that the price depends on current and expected future dividends. This is a present-discounted value model, with discount factor β. Future y's have less effect on p than current ones, as one would expect, because β^i is a decreasing function of the horizon i. And, even if no one plans on holding the asset indefinitely and collecting all the dividends, the price depends on them because they affect the value a buyer would attach to the asset, which in turn affects the value to current owners.

We now have a linear function of expectations of exogenous variables, so the final step is to replace them by forecasts. Thus

$$p_t = \lambda(y_t + \beta\rho y_t + \beta^2\rho^2 y_t + \ldots).$$

Your spidey sense should be tingling, for this infinite series converges (because both ρ and β are fractions) to:

$$p_t = \frac{\lambda y_t}{1 - \beta\rho},$$

which is the same solution we found with method A. It is simple to show that it satisfies the original difference equation; that check always is available. Finally, note that solving the equation by recursive substitution backwards (with an initial condition instead of a terminal condition) would not have yielded an answer with a convergent series or which made economic sense in this case.

Both these methods sometimes stymie very intelligent people. If you find them perplexing, try doing some numerical examples like those in the exercises.

(d) Bubbles

It is simple to show that the original difference equation has a host of solutions. For example, try adding a term like $(1/\beta)^t$ to our solution and show that we still have a solution.

These additional terms are called *bubbles*. They are named for the South Sea and Mississippi bubbles of the early eighteenth century, because these additional terms imply that the price must rise at the interest rate (at least), which seems to reflect pure speculation or self-fulfilling expectations. Imagine that you are holding an asset that is intrinsically useless. You hold it only because you expect it to rise in value. The opportunity cost of your funds is the interest rate, so the expected rate of appreciation of this asset must be at least as great as the interest rate. But $1/\beta$ is like a gross interest rate, so the bubble term grows over time like compound interest.

This term is called a bubble because it grows over time and also because nothing seems to support it. The extra term is unrelated to the dividends on the asset, sometimes called the fundamental. Nevertheless, the bubble is perfectly consistent with rational expectations. Along a bubble path, you hold the asset because you expect to sell it to someone else, who expects to sell it to someone else, and so on. This should remind you of pyramid sales or a Ponzi scheme.

The transversality condition rules out bubbles (explosive growth). If you've studied difference equations you may recall that we typically need an initial condition to get a unique solution. The transversality condition serves the same purpose, but as a terminal condition. However, in the exercises we'll see that, even when the transversality condition holds so that there are no bubbles, *apparent* bubbles are possible if we misspecify the behaviour of the fundamentals, $\{y_t\}$.

The idea of an apparent bubble is simple. Suppose that market participants expect the underlying fundamentals, such as dividends, to grow rapidly. The price will rise over time if their forecasts of future dividends are revised up repeatedly. In this case, there is no bubble: the rising price reflects expected future fundamentals. Meanwhile, a lowly economist observes that the current fundamentals have not been growing while the price has been growing, and so concludes that there must be a self-fulfilling speculative bubble. The implication of this argument is that detecting bubbles is very difficult, for we don't observe expected future fundamentals.

In this subsection we have used a very simple behavioural rule and probability law, but that should be enough to make some progress in the readings and exercises. With this background on modelling expectations, we now return to your regularly scheduled economic programming, and discuss some applications.

(e) Applications: Contracts, Exchange-Rates, Hyperinflation

• Our first application is to the role of monetary policy with forward-looking wage- or price-setting. Complete models can be found in papers by Fischer and Taylor and also in textbooks. But a version with only two-periods seems a good place to start.

Suppose that there are two periods denoted $t = 1, 2$. Money demand is given (in logarithms) by

$$m_t = y_t + p_t$$

where m is the log of money, y is the log of output, and p is the log of the price level. Thus this is simply the quantity theory of money.

For a given m one can think of this as an aggregate demand curve. Monetary policy will not affect output if there is a vertical aggregate supply curve. But if the short-run aggregate supply curve slopes up, then an increase in m will raise y, as well as p. So suppose that the supply side of the model is given by

$$y_t = y^* + \alpha(p_t - \mathrm{E}_{t-1}p_t) + \epsilon_t,$$

where ϵ_t is an iid shock with mean zero, $y^* + \epsilon_t$ is the average or natural rate of output, and the other term simply gives us a Phillips curve. There is no long-run tradeoff between inflation and unemployment in this model.

If m_t is exogenous then to solve the model we must find p_t and y_t. By substitution

$$m_t - p_t = y^* + \alpha(p_t - \mathrm{E}_{t-1}p_t) + \epsilon_t,$$

so that

$$p_t + \alpha(p_t - \mathrm{E}_{t-1}p_t) = m_t - y^* - \epsilon_t.$$

Thus

$$\mathrm{E}_{t-1}[p_t + \alpha(p_t - \mathrm{E}_{t-1}p_t)] = \mathrm{E}_{t-1}p_t = \mathrm{E}_{t-1}m_t - y^*,$$

because $\mathrm{E}_{t-1}\epsilon_t = 0$. Using this result to replace $\mathrm{E}_{t-1}p_t$ gives,

$$p_t(1+\alpha) - \alpha\mathrm{E}_{t-1}m_t + \alpha y^* = m_t - y^* - \epsilon_t,$$

or

$$p_t = \frac{m_t + \alpha\mathrm{E}_{t-1}m_t}{1+\alpha} - y^* - \frac{\epsilon_t}{1+\alpha}.$$

Thus in this model if the money supply is what was expected when prices are set then

$$p_t = m_t - y^* - \frac{\epsilon_t}{1+\alpha},$$

which means that $\delta p_t = \delta m_t$, and $\delta y_t = 0$. Unless there is a money 'surprise' a change in the money stock does not affect real output. In this sense the Phillips curve is vertical even in the short run.

Now we'll show that if prices (or wages) are set in advance then monetary policy is not neutral. Suppose that in period 1 firms set the price for period 2:

$$p_2 = \mathrm{E}_1 m_2 - y^*.$$

Then let us solve for y_2 given some arbitrary m_2:

$$y_2 = m_2 - p_2 = m_2 - \mathrm{E}_1 m_2 + y^*.$$

Thus any deviation in m_2 from the value expected when the contract or nominal price was set in the past will cause y_2 to deviate from y^*. Even if the change in monetary policy is widely known it can have strong real effects (on output and interest rates) because prices cannot adjust. This leaves open the important question of why such contracts are signed, but they certainly are. Anyway, the sluggishness or stickiness in prices gives the model a Keynesian flavour. It supports the observation that rational expectations models are not related to policy ineffectiveness. The Taylor model also has a long-run vertical Phillips curve, but the short run dynamics and policy implications are non-classical. It also is easy to illustrate this analysis in a diagram in (y, p) space: p is set, and then m may shift the AD curve.

• Our second application is to the monetary model of the exchange rate. This model is widely used in theoretical studies, though it is not very successful empirically. The building blocks are:

$$\begin{aligned} m_t - p_t &= -\alpha(E_t p_{t+1} - p_t) \\ m_t^* - p_t^* &= -\alpha(E_t p_{t+1}^* - p_t^*) \\ e_t - p_t + p_t^* &= 0 \\ i_t - i_t^* &= E_t e_{t+1} - e_t \end{aligned}$$

These are two money demand equations, purchasing power parity, and uncovered interest parity. Variables are in logarithms, with the exception of the two nominal interest rates. The two money supplies are exogenous. Combining the equations gives:

$$e_t = m_t - m_t^* + \alpha(E_t e_{t+1} - e_t),$$

which is in the form we studied in section (c). In several of the exercises you will be asked to work out the predictions of this model.

The overshooting model is an extension of this basic monetary model, and the extension involves making prices sticky. One reason that models with sticky goods prices are appealing in international finance is that real exchange rates tend to behave much like nominal ones. Why does this suggest prices might be sticky? A shock from monetary policy will affect the nominal exchange rate. If prices do not adjust (to PPP, say) then the real exchange rate also will be affected.

In a classic contribution, Dornbusch allowed prices to adjust to changes in aggregate demand, and worked out the implications of sluggish price adjustment. Suppose that there are goods and money markets. Consider an exogenous increase in the supply of money. At the moment of the increase output and prices are fixed. Equilibrium in the money market requires that the interest rate falls. Then from UIP the value of the currency jumps down (price of foreign exchange jumps up) so that the expected appreciation offsets the interest differential. We know that since the money supply has risen the long-run effect will be that the exchange rate must rise (depreciate). This is consistent only with an expected appreciation (from uip) if the rate initially over-depreciates.

Then as prices begin to rise the nominal and real exchange rates gradually appreciate. If prices are completely flexible then in the long run money is neutral. The idea is that the short run change in the exchange rate is greater than the long run change. This property arises because adjustment in asset markets is faster than in goods markets.

See the exercises for examples of the model. In solving these, remember to first work out the long-run (classical) effects of a change in the money supply. Remember also that the money market is always in equilibrium.

Some criticisms of the overshooting model would include the following. With an increase in the money supply the exchange rate over-depreciates, then gradually appreciates. This expected appreciation perhaps should be reflected in a forward premium on the currency. Yet such forward premiums typically are not observed in episodes in which overshooting allegedly occurs. However, one must be careful to note that the Dornbusch example is one in which there is an unexpected, permanent change. If there are a series of announcements of higher than expected money growth then the exchange rate need not start to depreciate (from its overshooting level) after the first one.

In Dornbusch's paper prices are sticky, so the real exchange rate fluctuates when there is a money supply (aggregate demand) shock. In this model the nominal and real exchange rates move together, as in reality, but the model returns to PPP. An alternative view is that changes in the real rate arise from real-side shocks.

• Our third application is to a simple linear model of hyperinflations. An economist named Philip Cagan argued in the 1950s that in hyperinflations, such as those experienced by some central European countries between the wars, one can study the relation between nominal variables while ignoring real variables. The idea simply is that money and prices are changing very fast relative to income. chapter 7, gives some background discussion.

Part of the historical context is that during the German hyperinflation officials of the Reichesbank argued that money growth was simply following, and not causing, the inflation. Similar arguments were heard in Russia during the early 1990s. Cagan hoped to show that money growth was causing inflation by using this simple model.

Cagan proposed the following, linear money demand expression:

$$m_t - p_t = -\alpha(\mathrm{E}_t p_{t+1} - p_t),$$

in which m_t is the log of the money supply and p_t is the log of the price level. The left-hand side is real balances or money demand and the right-hand side is the opportunity cost of holding money, given by the expected rate of inflation. The interest elasticity is α. Given a path for the money supply, this ends up explaining movements in the price level.

Next suppose that

$$m_t = \mu_0 + \mu_1 m_{t-1} + \epsilon_t,$$

where ϵ_t is an iid error with mean zero.

Exercise: Solve for the price level. Suppose that $\mu_1 = 1$ so that the money supply is growing (remember m is in logs) at average rate μ_0. Find the inflation rate.

Exercise: Suppose that the central bank announces that at a specific date in the future the money supply will be fixed. What happens to real balances along the adjustment path?

Further Reading

For introductions to linear models with rational expectations, I recommend Steven Sheffrin's *Rational Expectations* (1983) (chapter 1 and 27-54) or David Begg's *The Rational Expectations Revolution in Macroeconomics* (1982) (chapters 2-4).

Ph.D.-level readers could look at the more advanced book by Roger Farmer, *The Macroeconomics of Self-Fulfilling Prophecies* (1993). They also should be familiar with some of the techniques in the classic work on linear models by Thomas Sargent, *Macroeconomic Theory* (1987), Part II (not to be confused with his book called *Dynamic Macroeconomic Theory*).

A fine historical and economic discussion of bubbles is given by Peter Garber in "Famous first bubbles," *Journal of Economic Perspectives* (Spring 1990) 35-54. Garber argues that most bubbles are only apparent bubbles, as price changes can be explained by expected future fundamentals.

For the sticky-price model, see chapter 6, part B of David Romer's textbook *Advanced Macroeconomics* (1996). For some of the original research see John Taylor's "Staggered wage-setting in a macro model," *American Economic Review(P)* (1979) 108-113.

For the monetary model of the exchange rate, and its overshooting version with sticky prices, see Romer (1996), section 5.3, the original research by Rudiger Dornbusch, "Expectations and exchange rate dynamics," *Journal of Political Economy* (1976) 1161-1176, or chapter 8 of *Foundations of International Macroeconomics* (1996) by Kenneth Rogoff and Maurice Obstfeld.

Cagan's original article on hyperinflation is found in a book edited by Milton Friedman, *Studies in the Quantity Theory of Money* (1956). Many historical and statistical studies have examined this model in the past forty years. Chapters 7 and 8 of Bennett McCallum's *Monetary Economics* (1989) provide a very clear treatment. Thomas Sargent argues that hyperinflations can be ended without large output losses in "The Ends of Four Big Inflations" in his book *Rational Expectations and Inflation* (1986).

Exercises:

1. Suppose that the monetary authority controls short-term nominal interest rates by setting:

$$R_{1t} = R_{1t-1} + \lambda(R_{2t} - R_{1t}) + \epsilon_t, \tag{1}$$

with $\lambda > 0$. Here R_{1t} is the nominal yield on a bond with maturity 1 and R_{2t} is the yield on a bond with maturity 2. The error term ϵ_t is serially uncorrelated. Suppose also that the two-period bond yield satisfies

$$R_{2t} = \frac{R_{1t} + E_t[R_{1t+1}]}{2} \tag{2}$$

(a) Why might the authorities follow the policy in equation (1)?

(b) What theory leads to equation (2)?

(c) Show that the model can be solved for an interest-rate process of the form:

$$R_{1t} = R_{1t-1} + \epsilon_t.$$

(d) Describe how this theory could be tested econometrically.

(e) Is empirical evidence consistent with the theory in equation (2)?

Answer

(a) These are nominal interest rates, so an increase in long rates may reflect expected inflation. The central bank may raise short-term rates to prevent this inflation. More generally, its goal may be to stabilize the level of short-term rates.

(b) The expectations hypothesis of the term structure of interest rates produces this relationship. It is not an arbitrage relationship.

(c) Treat the answer as a guess. Using it in equation (2) gives $R_2 = R_1$. Substituting this in equation (1) reproduces the guess, which is thus confirmed.

(d) In this example, λ is not identified. But the random walk model for short rates could be tested, as could the flat yield curve.

(e) See Campbell *JEP* (1995) for some evidence on the expectations hypothesis.

2. Say we have the money market equilibrium equation

$$m_t - p_t = -\alpha E_t(p_{t+1} - p_t)$$

where $\alpha > 0$, and the money supply is governed by the process

$$m_{t+1} = \lambda \bar{m} + (1 - \lambda)m_t + e_{t+1}; \quad E_t e_{t+1} = 0,$$

where $\bar{m}$ is a constant. Using any method of solution, derive the solution for the current price level p_t.

Answer

Guess that the solution is of the form

$$p_t = k_0 + k_1 m_t.$$

Then the method of undetermined coefficients gives

$$k_0 = \frac{\alpha\lambda\bar{m}}{1+\alpha\lambda}$$

$$k_1 = \frac{1}{1+\alpha\lambda}$$

This can be checked in the difference equation to verify that it is a solution.

3. Consider the following linear macroeconomic model:

$$\begin{aligned}
m_t - p_t &= -\gamma R_t \quad \text{(money − demand)} \\
y_t &= \alpha(m_t - \mathrm{E}_{t-1}m_t) - \beta r_t \quad \text{(output)} \\
R_t &= r_t + \mathrm{E}_t p_{t+1} - p_t \quad \text{(Fisher relation)} \\
m_t &= m_{t-1} + \epsilon_t \quad \text{(money − supply rule)} \\
r_t &= \delta g_t \quad \text{(real interest rate)} \\
g_t &= g_{t-1} + \eta_t \quad \text{(fiscal policy rule)}
\end{aligned}$$

where m is (log) nominal money, p is the (log) price level, R is the nominal interest rate, r is the real interest rate, y is output, and g is government spending. All parameters are positive and $\lambda \in (0,1)$ with $\epsilon \sim IID(0,\sigma_\epsilon^2)$ and $\eta_t \sim IID(0,\sigma_\eta^2)$. Expectations are rational and E_t is based on observations of all variables at time t and earlier. Notice that $\mathrm{E}_t g_{t+i} = g_t$ and $\mathrm{E}_t m_{t+i} = m_t$.

(a) Find the reduced form for the model *i.e.* solve for p_t, y_t, r_t, and R_t when the money-supply and government spending evolve exogenously as shown.

(b) Suppose that the authorities want to stabilize the price level so that $p_{t+1} = p_t$ on average. If there is a fiscal shock η_{t+1} then how should the monetary authorities try to set ϵ_{t+1} in order to achieve this zero-inflation target?

4. Consider a dynamic linear model as follows:

$$\begin{aligned}
p_t &= \delta \mathrm{E}_t p_{t+1} + \alpha y_t \\
m_t - p_t &= \eta y_t \\
R_t &= r + \mathrm{E}_t p_{t+1} - p_t
\end{aligned}$$

where p is the log of the price level, y is the log of a deviation of output from trend, m is the log of the money supply, R is the nominal interest rate, and r is a constant real interest rate. E_t is a rational expectations operator.

(a) Suppose that the money supply is exogenous. Solve for each of p_t, y_t, and R_t in terms of current and expected future values of the money supply.

(b) Now suppose that the government is committed to a 'k% rule' whereby it is expected that $\mathrm{E}_t \Delta m_{t+i} = E_t(m_{t+i} - m_{t+i-1}) = k$ for $i = 1, 2, 3, \ldots$. Suppose $\delta \in (0,1)$. Show that the higher is the income elasticity of the demand for money the lower is the nominal interest rate.

Answer

(a)

$$p_t = \frac{\alpha}{\alpha+\eta}\mathrm{E}_t \sum_{i=0}^{\infty} m_{t+i}(\frac{\delta\eta}{\alpha+\eta})^i$$

$$y_t = \frac{m_t}{\eta} - \frac{p_t}{\eta}$$

$$R_t = r + \frac{\alpha}{\alpha+\eta}\mathrm{E}_t \sum_{i=0}^{\infty} \Delta m_{t+i+1}(\frac{\delta\eta}{\alpha+\eta})^i$$

(b) $R_t = r + [\alpha/(\eta+\alpha)][1 + \delta\eta/(\eta+\alpha) + \ldots]k = r + [\alpha/(\eta+\alpha-\delta\eta)]k$

5. Sometimes it can seem difficult to explain the prices of assets, such as houses, without reference to speculation and self-fulfilling price 'bubbles.' This question suggests that we may wrongly conclude that such speculation is occurring in a housing market if we do not forecast market fundamentals in the same way that market participants do.

Suppose that an index of house prices in a city is given by:

$$p_t = \beta \mathrm{E}_t p_{t+1} + h_t,$$

where E_t is the expectation at time t and h_t is an index of employment and economic activity in the city, which is a fundamental factor influencing house prices.

(a) By repeated substitution (and using a transversality condition), write the house price in terms of current and expected future economic fundamentals.

(b) From now on assume that $\beta = .5$. Suppose that an economist observes that $h_t = 1$ and forecasts that h will equal 1 for all future time periods. Solve for the house price index that this economist would predict as warranted by such fundamentals. Denote this p'.

(c) Now suppose that market participants actually expect that $h_t = 1$, $h_{t+1} = 1$, but that $h_{t+2} = 2$, and that h will take the value 2 thereafter. The actual price, denoted p, is determined by their expectations. Solve for the actual price at time t.

(d) Find the gap between the economist's prediction and the actual price in period t (using your answers to parts b and c) and also in period $t+1$.

(e) Now suppose that agents still expect that $h_t = 1$, $h_{t+1} = 1$, but that instead of expecting that $h = 2$ each period thereafter, market participants expect that $h_{t+2} = 1$ with probability 0.5 and $h_{t+2} = 3$ with probability 0.5. Suppose that they believe, correctly, that whatever value h takes at time $t+2$ it will take thereafter. Solve for p_t, and p_{t+1}. Then solve for p_{t+2} assuming that the outcome is $h_{t+2} = 1$. Would the economist suspect a speculative bubble was present?

Answer

(a) $p_t = h_t + \beta \mathrm{E}_t h_{t+1} + \ldots = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i h_{t+i}$.

(b) If $h = 1$ then $p'_t = 1/(1-\beta)$. With $\beta = .5$, $p' = 2$.

(c) $p_t = 1 + .5(1) + .25(2) + .125(2) + \ldots = 1.5 + 2 \cdot .25(1/(1-.5)) = 2.5$.

(d) The gap in period t is .5. In period $t+1$ the economist still finds $p'_t = 2$. The market gives

$$p_{t+1} = 1 + .5(2) + .25(2) + \ldots = 1 + .5 \cdot 4 = 3.$$

Thus the gap rises over time, which seems to mimic a bubble. But of course in this example this would be obvious to the economist when t+2 arrives. The economist would then observe $p_{t+2} = 4$ and $h_{t+2} = 2$.

(e) Hence part (e)! Again $p_t = 2.5$ and $p_{t+1} = 3$. But now instead $p_{t+2} = 2$. So it may appear to the economist that a bubble has grown and then burst in the housing market. [In fact, a strict bubble would have to grow like $(1/\beta)$, which this one does not.]

The most common error on this question is to fail to maintain the horizon starting at $i = 0$, as t varies. This problem will lead to declining prices, as early terms in the series are dropped, which is incorrect.

6. Stock prices are often described as being determined by dividends and by expected capital gains. Imagine that the price of a stock satisfies:

$$p_t = d_t + 0.9 E_t p_{t+1},$$

where d_t is the dividend. Suppose also that a statistician studies dividends and finds the following pattern:

$$d_{t+1} = 0.8 d_t + \epsilon_{t+1},$$

where $E_t \epsilon_{t+1} = 0$.

(a) If you ran an OLS regression of p_t on d_t, what coefficient would you expect to find?

(b) Suppose that the dividend policy becomes less persistent, in that the autoregressive parameter falls from 0.8 to 0.5. How does your prediction in part (a) change? Relate your answer to the Lucas critique.

(c) Suppose the original dividend policy applies. Another researcher regresses prices on dividends, and decides to include an exponential time trend in the regression to see if it is significant. This researcher finds:

$$p_t = \omega d_t + (1.1111)^t,$$

where ω is your answer to part (a). Does finding this additional term in the regression mean that the model is wrong? How can it be interpreted?

Answer

(a) 3.571. (b) 1.82 and Lucas critique discussion. (c) It's a bubble.

7. Suppose that an asset's price, p_t is related to its dividend, d_t, as follows:

$$p_t = d_t + 0.95 E_t p_{t+1},$$

where the dividend evolves as follows:

$$d_t = 0.2 + 0.8 d_{t-1} + \epsilon_t,$$

and where ϵ_t is unpredictable and has mean zero.

(a) Solve for p_t in terms of d_t.

(b) Is the price-earnings ratio (p_t/d_t) positively or negatively correlated with dividends?

(c) Describe how linear regressions could be used to estimate and test this model.

(d) Which has a larger variance, the price or the dividend?

Answer (a)

$$p_t = 15.8346 + 4.167 d_t$$

assuming that there are no bubbles.

(b) The price-earning ratio is: $15.8346/d_t + 4.167$. This is negatively correlated with earnings. This ratio will be low when earnings are high.

(c) The following system could be estimated:

$$d_t = a_0 + a_a d_{t-1}$$

$$p_t = k_o + k_1 d_t$$

Discuss the identification via cross-equation restrictions.

(d) The variance of the price is 17.36 times the variance of earnings.

8. Suppose that f_t denotes the temperature in central Florida, measured in degrees fahrenheit. It can take on two values, 80 or 60, in any period t, where t counts months. Each period the temperature will be the same as in the previous period with probability 0.7, and will change to the other value with probability 0.3.

The spot price of orange juice in cents per quart is given by:

$$p_t = 100 - 0.5(f_t + E_t f_{t+1}).$$

There also is a forward market in orange juice, in which a payment of x_t today entitles you to one quart of orange juice next month. The forward price satisfies:

$$x_t = E_t p_{t+1}.$$

(a) If the current temperature is 80, find the spot price.

(b) If the current temperature is 80, find the forward price.

(c) Suppose that the temperature drops to 60 in period $t+1$. Find the values of the spot and forward prices of orange juice in period $t+1$.

(d) If the current temperature is 80, what is the expected return on a one-period forward contract? Why do people hold such contracts?

Answer

(a) 23 (b) 27.2

(c) The spot price is 37. The futures price is 32.8.

(d) You pay 27.2 today. The payoff is 37 w.p. 0.3 and 23 w.p. 0.7, so $E(r) = 0$. But the forward price is less volatile, so it is held to reduce risk in that sense.

9. A recent graduate is considering the purchase of a used car. The car price satisfies:

$$p_t = \beta E_t p_{t+1} + q_t$$

where β is the owner's discount factor, and q_t measures the quality of the car. The quality evolves as follows:

$$q_t = 0.5 q_{t-1} + \epsilon_t,$$

where ϵ_t has mean zero and is unpredictable. Thus the quality tends to decline over time, reflecting depreciation of the car, while ϵ_t represents random repair costs.

(a) Solve for the car price as a function of q_t.

(b) It costs \$100 to dispose of a car, and so one should dispose of it when the resale price falls to this value. Suppose that $q_t = \$480$. If someone buys the car and plans to hold it for three years, then what must their discount factor be?

Answer

(a)

$$p_t = \frac{q_t}{1 - 0.5\beta}$$

(b) With $\beta = 0.8$ then $p_t = 800$. Then $400, 200, 100$.

10. This question examines some predictions for the nominal term structure of interest rates when investors are risk neutral. Call the discount factor β and the price level p_t.

(a) Use asset-pricing theory to derive an expression for Q_t^2, the price of a two-period bond at time t.

(b) One-period later, how will the price of this same asset (now with one-period until maturity) be related to the price of a one-period bond, Q_{t+1}^1?

(c) The ratio of the price of this bond at $t+1$ to its price at t is called the gross, holding period return. Why would the expected holding period return on the two-period bond ever differ from the return on a one-period bond? (Hint: Use a covariance decomposition.)

Answer

(a)

$$Q_t^2 = E_t \beta^2 \frac{p_t}{p_{t+2}}.$$

(b) The two prices will be identical, by arbitrage.

(c) The expected gross holding period return is:

$$E_t \frac{Q_{t+1}^1}{Q_t^2} = \frac{E_t \beta \frac{p_{t+1}}{p_{t+2}}}{E_t \beta^2 \frac{p_t}{p_{t+2}}}$$

using the law of iterated expectations. Applying the covariance decomposition to the denominator, we see that if inflation is unforecastable then the expected gross return is the same as that on a one-period bond. So the expected return differs from that yield because of an inflation risk premium.

11. It is often argued that asset prices are related to the expected present value of dividends. Suppose one begins with a relation between an asset price, p_t, its dividend, d_t, and the expected future price, $E_t p_{t+1}$:

$$p_t = 0.5 E_t p_{t+1} + d_t,$$

where 0.5 is a discount factor.

(a) Solve for p_t in terms of current and expected future dividends.

(b) Suppose that d_t is independently and identically distributed, with the following probability density:

Payoff	1	2	3
Probability	λ	$1-2\lambda$	λ

where the probabilities are between 0 and 1. Solve for the three possible values of p_t, one for each current value of the dividend.

(c) Calculate the variance of the asset price using these values and the associated probabilities.

(d) Some assets known as *derivatives* have payoffs which depend on the price of another asset. One example is an option, an asset which pays off if the price of an underlying asset reaches some specified level. Imagine an option contract which pays pays 1 in any time period in which $p_t > 4$ (note that the inequality is strict) and pays 0 if $p_t \leq 4$. Write out the stream of expected payoffs for this asset, and assume that its price, denoted q_t, is given by the expected, present value of its payoff stream, again with discount factor 0.5. Hence calculate the three possible values for q_t, one for each current value of the underlying asset price p_t.

(e) Recently, the risk involved in holding derivative assets has been much discussed in the news media. Consider two investment portfolios. One simply holds the underlying stock. You have calculated the variance of the value of this portfolio in part (c). The other portfolio holds 5 option contracts, with prices found in part (d). Which portfolio has a more volatile value?

Answer

(a)

$$p_t = d_t + 0.5E_t d_{t+1} + ... = E_t \sum_{i=0}^{\infty} 0.5^i d_{t+i}$$

(b) Clearly $E_t d_{t+i} = 2$. Thus using the usual power series the three values are 3, 4, and 5.

(c) The mean of the price is 4, so the variance is $\lambda \cdot 1^2 + (1-2\lambda) \cdot 0^2 + \lambda \cdot 1^2 = 2\lambda$.

(d) The payoffs start with 1, 0, or 0, and then have value λ in each later period. So the three values are λ, λ, and $1+\lambda$.

(e) I think that q has a mean of 2λ and a variance of $\lambda - \lambda^2$.

12. During the early 1990s Canadian interest rates exceeded U.S. interest rates, yet the Canadian dollar did not on average depreciate over that time. One possibility is that uncovered interest parity does not hold, but another possibility (sometimes called the 'peso problem') is that market expectations reflected a possibly large depreciation which was not observed. This question evaluates the plausibility of this argument.

(a) Suppose that the nominal exchange rate e is distributed independently each period as follows:

Value	Probability
1.22	0.45

1.28	0.45
x	0.10

where $x > 1.28$. The idea is that there can be infrequent, large depreciations. Investors know this distribution. Find $E(e)$ in terms of x.

(b) An economist collects a sample which includes only the values 1.22 and 1.28, each with sample frequency 0.5. Find the economist's estimate of $E(e)$.

(c) In the sample we observe an average current exchange rate given by your answer in (b). Suppose that the average international interest rate differential is 0.02 (200 basis points). What value for x could make this differential consistent with UIP?

(d) Does this suggest a danger in financing debt in U.S. dollars? Is there evidence that large depreciations are expected to be short-lived?

Answer

(a)

$$E(e) = .549 + .576 + .10x = 1.125 + .10x$$

(b) 1.25

(c)

$$.02 = \frac{1.125 + .10x - 1.25}{1.25}$$
$$x = 1.5$$

(d) Yes, it does suggest a danger in issuing debt denominated in U.S. dollars. But ... perhaps this value of x is not plausible ... The evidence suggests that investors believe depreciations will not suddenly be reversed, for the dollar does not go to a forward premium after depreciations (as the Dornbusch model would predict).

Another way to collect evidence on depreciation expectations would be to collect interest differentials for bonds of various maturities.

13. Dividing the government's budget constraint by national income gives:

$$b_t = (1 + r - g)b_{t-1} - s_t,$$

where b is the debt-income ratio, s is the surplus-income ratio, r is the interest rate on debt, and g is the growth rate of income. This question uses budget-constraint accounting to describe implications of a target for the debt-income ratio. The Maastricht treaty included such targets, and they also have been debated for Canada.

(a) Suppose that $b_t = 1.00$. Also suppose that $r = .06$ and $g = .03$. Let the target be $b_{t+15} = 0.5$. Suppose that $s_{t+i} = \overline{s}$ for $i = 1, 2, 3 \ldots, 15$. Solve for $\overline{s}$.

(b) Solve for $\overline{s}$ if the target is $b_{t+10} = 0.5$.

(c) In part (b) write $\overline{s}$ as a function of r and g to allow an investigation of the sensitivity of the surplus-income ratio to the projected growth and interest rates.

Answer

(a)

$$0.5 = (1.03)^i - (1 + 1.03 + 1.03^2 + \ldots + 1.03^{i-1})\overline{s}$$

Thus for $i = 15$ $\overline{s} = 0.056$.

(b) For $i = 10$, $\overline{s} = 0.073$. This is larger, as one would expect. Faster reduction in the debt-income ratio requires larger surpluses.

(c)

$$\overline{s} = \frac{(1 + r - g)^{10} - b_{t+10}}{\sum_{i=0}^{9}(1 + r - g)^i}$$

Then we can show using calculus that $d\overline{s}/dr > 0$ and $d\overline{s}/dg < 0$.

14. Sometimes explosive price behaviour is taken to be evidence of speculative bubbles, but it is actually explosive price behaviour relative to fundamentals that would be evidence of bubbles. This question explores this issue.

(a) Suppose that an asset price p_t satisfies

$$p_t = f_t + 0.5E_t p_{t+1},$$

where f_t is a fundamental. Solve for the price in terms of current and expected future fundamentals.

(b) Suppose that

$$f_t = 0.9f_{t-1} + \epsilon_t,$$

where ϵ_t is has mean zero and is unpredictable. If you regressed p_t on f_t what coefficient would you expect to find?

(c) Suppose instead that

$$f_t = (1.1)^t + \epsilon_t.$$

Now what regression is suggested by the theory?

(d) Suppose instead that

$$\begin{aligned} f_{t+i} &= 1, \ i = 0, 1 \\ &= 2, \ i = 2, 3, \ldots \end{aligned}$$

Will there be an apparent bubble?

Answer

(a)

$$p_t = f_t + 0.5E_t f_{t+1} + 0.25E_t f_{t+2} + \ldots$$

(b)

$$1 + 0.5 \cdot 0.9 + \ldots = \frac{1}{1 - 0.5 \cdot 0.9} = 1.82$$

(c) Now $p_t = 2.22 f_t + \epsilon_t$. Or one could write this a regression on time. The transversality condition does hold.

(d) Now the price is $\{2.5, 3.0, 4.0, 4.0, 4.0 \ldots\}$. A bubble involves the price rising like 2^t, so this is not explosive enough to look like a bubble.

15. A variable is lognormally distributed if its natural logarithm is normally distributed. For a lognormal random variable x,

$$ln\, x \sim N(\mu,\, \sigma^2).$$

For future reference, note that for such a variable

$$\mathrm{E}(x) = exp(\mu + \frac{\sigma^2}{2}).$$

Suppose that a representative agent has a time-separable utility functional with constant discount factor β. The period utility function is:

$$u(c_t) = \frac{c_t^{1-\alpha}}{1-\alpha}.$$

Also suppose that we know that endowments evolve as follows:

$$ln(\frac{y_t}{y_{t-1}}) = \mu + \epsilon_t,$$

with $\epsilon_t \sim IIN(0, \sigma^2)$. In equilibrium $c_t = y_t$.

(a) State the Euler equation linking consumption in adjacent periods.

(b) Solve for the return on a one-period, real discount bond.

(c) The endowment growth we have used so far does not allow for persistence (business cycles), and so the return in part (b) is constant. Suppose now that

$$ln(\frac{y_t}{y_{t-1}}) \sim IIN\big(\mu + \rho ln(\frac{y_{t-1}}{y_{t-2}}), \sigma^2\big).$$

Solve for the bond return as a function of the current state of the economy.

Answer

(a) The Euler equation is:

$$c_t^{-\alpha} = E_t \beta (1 + r_t) c_{t+1}^{-\alpha}.$$

(b) Use $x = (c_{t+1}/c_t)^{-\alpha}$. Then the price is:

$$q_t = \beta E(x) = \beta exp(-\alpha\mu + \alpha^2\sigma^2/2).$$

So the return is

$$r_t = \frac{1}{\beta} exp(\alpha\mu - \alpha^2\sigma^2/2) - 1,$$

which is constant over time.

(c) The return is

$$r_t = \frac{1}{\beta} exp(\alpha\mu + \alpha\rho ln(\frac{y_t}{y_{t-1}}) - \alpha^2\sigma^2/2) - 1.$$

16. Predictable exchange rate movements are possible even if investors have rational expectations. Suppose that the exchange rate between Canada and the United States satisfies:

$$e_t = 0.9 E_t e_{t+1} + f_t,$$

where e_t is the price of a U.S. dollar measured in Canadian dollars, f_t is a fundamental variable, and t counts years.

Suppose that the current value of the fundamental is 0.15 and that it evolves according to:

$$f_t = f_{t-1} + \epsilon_t,$$

where ϵ_t has mean zero and is unpredictable.

The authorities have announced that three years hence the exchange rate will be fixed at a value of 1.00.

(a) Using three steps of repeated substitution, find the current value of the exchange rate.

(b) Solve for the expected values of the exchange rate in each of the next two years.

(c) If uncovered interest-rate parity (UIP) held, what would you expect to see in the term structure of interest rates in Canada relative to that in the United States?

(d) Is there any empirical evidence that UIP holds?

Answer

(a)

$$e_t = f_t(1 + .9 + .81) + 0.729(1.00) = 2.71(0.15) + .729 = 1.1355$$

(b) 1.095 and 1.05

(c) With an expected appreciation the Canadian yield curve should be steeper (lower short rates).

(d) There is little evidence of this ... in fact the forward premium anomaly.

17. Suppose that domestic money demand is given by

$$m_t - p_t = -\delta R_t,$$

and foreign money demand by

$$m_t^* - p_t^* = -\delta R_t^*.$$

Suppose also that uncovered interest parity holds:

$$R_t - R_t^* = E_t(e_{t+1} - e_t),$$

as does purchasing power parity

$$e_t - p_t + p_t^* = 0.$$

Variables are in logarithms, except for the two interest rates.

(a) Find a first-order difference equation linking the exchange rate e_t to the exogenous relative money supply $m_t - m_t^*$.

(b) Suppose that the exchange rate is floating, and the relative money supply evolves this way:

$$m_t - m_t^* = \rho_0 + \rho_1(m_{t-1} - m_{t-1}^*) + \epsilon_t,$$

where $\epsilon_t \sim IIN(0, \sigma^2)$. Solve for the exchange rate.

(c) Explain how you could statistically test this model, for example by calculating moments or running linear regressions.

Answer (a)

$$e_t = \frac{1}{1+\delta} m_t - m_t^* + \frac{\delta}{1+\delta} E_t e_{t+1}.$$

(b)

$$e_t = \frac{\delta\rho_0}{1+\delta-\delta\rho_1} + \frac{1}{1+\delta-\delta\rho_1}(m_t - m_t^*).$$

(c) One could find the conditional or unconditional moments of e. Or consider the cross-equation restrictions, which involve four equations in three unknowns.

18. Let us study exchange-rate volatility in a model in which prices are flexible, the so-called monetary model of the exchange rate. The model is:

$$\begin{aligned} R_t - R_t^* &= \mathrm{E}_t(e_{t+1} - e_t) \\ e_t - p_t + p_t^* &= 0 \\ m_t - p_t &= -\alpha R_t \\ m_t^* - p_t^* &= -\alpha R_t^*. \end{aligned}$$

Here R is the interest rate, e is the (log) nominal exchange rate, p is the (log) price level, and m is the (log) money supply. Starred variables apply to a foreign country. The money supplies are exogenous.

(a) Solve for e_t in terms of current and expected future domestic and foreign money supplies.

(b) Suppose that

$$m_t = \rho m_{t-1} + \eta_t, \quad \eta_t \sim iid(0, \sigma^2),$$
$$m_t^* = \rho m_{t-1}^* + \eta *_t, \quad \eta_t^* \sim iid(0, \sigma^{*2}),$$

with $\mathrm{cov}(\eta_t, \eta_t^*) = 0$. Find the relationship between e_t and the current money supplies, m_t and m_t^*.

(c) The central bank wishes to raise e by raising m unexpectedly. Show briefly that the Lucas critique may apply to predicting the effect on e of an arbitrary change in m.

(d) We typically test whether an economic theory is a useful guide by studying its predictions for statistical moments. If $\rho = 0$ then what is the variance of e_t if this model is accurate?

(e) Can this model produce realistic movements in the real exchange rate?

Answer

(a) By substitution:

$$e_t = (m_t - m_t^*) + \alpha \mathrm{E}_t(e_{t+1} - e_t)$$
$$e_t = \frac{(m_t - m_t^*)}{1+\alpha} + \frac{\alpha}{1+\alpha}\mathrm{E}_t e_{t+1}$$

By solving forwards:

$$e_t = \frac{1}{1+\alpha} \cdot \mathrm{E}_t \sum_{i=0}^{\infty} (\frac{\alpha}{1+\alpha})^i (m_{t+i} - m_{t+i}^*)$$

(b) $e_t = (m_t - m_t^*)/(1 + \alpha - \alpha\rho)$

(c) Simple. The effect of m on e depends upon ρ, the forecastability of the policy, and whether agents know that. If ρ changes too then the effects may be difficult to predict.

(d) If $\rho = 0$ then $\mathrm{var}(e_t) = [\sigma^2 + \sigma^{*2}]/(1 + \alpha)$.

(e) No. PPP is built into it, so the variance of the real exchange rate is zero, obviously.

19. Nominal exchange rates are often modelled as follows:

$$e_t = f_t + \alpha \cdot E_t(e_{t+1} - e_t)$$

where e_t is the log of the nominal exchange rate and f_t is some fundamental, such as relative monetary policies in the two countries. This equation reflects the fact the the exchange rate is influenced by fundamentals and also by speculative future changes in its own value. It is used to describe exchange-rate behaviour both in free floats and in managed floats or target zones such as the European exchange rate mechanism.

(a) By repeated substitution and a transversality condition, write the exchange rate in terms of current and expected future fundamentals.

(b) Suppose that the fundamental is independently and identically distributed and can take on a value of 1.5 with probability .5 and a value of .5 with probability 0.5. Suppose that $\alpha = 0.25$. Solve for the two possible values of the exchange rate.

(c) Usually we think of nominal exchange rates as being more variable than fundamentals. Briefly list any small or large modifications to the model you can think of which would make the nominal exchange rate more variable.

Answer

(a)

$$e_t = (1+\alpha)^{-1}[f_t + \ E_t \sum_{i=1}^{\infty} (\frac{\alpha}{1+\alpha})^i \cdot f_{t+i}].$$

(b) $E_t\ f_{t+i} = 1$ for $i > 0$. The mean of the exchange rate therefore is 1. But at time t f_t is a random variable. So

$$e_t = 0.8[f_t + 1(.2 + .04 + .008 + ...)] = .8 \cdot f_t + .2$$

Thus the equiprobable values for e_t are 1.4 and .6.

(c) A small change might be to make f persistent, as in section 3 of the course. A larger change would be to completely change the model and introduce price stickiness and overshooting.

20. This question studies the monetary model of the nominal exchange rate, a standard model used to explain movements in this variable. The model is:

$$\begin{aligned} r_t &= r_t^* + E_t(e_{t+1} - e_t) \\ m_t - p_t &= y_t - \alpha r_t \\ m_t^* - p_t^* &= y_t^* - \alpha r_t^* \\ e_t &= p_t - p_t^* \end{aligned}$$

Here e is the exchange rate, p is the price level, y is output, m is the money supply (all in logs), and r is the level of the interest rate. Starred values denote foreign variables.

(a) Define a fundamental $f_t = (m_t - m_t^*) + (y_t^* - y_t)$. Write the exchange rate in terms of the current fundamental and the expected future exchange rate.

(b) Suppose that econometricians fit a time series model to the data on the fundamental and find that

$$f_t = \rho f_{t-1} + \epsilon_t$$

where $E_t(\epsilon_{t+1}) = 0$. Solve for the exchange rate in terms of curently observed fundamentals.

(c) We usually find that exchange rates are described by random walks, in other words that $\rho = 1$. That implies that the variance of the exchange rate is the same as that of the fundamental. Actual exchange rates seem to be much more volatile than this fundamental. Very briefly describe any alternatives to this model that might predict more volatility in e than in f.

(d) Now suppose that f does not follow the time series process in part (b). Instead the monetary authorities keep the fundamental within a narrow band to try to limit the variation in the exchange rate. Specifically, suppose that f_t can equal $1+\psi$ or $1-\psi$ each with probability 0.5. Thus $E_t(f_{t+i}) = 1$ for $i = 1, 2, \ldots$. Find the two possible values for the exchange rate, one for each current value of the fundamental.

(e) Find the range for the exchange rate (*i.e.* the difference between the two values for the exchange rate), relative to the range for the fundamental, which is 2ψ.

Answer

(a)

$$e_t = \frac{f_t}{1+\alpha} + (\frac{\alpha}{1+\alpha})E_t e_{t+1}$$

(b)

$$e_t = \frac{f_t}{1+\alpha-\alpha\rho}$$

(c) Two that we have discussed are (i) bubbles, and (ii) sticky prices and overshooting. These could be described in a few sentences.

(d) Simply substitute in the difference equation:

$$e = \frac{1+\psi}{1+\alpha} + \frac{\alpha}{1+\alpha}$$

or

$$e = \frac{1-\psi}{1+\alpha} + \frac{\alpha}{1+\alpha}$$

(e) The difference is

$$\frac{2\psi}{1+\alpha}$$

which is less than 2ϵ. We've discovered the so-called 'honeymoon' effect of exchange-rate target zones.

21. This question studies a potential difficulty with testing the well-known monetary model of the exchange rate. The model involves the following equations:

$$E_t e_{t+1} - e_t = r_t - r_t^*$$
$$m_t - p_t = -\alpha r_t$$
$$m_t^* - p_t^* = -\alpha r_t^*$$
$$p_t = p_t^* + e_t$$

where variables other than interest rates are in logarithms, stars denote foreign variables, e_t is the nominal exchange rate, m_t is the money supply, r_t is the interest rate, and p_t is the price level.

(a) Solve for the exchange rate in terms of current and expected future money supplies.

(b) Suppose that the foreign money supply is constant at a value of $\overline{m}$. Meanwhile, the domestic money supply is affected by the government's budget deficit. A large deficit is monetized, so deficits lead to changes in the money supply. Suppose that

$$m_t = \overline{m} + \rho d_{t-1} + \epsilon_t.$$

In turn, the deficit d_t evolves as follows:

$$d_t = d_{t-1} + \eta_t.$$

Both ϵ_t and η_t are independently distributed over time and have mean zero, so they play no role in forecasts. Solve for the relationship between the current exchange rate, e_t, and the lagged deficit, d_{t-1}.

(c) Do your findings suggest a difficulty in testing forward-looking models?

Answer

(a)

$$e_t = \frac{\alpha}{1+\alpha} E_t e_{t+1} + \frac{1}{\alpha}(m_t - m*_t)$$

so

$$e_t = \frac{1}{1+\alpha} E_t \sum_{i=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^i (m_{t+i} - m*_{t+i})$$

(b) Clearly

$$E_{t-1}(m_{t+i} - m*_{t+i}) = \rho d_{t-1}.$$

Thus

$$e_t = \rho d_{t-1} + \epsilon_t.$$

(c) There is a difficulty (called 'observational equivalence') because the exchange rate may be statistically related to variables that help predict the true causes of its variation. But this relationship is easy to distinguish from a structural effect of the deficit on the exchange rate, simply be seeing whether d forecasts m, and by means of stability tests.

22. While no model of the nominal exchange rate is very successful, the monetary model has some success at explaining long-term movemements in that variable. Suppose that the model is given by:

$$e_t = f_t + \beta E(e_{t+1}|I_t),$$

where e is the nominal exchange rate, f is a 'fundamental' such as the current relative money supply, and I_t is the information of participants in the foreign exchange market. Suppose also that f_t evolves as follows:

$$f_t = \rho_0 + \rho_1 f_{t-1} + \epsilon_t,$$

where $E(\epsilon_t|I_{t-1}) = 0$, and $0 < \rho_1 < 1$.

(a) Solve for the exchange rate as a function of the current fundamentals. [If you cannot solve this, at least solve the case with $\rho_0 = 0$.]

(b) Imagine studying your answer in part (a) along with the law of motion for f_t. Are the parameters identified?

(c) Suppose that you estimated the equations in part (b), and found estimates $\hat{\rho}_1 = 0.7$ and $\hat{\beta} = 0.9$. A skeptical critic argues that your monetary model cannot be correct because the variance of e is much larger than the variance of f. What is your prediction for the ratio of these two variances?

(d) What does the Lucas Critique have to say about the linear regression of e_t on f_t?

Answer

(a) I used the method of undetermined coefficients, and found

$$e_t = \frac{\beta\rho_0}{(1-\beta)(1-\beta\rho_1)} + \frac{1}{(1-\beta\rho_1)} f_t.$$

(b) All three parameters β, ρ_0, and ρ_1 are overidentified.

(c) $e = \text{constant} + 2.70f$ so the predicted variance ratio is 7.29.

(d) [Your answer here.]

23. This question uses the monetary model of the nominal exchange rate to study the effect of differential money growth. The model is:

$$\begin{aligned}
m_t - p_t &= -\alpha i_t \\
m_t^* - p_t^* &= -\alpha i_t^* \\
i_t - i_t^* &= E_t s_{t+1} - s_t \\
s_t - p_t + p_t^* &= 0
\end{aligned}$$

where starred variables are foreign, m is the log of the money supply, p is the log of the price level, s is the log of the exchange rate, and i is the nominal interest rate.

(a) Derive a first-order, expectational difference equation linking s_t to $E_t s_{t+1}$ and the difference between the two money supplies.

(b) Suppose that domestic monetary policy evolves as follows:

$$m_t = m_t^* + \beta t + \epsilon_t,$$

where $\epsilon_t \sim IIN(0, \sigma^2)$ is unpredictable. Solve for the exchange rate. (Hint: Include a constant term in your guess.)

(c) A macroeconomist writes: "The monetary model of the exchange rate is refuted, because the variance of the detrended exchange rate is far less than the variance of detrended relative money supplies." Discuss.

Answer

(a)

$$s_t = \frac{m_t - m_t^*}{1+\alpha} + \frac{\alpha}{1+\alpha} E_t s_{t+1}.$$

(b) The method of undetermined coefficients gives:

$$s_t = \alpha\beta + \beta t + \frac{1}{1+\alpha}\epsilon_t.$$

(c) Here the variance of the detrended exchange rate is

$$\frac{\sigma^2}{(1+\alpha)^2},$$

which is less than the corresponding variance in the fundamental when the theory holds. So that qualitative ratio cannot serve as a test of the theory.

24. Consider the following discrete-time version of the Dornbusch model of exchange-rate overshooting:

$$\begin{aligned} y_t &= g_t + 0.1(e_t - p_t) \\ m_t - p_t &= -0.5 R_t \\ R_t &= R_t^* + \mathrm{E}(e_{t+1} - e_t) \\ E(e_{t+1} - e_t) &= \theta(\overline{e} - e_t) \\ p_t - p_{t-1} &= 0.3(y_{t-1} - \overline{y}) \end{aligned}$$

(a) Suppose that the long-run value of the nominal exchange rate is $\overline{e} = 2$ that $R^* = .10$ and that in the long run $m = 0.95$ and $g = 1$. Find the long-run values for p and y, denoted $\overline{p}$ and $\overline{y}$.

(b) Now consider an unexpected increase in the money supply to a new value of 1.2, starting from the steady state of part (a). Suppose that $\theta = 0.5$. In a graph or table describe the subsequent time paths for the nominal exchange rate, the real exchange rate, and output. Remember that $\overline{e}$ changes.

(c) How would different value of θ affect (qualitatively) the outcomes in part (b)?

25. Volatile nominal exchange rates need not reflect volatile domestic monetary policy. Consider the following model of a small, open economy:

$$y_t = g_t + \eta(e_t - p_t)$$
$$m_t - p_t = -\lambda R_t$$
$$R_t = R_t^* + \mathrm{E}_t(e_{t+1} - e_t)$$
$$p_t - p_{t-1} = \gamma y_t$$

Here y_t is log domestic output, g_t is log government spending, e_t is the log nominal exchange rate, p_t is the log price level, R_t is the interest rate, and R_t^* is the foreign interest rate. The parameters η, λ, and γ are positive, and R^*, m, and g are exogenous.

(a) In long-run equilibrium suppose that $R = R^*$ and $p_t = p_{t-1} = \overline{p}$. In long-run equilibrium suppose that exogenous variables are constant at $\overline{m}$, $\overline{g}$. Solve for the long-run values of p and e (denoted $\overline{p}$ and $\overline{e}$), given exogenous variables.

(b) To study the very short-run properties of this economy, suppose that prices are fixed and that one can model expectations using your answer to part (a). Specifically, suppose that

$$\mathrm{E}_t(e_{t+1} - e_t) = \theta(\overline{e} - e_t)$$

where $\theta > 0$, and $\overline{e}$ is your answer to part (a), written in terms of exogenous variables. Use this substitution and the fixity of prices to find the short-run reduced form equation for the exchange rate, given the price level. Recall that the goods market is in disequilibrium.

(c) Find the short and long-run effects on the nominal exchange rate of a permanent change in the world interest rate R^*. If we observed exchange-rate overshooting would that necessarily imply that monetary policy had changed?

Answer

(a) $p_t - p_{t-1} = 0$. So $y_t = 0$, and $p = m + \lambda R^*$. And $e = p - g/\eta$, so $e = m + \lambda R^* - g/\eta$.

(b) The system now is: $0 = g_t + \eta(e_t - p_t)$ But there is goods market disequilibrium, so ignore this equation, cannot determine p_t. Use these two equations:

$$m_t - p_t = -\lambda Rt$$
$$R_t = R_t^* + \theta(\overline{e} - e_t)$$

Combine these:

$$R_t = p_t/\lambda - m_t/\lambda$$
$$e_t = R_t^*/\theta + m_t/\lambda\theta - p_t/\lambda\theta + m + \lambda R^* - g/\eta$$

(c) Short run effect of a permanent change in R^* is $1/\theta + \lambda$. Long run effect is λ. So overshooting is induced by this as well.

26. Consider the following discrete-time version of the Dornbusch model of exchange-rate overshooting:

$$y_t = g_t + 0.1(e_t - p_t)$$

$$m_t - p_t = -0.5R_t$$

$$R_t = R_t^* + E(e_{t+1} - e_t)$$

$$E(e_{t+1} - e_t) = \theta(\bar{e} - e_t)$$

$$p_t - p_{t-1} = 0.3(y_{t-1} - \bar{y})$$

(a) Suppose that the long-run value of the nominal exchange rate is $\bar{e} = 2$, that $R^* = .10$ and that in the long run $\bar{m} = 0.95$ and $\bar{g} = 1$. Find the long-run values for p and y, denoted $\bar{p}$ and $\bar{y}$.

(b) Now consider an unexpected increase in the money supply to a new value of 1.2, starting from the steady state of part (a). Suppose that $\theta = 0.5$. In a graph or table describe the subsequent time paths for the nominal exchange rate, the real exchange rate, and output. Remember that $\bar{e}$ changes.

(c) How would different value of θ affect (qualitatively) the outcomes in part (b)?

27. In this question we shall examine the usefulness of the Cagan model of money-demand when money growth is increasing (the money stock is accelerating) as in a hyperinflation. Suppose that money-demand is given by:

$$m_t - p_t = -\alpha(p_{t+1} - p_t),$$

where m_t is the log of the money supply, p_t is the log of the price level, $\alpha > 0$, and there is perfect foresight (just for simplicity here). Call the inflation rate $\pi_t = p_t - p_{t-1}$. Suppose further that the growth rate of the money stock is denoted $\Delta log M_t = m_t - m_{t-1} = g_t$. Finally, assume that

$$g_t = (1 + \mu)g_{t-1},$$

with $\alpha\mu < 1$.

(a) Solve for the price level in terms of future money supplies, with a transversality condition so that there are no bubbles allowed.

(b) During the German hyperinflation, the central bank argued that because $\pi_t > g_t$ the expansion of the money supply could not be causing the inflation. Assess this view by finding the relation between π_t and g_t in the model given here.

Answer

(a)

$$p_t = \frac{1}{1+\alpha}[m_t + (\frac{\alpha}{1+\alpha})m_{t+1} + \ldots]$$

(b)

$$\pi_t = \frac{g_t}{1-\alpha\mu}$$

so that the model is perfectly consistent with $\pi_t > g_t$ and the central bank's view may have been wrong.

We do not observe the seigniorage maximizing g perhaps because of bubbles; expectational errors which the monetary authority tries to exploit (and empirically we know the growth rate regression has much autocorrelation); other sources of revenue (but inflation erodes those in fact, thus either they did not max S or the model is wrong). So the Cagan models seems a reasonable guide but max S does not, to actual events. The central bank may have had a 'needs of trade' view. The model could be wrong too; matching the property in part (b) does not mean that the model is correct in all respects.

28. Recent disinflation in Canada has brought about so-called monetary 're-entry'. The idea is that lower inflation reduces the opportunity cost of holding money so that velocity declines and rapid growth in nominal monetary aggregates (relative to growth in the price level) may be observed. We can describe this simply with the Cagan model of hyperinflations. Suppose that

$$m_t - p_t = -\alpha(E_t p_{t+1} - p_t)$$

and that agents have rational expectations. Moreover, suppose that

$$m_t = (1+\mu)m_{t-1} + \epsilon_t$$

where ϵ is unpredictable and has zero mean.

(a) Solve for the log price level p_t in terms of the current log money supply m_t.

(b) What is the inflation rate?

(c) Show the long-run effect of a permanent, unexpected decline in μ, the monetary growth rate, on log real balances $m_t - p_t$

Answer

(a) By the method of undetermined coefficients:

$$p_t = \frac{m_t}{1-\alpha\mu}$$

(b) The inflation rate is μ.

(c)

$$m_t - p_t = m_t(1 - \frac{1}{1-\alpha\mu})$$

so a decline in μ raises real balances. Or see the original equation plus the answer to (b).

29. Suppose that money demand is given by:

$$m_t - p_t = -0.5\mathrm{E}_t(p_{t+1} - p_t),$$

where m is the log of the money supply and p is the log of the price level. Suppose that

$$\Delta m_t = 10 + \Delta m_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim \mathrm{iid}(0, \sigma^2)$.

(a) Solve for the current inflation rate in terms of the money growth rate.

(b) Suppose that the central bank credibly announces at t that at $t+4$ the money supply process will shift to

$$\Delta m_t = \Delta m_{t-1} + \epsilon_t.$$

Describe the effect on inflation.

(c) Thus describe the path of real balances.

Answer

(a) The method of undetermined coefficients gives:

$$p_t = 7.5 + 1.5m_t - 0.5m_{t-1},$$

so that

$$\Delta p_t = 1.5\Delta m_t - 0.5\Delta m_{t-1}.$$

(b) Inflation jumps down. Then it gradually falls, smoothly pasting to the new average rate of Δm_t.

(c) During the transition real balances rise, because inflation rises more slowly than money growth does. Then at $t+4$ real balances jump down as Δm does.

30. This question uses a simple-money demand function to predict the inflation rate when the money supply growth rate declines over time. Suppose that the money-demand equation is:

$$m_t - p_t = -\alpha(E_t p_{t+1} - p_t),$$

and that the central bank sets the growth rate of the money supply so that

$$\Delta m_t = 5.0 - 0.10 \cdot t.$$

(a) What is $E_t m_{t+1}$?

(b) Use the method of undetermined coefficients to find an expression for the price level. [Hint: Guess that $p_t = a + bm_t + ct$.]

Answer

(a)

$$E_t m_{t+1} = 5.0 + m_t - 0.10(t+1)$$

(b)

$$p_t = (4.9\alpha - 0.10\alpha^2) + m_t - 0.10\alpha t$$

31. To study inflation (denoted π_t) and money growth (denoted x_t) in hyperinflations, economists often use a relationship like:

$$x_t - \pi_t = -0.5(E_t \pi_{t+1} - \pi_t).$$

Suppose that the current money growth rate is $x_t = 100$. Next year the possible values are $x_{t+1} = 100$, with probability λ and $x_{t+1} = 0$ with probability $1 - \lambda$. From year to year the money growth rate remains at 100 with probability λ, but once it switches to 0 it never switches back to 100.

(a) Solve for $E_t x_{t+i}$ for any $i > 0$.

(b) Solve for the inflation rate as a function of λ, while the hyperinflation persists.

(c) Does historical evidence suggest that expecting an end to rapid money growth can lower the current inflation rate?

Answer

(a)

$$E_t x_{t+i} = \lambda^i (100)$$

(b) Using the method of undetermined coefficients gives:

$$\pi_t = \frac{2}{3-\lambda} x_t = \frac{200}{3-\lambda}.$$

(c) According to Sargent, during interwar hyperinflations the rate of inflation began to decrease even before the money growth rate did, when reforms were announced. But these were fiscal reforms, and not just money growth rate announcements.

32. Some commentators have observed that the real money supply has been growing rapidly in Canada. This question studies the predictions of the Cagan model for this variable. Suppose that log money m_t and log prices p_t are related according to:

$$m_t - p_t = -\alpha[E_t p_{t+1} - p_t].$$

(a) Imagine that the money supply is growing according to

$$m_t = 0.10 + m_{t-1} + \epsilon_t$$

where ϵ_t is an unforecastable error term. Solve for the inflation rate.

(b) Now suppose that the central bank unexpectedly changes the process for money to

$$m_t = 0.05 + m_{t-1} + \epsilon_t.$$

Describe (perhaps using a diagram) what happens to real balances.

(c) How does your description of the path of real balances change if the policy change is announced in advance?

Answer

(a) The method of undetermined coefficients gives

$$p_t = \mu\alpha + m_t,$$

so the inflation rate is equal to the money growth rate: 10 percent.

(b) From the solution, p equals m plus a constant which now is a smaller number. So for a time in transition m must grow faster than p so real balances will rise.

(c) If the policy is announced in advance, then real balances will begin to rise before the policy takes effect, because prices reflect expected future money growth.

33. Suppose that you plan to study data from a high-inflation country using the Cagan money-demand function:

$$m_t - p_t = -\alpha E_t(p_{t+1} - p_t),$$

where m is the log of the money supply and p is the log of the price level. Suppose that a good forecasting model for the money supply is:

$$\Delta m_t = \rho \Delta m_{t-1} + \epsilon_t,$$

where ϵ_t is white noise.

(a) Outline the rational expectations cross-equation restrictions which could be used to test this model.

(b) What would an inflationary bubble look like? Can we test for such a bubble?

Answer

(a)

$$p_t = \frac{1}{1 + \alpha - \alpha\rho}[(1 + \alpha)m_t - \alpha\rho m_{t-1}]$$

Thus the two equations can serve to identify α and ρ. There are no overidentifying restrictions, but we could test to see if $\alpha > 0$.

(b) A bubble in inflation would be an additive term:

$$b_t = \left(\frac{1+\alpha}{\alpha}\right)^t.$$

(Stochastic bubbles with this same average behaviour also are possible.) We could test for this by seeing whether there is an explosive trend in inflation that is not present in money growth. But misspecification of the money-growth forecasts might sometimes make it falsely seem that a bubble is present.

34. This question uses the Cagan money demand function to study the dynamics of money and prices. Let m_t denote the logarithm of the money stock and p_t the log of the price level. The demand for money is given by:

$$m_t - p_t = -0.5(E_t p_{t+1} - p_t).$$

The money supply evolves this way:

$$m_t = 0.3 + 0.8 m_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim IIN(0,1)$ so that $E_{t-1}\epsilon_t = 0$.

(a) What is the nominal interest elasticity of money demand?

(b) Is the series $\{m_t\}$ stationary? If it is, how could you prove this?

(c) Solve for the price level.

(d) Suppose that a forecaster tries to predict the price level using lagged information on money:

$$p_t = \beta_0 + \beta_1 m_{t-1} + \eta_t.$$

If the theory you used in parts (a)-(c) is correct then what should be the values of the parameters (including the residual variance) in this statistical model?

Answer (a) 0.5

(b) It is stationary since ϵ is stationary and the root of the difference equation is less than one in absolute value. You could prove this by solving backwards and finding the unconditional moments.

(c) The method of undetermined coefficients gives:

$$p_t = 0.136 + 0.909 m_t.$$

(d) Substituting the law of motion for m in the answer in (c) gives:

$$p_t = 0.409 + 0.727 m_{t-1} + 0.909\epsilon_t.$$

Notice that the error variance is 0.826 and that this residual is perfectly correlated with the residual in the money equation. Also notice that if we regressed p_t on p_{t-1} we would not find a separate role for m_{t-1} because the two are deterministically related. This question gives an example of rational expectations cross-equation restrictions.